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Energy transfer between two beams in writing a reflection volume hologram in a dynamic medium

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A theoretical study is presented of the stationary energy transfer between two beams in the writing of a reflection hologram in a dynamic medium. The intensities of the two beams both inside the medium and at the surfaces are determined as a function of absorption coefficient, the effective coupling constant, the initial intensity ratio and the thickness of the medium. Numerical results obtained from the computer calculations are presented in graphical form.

1. Introduction

The phenomenon of energy transfer (beam coupling) between two beams in writing a volume hologram in photorefractive crystals has been known for some time. It was first observed by Staebler and Amodei [1] in the early seventies. Recent years have seen an increasing interest and intensive studies of this phenomenon [2–8] and, in particular, the closely related field of the generation of a phase-conjugate wave through so-called four-wave mixing (FWM). For instance, Huignard and Marrakchi [7] have succeeded in signal beam amplification in $Bi_{12}SiO_{20}$(BSO) crystals through the energy transfer between two interfering beams. In their two-wave mixing experiments, an additional $\pi/2$ phase shift was introduced by moving the crystal or the interference fringes at constant speed, and a high external electric field ($> 8\,\mathrm{kV\,cm^{-1}}$) was used to obtain signal beam amplification. It was shown that the phase-shift of the holographic grating (HG) with respect to the fringe pattern (FP) is responsible for the energy transfer. When the phase shift ψ is equal to $\pi/2$ due to the diffusion field, the energy transfer will reach its maximum [1].

However, in most theoretical and experimental investigations of the beam-coupling [2–8], only transmission geometry was used and very few works have dealt with a reflection geometry. The work of Kukhtarev and Odulov [9] on four-wave mixing in electro-optic crystals includes both the transmission and reflection geometries, based on a dynamic theory [5]. Their theory is also applicable to the two-wave mixing provided only one pump beam exists. In their treatment, the absorption in the crystal was ignored and the condition of non-depleting pump beams was assumed. However, in practice, the absorption may not be ignored in highly sensitive photorefractive crystals such as BSO and $Bi_{12}GeO_{20}$(BGO) and the condition of non-depletion may not always be met.

In this paper, theoretical work on the stationary energy transfer [3] between two beams in writing a reflection volume hologram in a dynamic medium is reported. The work was stimulated by the generation of the phase-conjugate beam through a reflection volume hologram formed in BGO crystals in the FWM experiment in which reasonably high wavefront reflectivity was obtained [10].

When the absorption is taken into account the mathematics involved in solving the coupled wave equations in a reflection hologram is much more complicated than that in a transmission hologram [7], because we are now dealing with a two-point boundary condition (bc) and the simple and useful relation

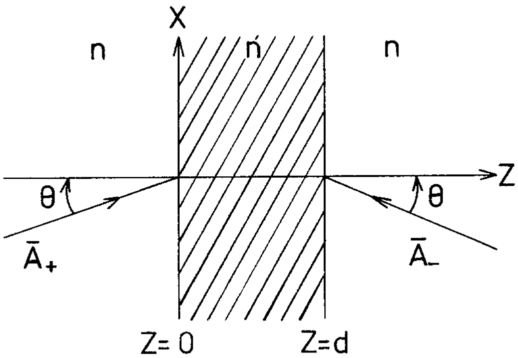


Figure 1 Two-wave mixing configuration for a reflection geometry. The shadow area denotes the dynamic medium.

of energy conservation of the wave intensities, such as that given by Huignard and Marrakchi [7] does not exist. In our studies, it has been found that, in general, numerical methods have to be used to obtain the solutions of the coupled wave equations.

2. Theory

The two-wave mixing configuration in a reflection geometry is shown in Fig. 1. The two plane waves $\bar{A}_+$ (reference) and $\bar{A}_-$ (signal) impinge on the dynamic medium from the opposite sides. For simplicity, we assume that the incident plane is in the X–Z plane to which the polarization vectors of the light are perpendicular (H-mode) and $\bar{A}_+$ and $\bar{A}_-$ both make an angle θ with the Z-axis and that the average refractive index n inside or outside the medium is the same.

In order to shed some light onto this problem, we first ignore the absorption inside the medium. Based on the coupled wave theory [2, 11], and the dynamic theory [5], the coupled wave equations are readily derived as follows:

$$\frac{dI_-}{dZ} + 2g\frac{I_-I_+}{I_- + I_+} = 0 \tag{1a}$$

$$\frac{dI_+}{dZ} + 2g\frac{I_-I_+}{I_- + I_+} = 0 \tag{1b}$$

where $g = \Gamma \sin \psi$,

$$\Gamma = \frac{\pi n^3 \gamma (A^2 + B^2)^{1/2}}{\lambda \cos \theta} \tag{1c}$$

$$\psi = \tan^{-1}(B/A) \tag{1d}$$

where Γ is the coupling constant, ψ the phase shift of HG with respect to FP [2, 5, 7], λ the light wavelength in free space, A and B the unshifted and the $\pi/2$ shifted components of the electric field grating [5], I_- and I_+ the intensities of the signal and reference waves, respectively, and γ the appropriate electro-optic coefficient. From Equation 1 it is obvious that parameter g is proportional to the $\pi/2$ shifted component of the refractive index modulation [7], which is responsible for the energy transfer, and it may be termed the effective coupling constant.

Subtracting Equation 1a from Equation 1b and integrating we have

$$I_+ - I_- = \Delta \equiv \text{constant}. \tag{2}$$

Equation 2 indicates that in a reflection geometry the relation of energy conservation of the wave intensities is different from that in a transmission geometry (where $I_+ + I_- \equiv$ constant) because in the former case the energy flow of the waves $\bar{A}_+$ and $\bar{A}_-$ is along opposite directions.

Adding Equation 1a to Equation 1b and using Equation 2, the resultant differential equation is readily solved and after using the following bc at $Z = 0$

$$\frac{I_-(0)}{I_+(0)} = \frac{I_{-0}}{I_{+0}} = M_0 \tag{3}$$

we obtain

$$I_-(Z) = \tfrac{1}{2}I_+(0)\{[1 + 2M_0(2e^{-2gZ} - 1) + M_0^2]^{1/2} - (1 - M_0)\} \tag{4a}$$

$$I_+(Z) = \tfrac{1}{2}I_+(0)\{[1 + 2M_0(2e^{-2gZ} - 1) + M_0^2]^{1/2} + (1 - M_0)\}. \tag{4b}$$

where M_0 is the resultant beam ratio at the boundary $Z = 0$.
For the two-point bc, $I_+(0)$ and $I_-(d)$ are given, consequently the coefficient M_0 in Equation 3 is still unknown. Using the bc at $Z = d$ and Equation 4a, it yields

$$M = \frac{I_-(d)}{I_+(0)} = \tfrac{1}{2}\{[1 + 2M_0(2e^{-2gd} - 1) + M_0^2]^{1/2} - (1 - M_0)\}. \tag{5}$$

From Equation 5, we have

$$M_0 = \frac{M(M+1)}{M + e^{-2gd}}. \tag{6}$$

where M is the initial intensity ratio of the two beams.
An inspection of Equation 6 indicates that when $g > 0$, then $M_0 > M$, the energy transfers from $\bar{A}_+$ to $\bar{A}_-$, and when $g < 0$, then $M_0 < M$, the direction of energy transfer is reversed and this is independent of M. That is, the direction of energy transfer depends on the direction of the phase shift of HG and the sign of the change of the refractive index. Thus, the energy can even transfer from the weaker beam to the stronger beam.

Comparing Equations 4 and 6 with Equation 9 in [7], with the absorption coefficient $\alpha = 0$ for a transmission geometry, one can see that the relations in a reflection geometry are much more complicated.

Regarding the phases of the $\bar{A}_+$ and $\bar{A}_-$ waves, similarly we have

$$\frac{d\phi_-}{dZ} + \frac{\Gamma \cos \psi I_+}{I_+ + I_-} = 0 \tag{7a}$$

$$\frac{d\phi_+}{dZ} - \frac{\Gamma \cos \psi I_-}{I_+ + I_-} = 0. \tag{7b}$$

Subtracting Equation 7a from Equation 7b and using the bc

$$\phi(0) = \phi_+(0) - \phi_-(0) = \phi_0,$$

we obtain

$$\phi(Z) = \phi_+(Z) - \phi_-(Z) = \phi_0 + \Gamma \cos \psi Z. \tag{8}$$

It can be seen from Equation 8 that the phase difference between the two interfering waves changes linearly with Z and depends on the unshifted components of the grating. For a reflection grating formed inside a photorefractive crystal such as BGO when no external electric field is applied [10], $\Gamma \cos \psi = 0$ then $\phi(Z) \equiv \phi_0$.

When $\alpha \neq 0$, the coupled wave equations become

$$\frac{dI_+}{dZ} + \alpha I_+ + 2g\frac{I_+ I_-}{I_+ + I_-} = 0 \tag{9a}$$

$$\frac{dI_-}{dZ} - \alpha I_- + 2g\frac{I_+ I_-}{I_+ + I_-} = 0 \tag{9b}$$

where $\alpha = \alpha_i/\cos\theta$ and α_i is the intensity absorption coefficient.

Notice the different signs in αI_+ and αI_-, because $\bar{A}_+$ and $\bar{A}_-$ attenuate along opposite directions due to the absorption inside the medium. Subtracting Equation 9b from Equation 9a and denoting

$\Sigma = I_+ + I_-$ and using Equation 2, it follows that:

$$\frac{d\Delta}{dZ} + \alpha\Sigma = 0. \tag{10}$$

Adding Equations 9b and a we have

$$\frac{d\Sigma}{dZ} + \alpha\Delta + g\frac{(\Sigma^2 - \Delta^2)}{\Sigma} = 0. \tag{11}$$

Multiplying Equation 11 by Σ and using Equation 10, Equation 11 becomes

$$\frac{d(\Sigma^2 - \Delta^2)}{dZ} = -2g(\Sigma^2 - \Delta^2). \tag{12}$$

After integration and using the bc at $Z = 0$, we obtain

$$I_+I_- = M_0 I_+^2(0)\, e^{-2gZ} = M_0 I_{+0}^2\, e^{-2gZ}. \tag{13}$$

Multiplying Equations 9a and b by I_- and I_+, respectively, and then subtracting one from the other, we have

$$\frac{1+Q}{2Q^2(\alpha+g) + 2Q(\alpha-g)}\, dQ = dZ \tag{14a}$$

where

$$Q = I_-/I_+. \tag{14b}$$

Depending on the relative values of α and g, the solutions of Equation 14 have different forms as follows:

(a) for $|g| \neq \alpha$ (the general case)

$$\ln\left\{\left[\frac{(\alpha-g)+(\alpha+g)Q}{(\alpha-g)+(\alpha+g)M_0}\right]^{-2g}\left(\frac{Q}{M_0}\right)^{\alpha+g}\right\} = 2(\alpha^2 - g^2)Z \tag{15a}$$

except at the singular points where $\alpha - g + (\alpha + g)M_0 = 0$ (solutions at these singular points can be obtained using the continuity of the function M_0).

(b) for $g = \alpha$

$$\ln\left(\frac{Q}{M_0}\right) = 4gZ + \frac{1}{Q} - \frac{1}{M_0} \tag{15b}$$

(c) for $g = -\alpha$

$$\ln\left(\frac{Q}{M_0}\right) = -4gZ - Q + M_0. \tag{15c}$$

Applying the bc at $Z = d$ to Equations 13 and 15, we then obtain

(a) for $|g| \neq \alpha$

$$\ln\left\{\left[\frac{(\alpha-g)M_0 + (\alpha+g)M^2\, e^{2gd}}{(\alpha-g)M_0 + (\alpha+g)M_0^2}\right]^{-2g}\left[\frac{M\, e^{gd}}{M_0}\right]^{2(\alpha+g)}\right\} = 2(\alpha^2 - g^2)d \tag{16a}$$

(b) for $g = \alpha$

$$2\ln\left(\frac{M}{M_0}\right) = 2gd - \frac{1}{M_0} + \frac{M_0\, e^{-2gd}}{M^2} \tag{16b}$$

(c) for $g = -\alpha$

$$2\ln\left(\frac{M}{M_0}\right) = -6gd + M_0 - \frac{M^2\, e^{2gd}}{M_0}. \tag{16c}$$

Equation 16 establishes the relation between the unknown M_0 and the known M. Unfortunately Equations 15 and 16 are nonlinear equations and in general an analytical solution is difficult to obtain and a numerical method has to be used. Using a computer the solutions $I_+(Z)$ and $I_-(Z)$ are easily obtained. The procedure is as follows: (a) first solve Equation 16 to obtain the value of M_0 from the given values of α, g, d and M; and (b) substitute the obtained value of M_0 into Equations 15 and 13, and

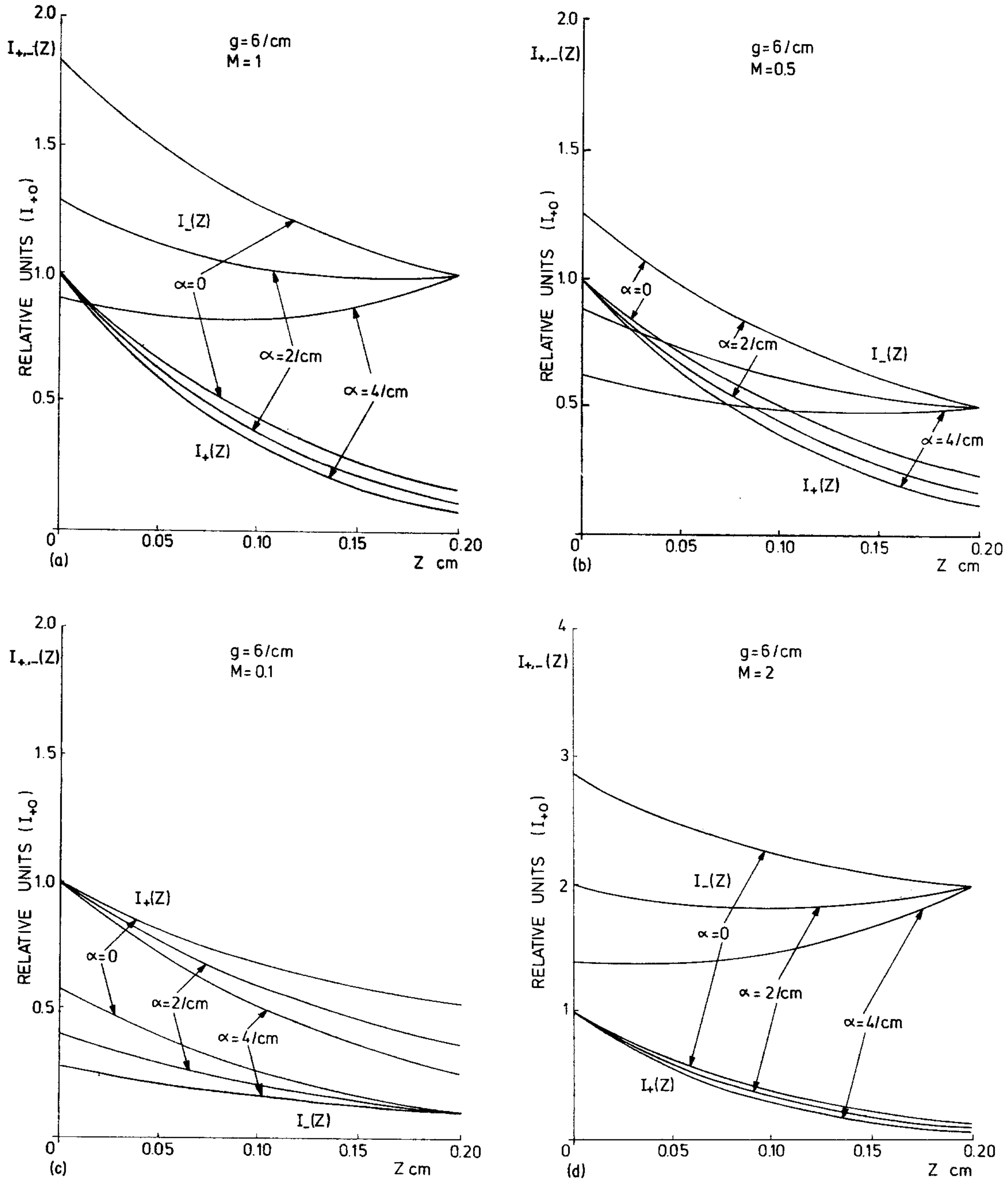
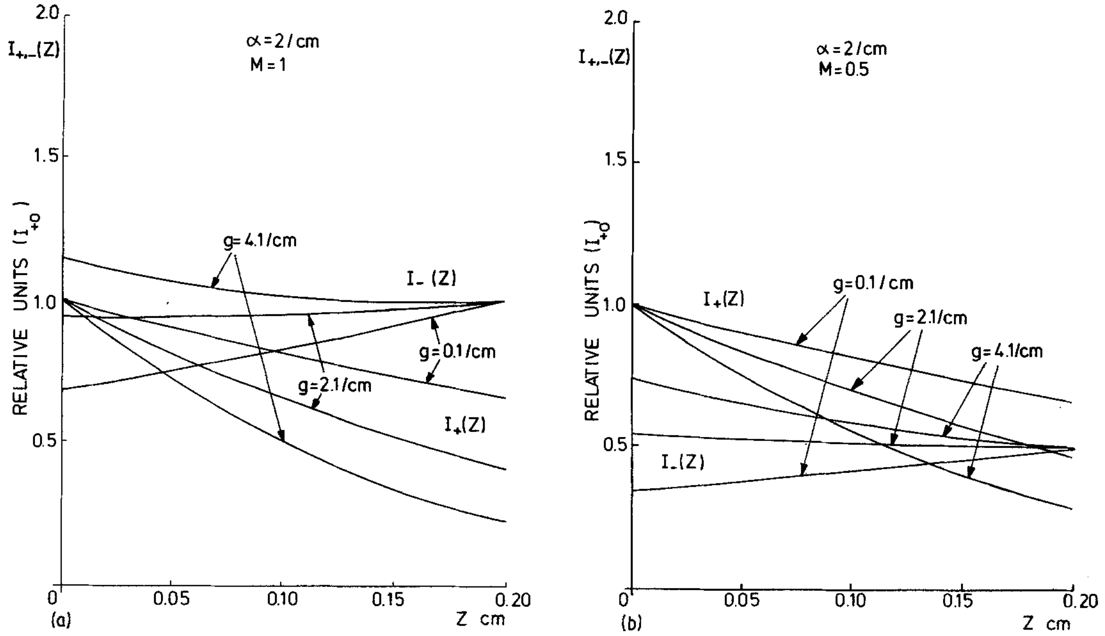


Figure 2 Intensity of reference and signal wave $I_+(Z)$ and $I_-(Z)$ as a function of Z, depth in the medium with their initial intensity ratio M and the actual absorption coefficient α as parameters. Thickness of the medium $d = 0.2$ cm and the effective coupling constant $g = 6\,\text{cm}^{-1}$.

use Equation 14b then solve them to obtain $I_-(Z)$ and $I_+(Z)$. Note that when $\alpha = 0$, Equation 16a reduces to Equation 6 and so forth.

The absorption in the medium will not affect the phase difference ϕ in Equation 8 between $\bar{A}_+$ and $\bar{A}_-$, because the absorption has no contribution to the imaginary part of the nonlinear wave equation which determines the phase difference ϕ.



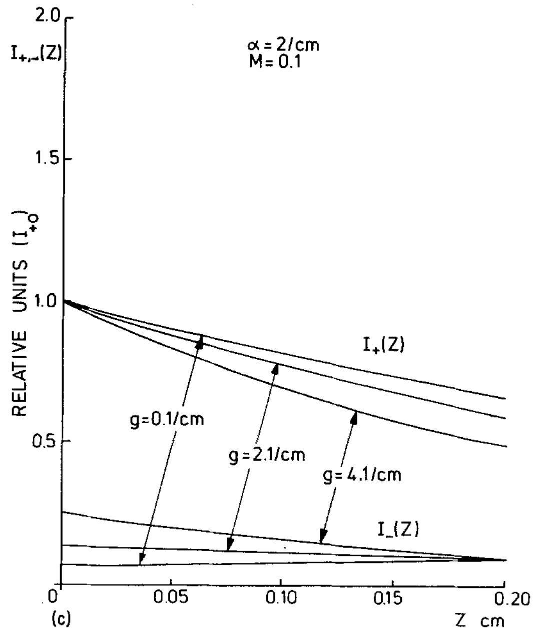


Figure 3 $I_+(Z)$ and $I_-(Z)$ as a function of Z with M and g as parameters, and $d = 0.2$ cm and $\alpha = 2\,\text{cm}^{-1}$.

3. Computed results and discussions

Using Equations 15 and 16, the intensities of the signal and the reference wave $I_-(Z)$ and $I_+(Z)$ at different depths inside the medium are obtained and plotted in Figs. 2 and 3 for a given thickness d of a medium with α the actual intensity absorption coefficient, g the effective coupling constant and M the initial intensity ratio of these two waves (in other words, the bc) as parameters. In all the figures $I_{+,-}$ is shown in the units of I_{+0}.

As a whole, one can see from Figs. 2 and 3 that

(a) the curves in Figs. 2 and 3 are similar. This indicates that the effect of α and g on $I_{+,-}(Z)$ is similar, at least to some degree.

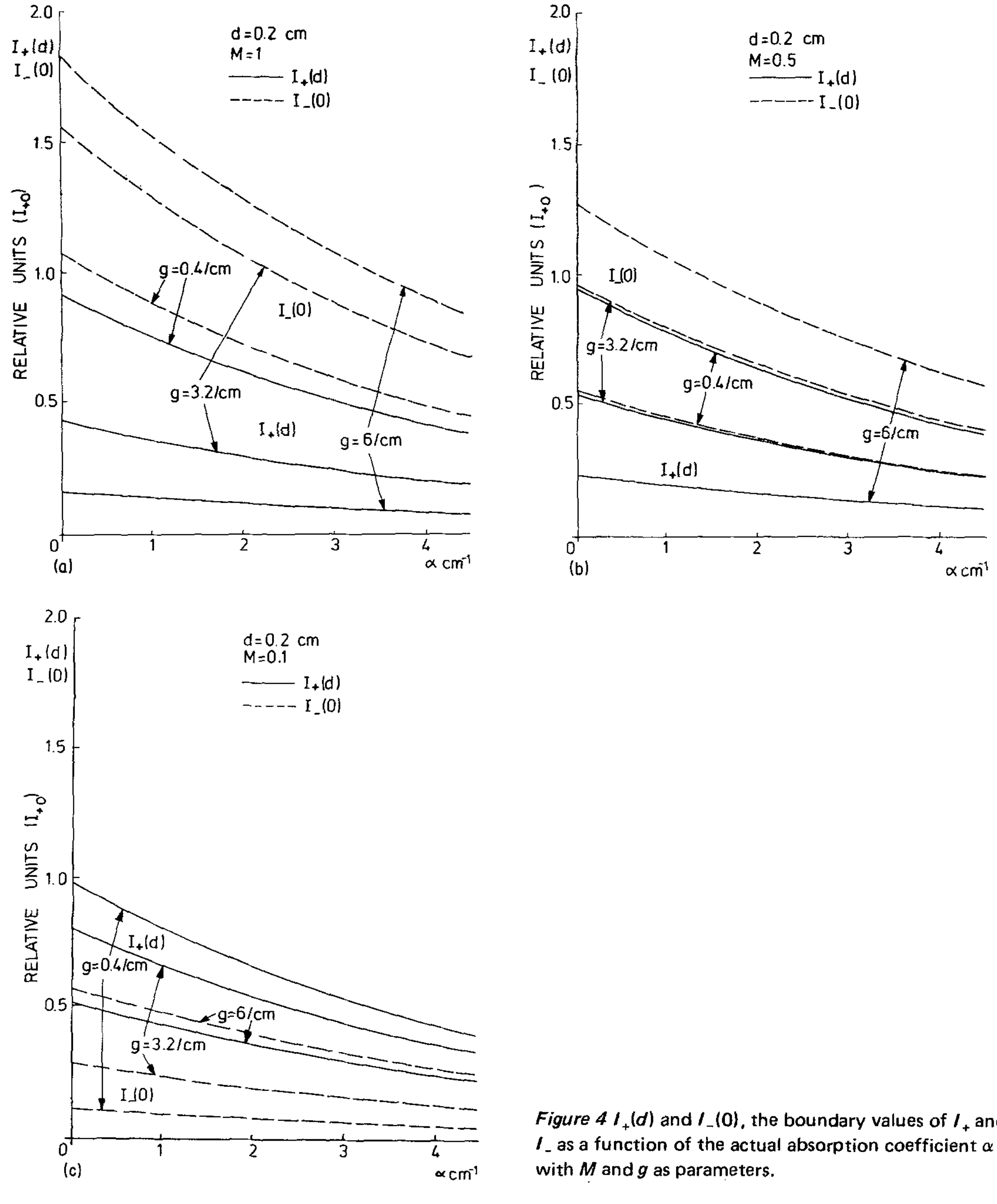


Figure 4 $I_+(d)$ and $I_-(0)$, the boundary values of I_+ and I_- as a function of the actual absorption coefficient α with M and g as parameters.

(b) When $g > 0$, energy always transfers from the reference wave $\bar{A}_+$ to the signal wave $\bar{A}_-$ regardless of whether $\bar{A}_+$ is stronger or weaker than $\bar{A}_-$. This can be clearly seen from Fig. 2, especially from Fig. 2d, in which I_+ is always smaller than I_- ($M = 2$).

(c) When g is large enough to compensate for the energy loss of the signal wave due to the absorption of the medium, say $g \gtrsim \alpha$, signal amplification is achieved as shown by most of the curves of $I_-(Z)$. The competition between these two factors is shown in most of the curves of $I_-(Z)$, in particular, the third curve of $I_-(Z)$ in Fig. 2a and the second curve of $I_-(Z)$ in Fig. 2d where $I_-(Z)$ decreases at first after entering the medium at $Z = d(= 0.2\,\text{cm})$, due to the dominant role of the absorption, then it increases

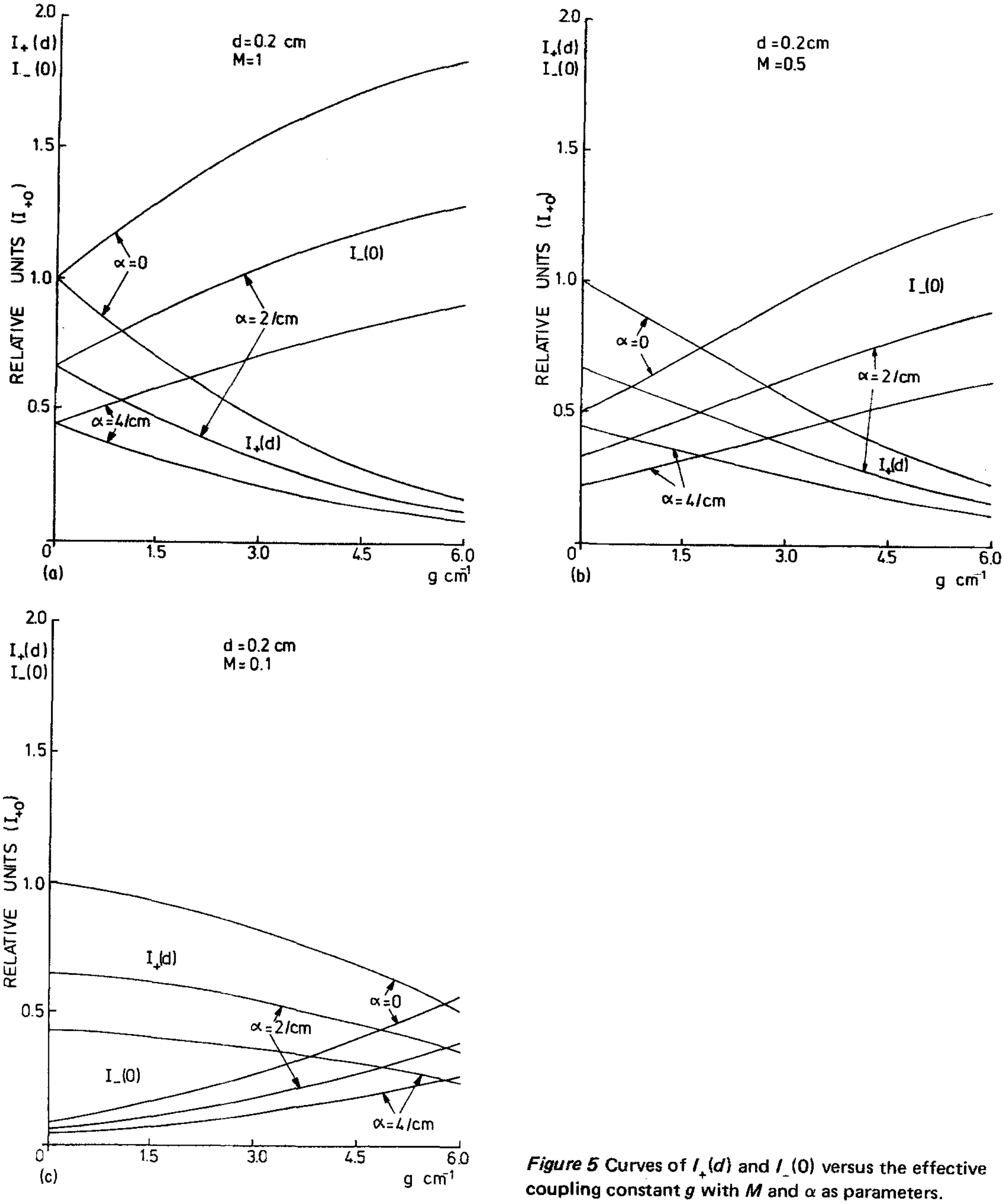


Figure 5 Curves of $I_+(d)$ and $I_-(0)$ versus the effective coupling constant g with M and α as parameters.

gradually because it obtains more energy from the reference wave $\bar{A}_+$ than that it loses due to the absorption.

(d) The effect of α is always to reduce both $I_+(Z)$ and $I_-(Z)$. Therefore it is desirable to use a medium with less absorption in order to obtain a higher net gain.

(e) When $\alpha = 0$, the relation of energy conservation (Equation 2) is clearly shown as the distance between the curves of $I_+(Z)$ and $I_-(Z)$ along the co-ordinate axis in Fig. 2 is always constant.

In Figs. 4 and 5, $I_+(d)$ and $I_-(0)$, the intensities of the reference and signal waves at the surfaces of the medium, are plotted as the functions of α or g for a given thickness d and with M, g or α as parameters.

Since $M_0 = I_-(0)/I_+(0)$ is proportional to $I_-(0)$ and is a function of M, d, g and α (see Equation 16), curves of $I_-(0)$ versus α or g also represent the dependence of M_0 on α or g.

Inspection of Figs. 4 and 5 indicates that

(a) The relationships between $I_+(d), I_-(0)$ and α or g are nonlinear.

(b) With a large g, then $I_-(0)$, the end value of I_- when emerging from the medium is larger than $I_+(d)$ in many cases even if M is quite small (e.g. curves in the middle of Fig. 4c).

(c) The value of g at which the two curves of $I_+(d)$ and $I_-(0)$ with the same value of α intersect, i.e. $I_+(d) = I_-(0)$, is independent of α (Fig. 5). This interesting feature is easily explained using Equation 13 with the condition $I_+(d) = I_-(0)$. Then we have

$$\frac{I_-(d)}{I_{+0}} = M = e^{-2gd}. \tag{17}$$

The value of g in Equation 17 obviously does not depend on α.

In deriving Equation 9 and solving it with the bc at $Z = 0$ and d, the following approximations have been introduced, i.e. neglecting higher-order waves (only the zeroth and the first order are retained), the boundary diffracted waves and the second derivatives of I_+ and I_- [11–13]. Thus, Equation 9 is not exact. In the most interesting dynamic media such as photorefractive crystals of $LiNbO_3$, $KNbO_3$, BSO and BGO etc., the absorption loss per wavelength is small ($2\pi n/\lambda \gg \alpha$) and the energy interchange (per wavelength) between the two interfering beams is slow (the exponential gains are in the order of 10/cm in $LiNbO_3$ [6] and BSO [7]). Then it may be justified to neglect the second derivatives of I_+ and I_-. However, in a recent paper in which rigorous coupled-wave analysis of planar-grating diffraction has been used, Moharam and Gaylord [13] pointed out that in a reflection grating the second derivatives of the field amplitudes and the boundary diffraction need to be included in the coupled-wave equation to produce accurate results. Although they analysed the conventional volume grating only, it is expected that their conclusion may be applied to the dynamic volume grating as well, perhaps with some modifications. Further work is being carried out in this direction. Using sensitive photorefractive BGO crystals in a reflection geometry, energy transfer between two interfering beams in a two-wave mixing experiment has been observed experimentally. Details of the experimental work and its comparison with the theoretical results given in this paper will be reported in due course.

4. Conclusion

A theoretical study of the energy transfer between two waves in writing a reflection hologram in a dynamic medium has been carried out. Differing from the case of a transmission geometry [7], numerical methods have to be used to obtain the solutions of the coupled-wave equation. It has been found that signal beam amplification is possible providing the effective coupling constant is large enough and the absorption is not too large.

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