AXISYMMETRIC DEFORMATION OF ANISOTROPIC MULTILAYER SHELLS OF REVOLUTION OF INTRICATE SHAPES

É. I. Grigolyuk and G. M. Kulikov

UDC 678.067.5:624.074

The resolvent equations in the simplest variant of the geometrically nonlinear theory of anisotropic multilayer Timoshenko-type shells are derived here, and put in a form convenient for numerical solution with the aid of a computer. An analysis of studies made on this subject can be found elsewhere [1] and, therefore, here reference will be made only to some published items not included in that survey [2-4].

Calculating the geometrical parameters of a shell of revolution whose original surface has been generated by rotation of a plane curve of an arbitrary form is a difficult and often unsolvable problem, inasmuch as the abscissas and the ordinates of points on the meridian are given approximately only. It is well known [5] that a small error of the ordinates can lead to large errors in the curvatures when the latter are calculated by interpolation or finite-difference methods. The search for a method of effectively solving this class of problems has produced the algorithm of smoothing experimental data with cubic splines [6, 7]. This method and the methodological basis of the algorithm will be covered completely in the second part of this article.

In the third part of this article some results of numerical calculations pertaining to corded-rubber shells of revolution, calculation made according to the specially developed ANSTIM program, will be discussed. This program is suitable for evaluating the combined effect of anisotropy and geometrical nonlinearity of the stress–strain state of multilayer shells of revolution with an analytically describable reference surface. The effect of anisotropy has already been analyzed within the scope of the linear theory of shells [8-11].

1. We consider a thin multilayer shell of revolution consisting of N anisotropic layers. As the reference surface we will use the inside surface of any k-th layer or the contact surface between layers, in a curvilinear orthogonal system of coordinates α_1, α_2. The transverse coordinate z will be read toward an ascending normal to the original surface. Compression of the shell across its thickness will be disregarded. Let h be the thickness of shell; h_k, thickness of the k-th layer; δ_k, distance from the reference surface to the upper boundary of the k-th layer; A_i, Lame constants; k_i, curvatures of the coordinate lines; u_i and w, respectively, tangential and the normal displacements of points on the original surface; u_i^k, tangential displacements of points in the k-th layer; β_i and μ_i, functions characterizing the transverse shear; and q, a normal load. Here and henceforth $i = 1, 2$ and $k = 1, 2, \ldots, N$.

According to the Timoshenko hypothesis about the kinematics, we have for the entire stack of layers

$$u_i^k = u_i + z\beta_i.$$

For the shearing stresses we use the independent approximation

$$\sigma_{i3}^k = f(z)\,\mu_i. \tag{1.1}$$

This approximation introduces an only formal contradiction into a Timoshenko-type theory, inasmuch as the elasticity relations for shearing stresses are satisfied here integrally over the stack thickness.

We will now proceed to the nonlinear relations for strains [12, 13]. The expressions defining the strain tensor for the k-th layer will be written in the quadratic approximation for small elongations and displacements according to the simplest variant of the nonlinear theory of shells, viz.,

---

Moscow Institute of Automotive Mechanics. Translated from Mekhanika Kompozitnykh Materialov, No. 4, pp. 637-645, July-August, 1981. Original article submitted November 17, 1980.

0191-5665/81/1704-0437$07.50

$$\varepsilon_{ii}^k = E_{ii} + zK_{ii};\quad \varepsilon_{12}^k = E_{12} + zK_{12};\quad \varepsilon_{i3} = \beta_i - \theta_i;\quad \varepsilon_{33} = 0;$$
$$E_{ii} = \varepsilon_i + \frac{1}{2}\theta_i^2;\quad E_{12} = \omega + \theta_1\theta_2, \tag{1.2}$$

where

$$\theta_1 = k_1 u_1 - \frac{1}{A_1}\frac{dw}{d\alpha_1};\quad \theta_2 = k_2 u_2;\quad \varepsilon_1 = \frac{1}{A_1}\frac{du_1}{d\alpha_1} + k_1 w;\quad \varepsilon_2 = k_2 w - \rho u_1;$$
$$\omega = \frac{1}{A_1}\frac{du_2}{d\alpha_1} + \rho u_2; \tag{1.3}$$
$$K_{11} = \frac{1}{A_1}\frac{d\beta_1}{d\alpha_1};\quad K_{22} = -\rho\beta_1;\quad K_{12} = \frac{1}{A_1}\frac{d\beta_2}{d\alpha_1} + \rho\beta_2 + \frac{k_2}{A_1}\frac{du_2}{d\alpha_1} + k_1\rho u_2;$$
$$\rho = -\frac{1}{A_1 A_2}\frac{dA_2}{d\alpha_1}.$$

The relation between stresses and strains in the k-th layer follows the generalized Hooke's law

$$\sigma_{11}^k = b_{11}^k\varepsilon_{11}^k + b_{12}^k\varepsilon_{22}^k + b_{16}^k\varepsilon_{12}^k;\quad \sigma_{22}^k = b_{12}^k\varepsilon_{11}^k + b_{22}^k\varepsilon_{22}^k + b_{26}^k\varepsilon_{12}^k;$$
$$\sigma_{12}^k = b_{16}^k\varepsilon_{11}^k + b_{26}^k\varepsilon_{22}^k + b_{66}^k\varepsilon_{12}^k. \tag{1.4}$$

Upon introducing here specific forces and specific moments [14], then integrating with relations (1.4) taken into account, we obtain

$$[T_1, T_2, S, M_1, M_2, H]^T = \begin{bmatrix} A & B \\ B & C \end{bmatrix} [E_{11}, E_{22}, E_{12}, K_{11}, K_{22}, K_{12}]^T.$$

The components of the stiffness matrix can be calculated according to the expressions

$$A_{mn} = \sum_{k=1}^{N} (\delta_k - \delta_{k-1}) b_{mn}^k;\quad B_{mn} = \frac{1}{2}\sum_{k=1}^{N} (\delta_k^2 - \delta_{k-1}^2) b_{mn}^k;$$
$$C_{mn} = \frac{1}{3}\sum_{k=1}^{N} (\delta_k^3 - \delta_{k-1}^3) b_{mn}^k \quad (m, n = 1, 2, 6). \tag{1.5}$$

The mixed variational principle yields, after standard transformations, the equations of equilibrium

$$\frac{dT_1}{dt} = \rho(T_1 - T_2) - k_1 N_1;\quad \frac{dN_1}{dt} = \rho N_1 + k_1 T_1 + k_2 T_2 - q;$$
$$\frac{dM_1}{dt} = \rho(M_1 - M_2) + Q_1;\quad \frac{dS^*}{dt} = 2\rho S^* + k_2(T_2\theta_2 + S\theta_1); \tag{1.6}$$
$$\frac{dH}{dt} = 2\rho H + Q_2;\quad N_1 = Q_1 - T_1\theta_1 - S\theta_2;\quad S^* = S + 2k_2 H;\quad \frac{d}{dt} = \frac{1}{A_1}\frac{d}{d\alpha_1}$$

and the additional relations

$$\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} (\varepsilon_{13} - a_{45}^k\sigma_{23}^k - a_{55}^k\sigma_{13}^k) f(z)\,dz = 0 \quad (1 \rightleftarrows 2;\ 4 \rightleftarrows 5), \tag{1.7}$$

characteristic of a Timoshenko-type theory. Here a_{44}^k, a_{55}^k, a_{45}^k are the elasticity constants of the k-th layer [15]. Relations (1.7) signify that the expressions for the shearing stresses in the theory of elasticity are valid integrally over the thickness of a ponderable shell f(z). We now take expression (1.2) for ε_{i3} and expression (1.1) for σ_{i3}^k, also σ_{i3}^k from expression (1.7), and after transformations obtain

$$\mu_1 = \tilde{q}_{44}(\beta_1 - \theta_1) - \tilde{q}_{45}(\beta_2 - \theta_2) \quad (1 \rightleftarrows 2;\ 4 \rightleftarrows 5);\quad \tilde{q}_{mn} = \frac{\tau\tau_{mn}}{\tau_{44}\tau_{55} - \tau_{45}^2}$$
$$(m, n = 4, 5); \tag{1.8}$$

$$\tau = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} f(z)\,dz; \quad \tau_{mn} = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} a_{mn}{}^k f^2(z)\,dz \qquad (m, n = 4, 5).$$

When the distribution of shearing stresses over the stack thickness is specified, then relations (1.8) yield the μ_i functions and thus the shearing forces $Q_i = \tau u_i$.

For a numerical solution of this problem, we rewrite expressions (1.3) for the strains in the form

$$\frac{du_1}{dt} = E_{11} - k_1 w - \frac{1}{2}\theta_1^2; \quad \frac{dw}{dt} = k_1 u_1 - \theta_1; \quad \frac{d\beta_1}{dt} = K_{11}; \quad \frac{du_2}{dt} = E_{12} - \rho u_2 - \theta_1\theta_2;$$
$$\frac{d\varepsilon_{23}}{dt} = K_{12} - \rho\varepsilon_{23} - 2k_2(E_{12} - \theta_1\theta_2). \qquad (1.9)$$

We have thus obtained a resolvent system of ten nonlinear ordinary first-order differential equations (1.6) and (1.9). The canonical systems of differential equation will be supplemented with boundary conditions, five at each end of the shell: $T_1 = T^*_1$ or $u_1 = 0$, $N_1 = Q^*_1$ or $w = 0$, $M_1 = M^*_1$ or $\beta_1 = 0$, $S^* = T^*_{12}$ or $u_2 = 0$, and $H = M^*_{12}$ or $\varepsilon_{23} = 0$.

The thus-constructed Timoshenko-type theory of anisotropic multilayer shells allows for a natural transition to the classical theory. Letting $\beta_i = \theta_i$ and disregarding the nonlinear terms in all expressions, we arrive at the well-known relations in the linear theory of anisotropic shells such as $K_{12} = 2k_2\omega$.

We now introduce the vector of solutions $Y = [T_1, N_1, M_1, S^*, H, u_1, w, \beta_1, u_2, \varepsilon_{23}]^T$. The given nonlinear boundary-value problem was numerically solved by a process of successive approximation according to the modified Newton method [16]. According to that method, the system of equations (1.6) and (1.9) can be linearized and rewritten as

$$\frac{dY^{(n+1)}}{dt} = F(t, Y^{(n)}, Y^{(n+1)}). \qquad (1.10)$$

We do not show the right-hand sides of system (1.10) and for details we refer the reader to [16], where the distinctive features have been described which characterize implementation of the algorithm of numerical solution in Kirchhoff–Love problems pertaining to the strength of orthotropic shells.

2. Splines are known to have been used principally for interpolation [6]. Interpolation splines yield excellent results when the ordinates are given with a sufficiently close accuracy. Otherwise, an interpolation should be replaced with smoothing. Here the algorithm of data smoothing with cubic splines [7] will be examined and extended to a larger class of problems.

A cubic spline for smoothing data points (t_j, y_j) on a Δt: $t_0 < t_1 < \ldots < t_n$ grid will be constructed in the conventional manner. We find a function with an absolutely continuous first derivative and summable with the square of its second derivative which minimizes the functional

$$E(s) = \int_{t_0}^{t_n} (s'')^2 dt \qquad (2.1)$$

in the class of functions satisfying the condition

$$\sum_{j=0}^{n} p_j^2 [s(t_j) - y_j]^2 \leqslant \Omega. \qquad (2.2)$$

Here $\Omega = (n + 1)\sigma^2$; and σ is the mean deviation of ordinates y_j from exact values of the function. The weights $p_j > 0$ are used as parameters with which a spline can be fixed at some given points. Usually $p_j = 1$, but larger values are assigned to the weights for points through which the spline is to pass.

It is well known from the general theory that a cubic spline minimizes the functional (2.1) under constraint (2.2) and that it is the only one that does. Inasmuch as the exact value of the parameter σ is usually not known, it is entirely permissible to require that equality (2.2) be satisfied and to replace the original problem with minimization of the functional

$$\Phi(\lambda, s)=E(s)+2\lambda\left[\sum_{j=0}^{N} p_j^2(a_j-y_j)^2-\Omega\right], \tag{2.3}$$

where $a_j = s(t_j)$ ($j = 0, 1, \ldots, n$) are unknown, for the time being, and λ is the Lagrange multiplier (smoothing parameter). Such a formulation of the problem constitutes a compromise between smoothing and approximation.

Unlike in another study [7], where the natural boundary conditions $s''(t_0 + 0) = s''(t_n - 0) = 0$ of the variational problem (2.3) were considered, we will consider the more general boundary conditions (f'_0 and f'_n are given numbers)

$$s'(t_0+0)=f'_0, \quad s'(t_n-0)=f'_n. \tag{2.4}$$

The cubic spline will be sought in the form

$$s(t)=a_l+b_l(t-t_l)+c_l(t-t_l)^2+d_l(t-t_l)^3; \quad t\in[t_l, t_{l+1}] \quad (l=0,1,\ldots,n-1). \tag{2.5}$$

The conditions of continuity of the cubic spline, its first and second derivatives at nodes of the grid, and the boundary conditions (2.4) yield

$$d_l=\frac{c_{l+1}-c_l}{3h_l}; \quad b_l=\frac{a_{l+1}-a_l}{h_l}-h_lc_l-h_l^2d_l; \quad h_l=t_{l+1}-t_l$$
$$(l=0,1,\ldots,n-1); \tag{2.6}$$

$$Tc=Qa+W; \quad W=[-f'_0, 0, \ldots, 0, f'_n]^T; \quad c=[c_0, c_1, \ldots, c_n]^T;$$
$$a=[a_0, a_1, \ldots, a_n]^T. \tag{2.7}$$

Here $2c_j = s''(t_j)$ ($j = 0, 1, \ldots, n$) and T, Q are symmetric matrices with three nonzero diagonals each: $T_{l,l+1}=T_{l+1,l}=\frac{h_l}{3}$; $Q_{l,l+1}=Q_{l+1,l}=\frac{1}{h_l}$ $(l=0,1,\ldots,n-1)$; $T_{ll}=\frac{2}{3}(h_{l-1}+h_l)$; $Q_{ll}=-\frac{1}{h_{l-1}}-\frac{1}{h_l}$ $(l=1,2,\ldots,n-1)$; $T_{00}=\frac{2}{3}h_0$; $T_{nn}=\frac{2}{3}h_{n-1}$; $Q_{00}=-\frac{1}{h_0}$; $Q_{nn}=-\frac{1}{h_{n-1}}$. Minimization of functional (2.3), with relations (2.5)-(2.7) taken into account, leads to the matrix equation

$$a=y-\lambda^{-1}P^{-2}Qc; \quad y=(y_0, y_1, \ldots, y_n)^T; \quad P=\operatorname{diag}(p_0, p_1, \ldots, p_n). \tag{2.8}$$

Inserting this equation into relations (2.7) yields the system of linear algebraic equations in the vector c

$$(\lambda T+QP^{-2}Q)c=\lambda(Qy+W). \tag{2.9}$$

For a determination of the smoothing parameter we turn to equality (2.2), the left-hand side of which will be rewritten as

$$F^2(\lambda)=(a-y)^TP^2(a-y).$$

An analysis made in another study [17] has shown that the function $G(\lambda) = 1/F(\lambda)$ is a rigorously rising and convex one, which fully justifies application of the Newton method for finding the root of the equation $G(\lambda)=\Omega^{-\frac{1}{2}}$. Iterations will be performed according to the relation

$$\lambda^{(m+1)}=\lambda^{(m)}+\frac{F(\lambda^{(m)})\left[\Omega^{1/2}-F(\lambda^{(m)})\right]}{\Omega^{1/2}F'(\lambda^{(m)})}. \tag{2.10}$$

The iteration process (2.10) is globally convergent. However, a case is possible where $F(0) \leqslant \Omega^{1/2}$. There the spline degenerates into a broken line.

We have thus constructed the algorithm of data smoothing with cubic splines. The sequence of calculations is as follows: we find the vector c from the system of linear algebraic equations (2.9), whereupon relations (2.6) and (2.8) yield the remaining coefficients (2.5) of the smoothing spline.

Let us now apply the smoothing algorithm to the problem of calculating the geometrical parameters of a shell of revolution, with the equation of its meridian written in the parametric form $x = x(t)$, $y = y(t)$. Here t is the arcuate coordinate read along the meridian

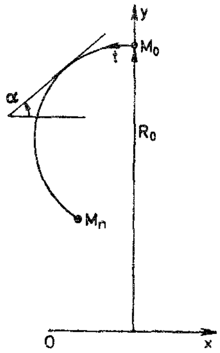

Fig. 1

(Fig. 1) from the starting point M_0 to point M_n. The derivatives are determined from the well known relations

$$\frac{dx}{dt} = -\cos\alpha; \quad \frac{dy}{dt} = -\sin\alpha, \tag{2.11}$$

where α is the angle which a tangent to the meridian forms with the axis of rotation ox.

Let the angle α_0 of the tangent at point M_0 and the angle α_n of the tangent at point M_n be given, also let the coordinates x_j, y_j (j = 0, 1, ..., n) on the Δt: $t_0 < t_1 < \ldots < t_n$ grid be given. With the aid of relations (2.11) one can easily write down the boundary conditions (2.4) needed for a numerical implementation of the algorithm. Let s(t, x) and s(t, y) be the cubic splines for smoothing the data points x_j and y_j (j = 0, 1, ..., n), respectively. Using this notation, we calculate the Lame constants and the curvatures of the coordinate lines as well as the parameter ρ (1.3) from the relations $A_1 = 1$;

$A_2 = s(t, y)$; $k_2 = -\dfrac{s'(t,x)}{A_2}$; $\rho = -\dfrac{s'(t,y)}{A_2}$; $k_1 = s'(t,x)s''(t,y) - s'(t,y)s''(t,x)$.

We will show a numerical example indicating the capabilities of the algorithm developed here. We will consider a toroidal shell whose original surface has been generated by rotation of a circle with the radius R_1 = 20 cm. The distance from the axis of rotation to the equator is R_0 = 40 cm (Fig. 1).

We use $\Delta\varphi$: 0°, 3°, ..., 117°, 120° grid where $t = \pi\varphi R_1/180$. The abscissas and the ordinates of points on the meridian, given at the nodes of this grid, will be rounded off to the first decimal figure so that the maximum deviation of the coordinates from their exact values will not exceed 0.05 cm. The angle of the tangent at the equator is $\alpha_0 = 0°$ and for the angle α_n we select several values: 120 (exact), 119, and 118°. The results of numerical calculations are given in Table 1, where the fraction of the error due to rough stipulation of the coordinates is seen to be insignificant in the first variant (α_0 = 120°). In the α_n = 118° variant, on the other hand, the approximation of the curvatures is getting somewhat worse. In all fairness, however, we must note that in practice the error of an α_n determination has never exceeded ±1° and, therefore, the data in Table 1 should be regarded as entirely satisfactory. Reducing the number of nodes to 21 gives rise to a negligible error, not larger than 5%.

It is to be noted, furthermore, that interpolation splines and finite-difference relations yield, in principle, incorrect results up to where the curvature changes sign. Attempts to use the method of least squares also produced unsatisfactory results.

3. The algorithms shown here have been implemented in the form of the ANSTIM program, suitable for analyzing the combined effect of anisotropy and geometrical nonlinearity on the state of stress and strain of multilayer shells of revolution with a meridian of an arbitrary shape. The capabilities of the ANSTIM program will be demonstrated here on the design of corded-rubber shells of revolution.

The earliest stage of studies pertaining to the theory of rubber-cord shells has been well documented in [18]. The greatest progress in the theory of corded-rubber shells has been made in applications to mechanics of pneumatic tires [19-23].

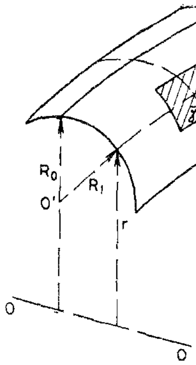

Fig. 2

TABLE 1

Variant	Parameter	φ 0°	15°	30°	45°	60°	75°	90°	105°	120°
Exact value	$10^{-1}\cdot A_2$	4,0000	3,9659	3,8660	3,7071	3,5000	3,2588	3,0000	2,7412	2,5000
	$10^2\cdot\rho$	0,0000	0,6526	1,2933	1,9074	2,4744	2,9640	3,3333	3,5238	3,4641
	$10^2\cdot k_2$	2,5000	2,4356	2,2401	1,9074	1,4286	0,7942	0,0000	−0,9442	−2,0000
	$10\cdot k_1$	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
$\alpha_n=120°$	$10^{-1}\cdot A_2$	3,9992	3,9651	3,8655	3,7069	3,4997	3,2589	3,0002	2,7424	2,5029
	$10^2\cdot\rho$	0,0000	0,6534	1,2874	1,9080	2,4719	2,9624	3,3272	3,4962	3,4601
	$10^2\cdot k_2$	2,5005	2,4203	2,2335	1,9094	1,4212	0,7956	0,0034	−0,9355	−1,9977
	$10\cdot k_1$	1,0002	0,9991	0,9860	1,0133	0,9998	1,0087	0,9894	1,0040	1,0026
$\alpha_n=119°$	$10^{-1}\cdot A_2$	3,9992	3,9651	3,8655	3,7068	3,4997	3,2589	3,0004	2,7426	2,5020
	$10^2\cdot\rho$	0,0000	0,6534	1,2875	1,9080	2,4718	2,9611	3,3245	3,4999	3,4957
	$10^2\cdot k_2$	2,5005	2,4210	2,2339	1,9097	1,4210	0,7943	−0,0007	−0,9306	−1,9377
	$10\cdot k_1$	1,0002	0,9994	0,9858	1,0137	1,0005	1,0110	0,9895	0,9755	0,9209
$\alpha_n=118°$	$10^{-1}\cdot A_2$	3,9992	3,9651	3,8655	3,7068	3,4997	3,2590	3,0005	2,7428	2,5011
	$10^2\cdot\rho$	0,0000	0,6534	1,2876	1,9080	2,4716	2,9600	3,3223	3,5041	3,5303
	$10^2\cdot k_2$	2,5005	2,4228	2,2347	1,9104	1,4205	0,7938	−0,0049	−0,9284	−1,8771
	$10\cdot k_1$	1,0002	1,0007	0,9856	1,0151	1,0015	1,0144	0,9915	0,9498	0,8344

Note. σ = 0.029 cm.

We consider a shell of revolution consisting of an even number of antisymmetric corded-rubber layers with a reference surface which cannot be described analytically. This problem is undoubtedly a very interesting one: firstly, because such shells are nowadays designed on the basis of the theory of orthotropic shells and, secondly, because no clear conception can be found in the technical literature on how the shape of the meridian should be approximated. Owing to the intricate shape of the meridian (possibly a combination of convex and concave segments), an approximation of the latter with analytical functions will appreciably distort the true state of stress and strain of the structure. The algorithm of smoothing by means of splines, which is included in the ANSTIM program, does not have these drawbacks.

Let the shell be made of 2L corded-rubber layers and let the contact surface between the layers L and L + 1 serve as the reference surface. We will assume that all layers of this shell are of the same constitution and differ only in the angles γ_k which their cord filaments form with the meridian, these angles being $\gamma_k = (-1)^k\gamma$ (k = 1, 2, ..., 2L). This arrangement is close to a real one and is used in the construction of diagonal tires [24].

The elasticity relations for the k-th corded-rubber layer will be expressed as

$$\sigma_{1'1'}{}^k = c_{11}\varepsilon_{1'1'}{}^k + c_{12}\varepsilon_{2'2'}{}^k;\quad \sigma_{2'2'}{}^k = c_{12}\varepsilon_{1'1'}{}^k + c_{22}\varepsilon_{2'2'}{}^k;\quad \sigma_{1'2'}{}^k = c_{66}\varepsilon_{1'2'}{}^k;$$
$$\varepsilon_{1'3}{}^k = G_{13}^{-1}\sigma_{1'3}{}^k;\quad \varepsilon_{2'3}{}^k = G_{23}^{-1}\sigma_{2'3}{}^k$$

(axes 1', 2', 3 running, respectively, along the cord filaments, across the cord filaments in the plane of a layer, and normally to the plane of a layer, as shown in Fig. 2). The components of the mean-stiffness matrix for a layer are

$$c_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}};\quad c_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}\cdot\quad c_{12} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}};\quad c_{66} = G_{12}.$$

TABLE 2

Variant	Parameter	φ								
		0°	15°	30°	45°	60°	75°	90°	105°	120°
Exact value	$10^{-1} \cdot T_1$	4,085	4,056	3,985	3,915	3,897	3,977	4,184	4,529	5,000
	$10^{-1} \cdot T_2$	5,669	5,447	4,859	4,089	3,325	2,676	2,179	1,832	1,619
	$10 \cdot S$	1,760	1,715	1,592	1,422	1,243	1,079	0,940	0,815	0,648
	M_1	−0,402	−0,310	−0,037	0,386	0,842	1,080	0,658	−1,267	−6,653
	M_2	−0,545	−0,404	−0,038	0,398	0,703	0,710	0,326	−0,537	−2,154
	H	−2,444	−2,401	−2,282	−2,114	−1,937	−1,785	−1,681	−1,638	−1,752
	$10 \cdot u_1$	0,000	1,360	2,243	2,408	1,925	1,089	0,269	−0,184	0,000
	$10 \cdot w$	1,942	2,510	3,981	5,727	6,958	7,007	5,594	2,977	0,000
ANSTIM	$10^{-1} \cdot T_1$	4,103	4,073	4,004	3,938	3,920	3,999	4,205	4,549	5,017
	$10^{-1} \cdot T_2$	5,697	5,473	4,884	4,113	3,344	2,692	2,191	1,843	1,629
	$10 \cdot S$	1,759	1,706	1,587	1,420	1,235	1,072	0,929	0,802	0,627
	M_1	−0,401	−0,294	−0,019	0,344	0,821	1,068	0,670	−1,250	−6,641
	M_2	−0,543	−0,382	−0,017	0,355	0,685	0,702	0,333	−0,530	−2,156
	H	−2,457	−2,413	−2,294	−2,126	−1,948	−1,795	−1,690	−1,646	−1,759
	$10 \cdot u_1$	0,000	1,314	2,178	2,337	1,853	1,052	0,247	−0,192	0,000
	$10 \cdot w$	2,099	2,664	4,074	5,720	6,930	6,975	5,581	2,977	0,000

Note. σ = 0.029 cm.

The five independent elasticity constants E_1, E_2, ν_{12}, G_{12}, and G_{23} of an elementary corded-rubber layer will be calculated according to the well known relations in the theory of reinforced plastics. Since no method based on a structural analysis of a corded-rubber layer and taking into account the constitution of such a layer as well as the mechanical properties of its components is available, we will use the relations in study [8] as the first approximation.

In rotated coordinates a corded-rubber layer has anisotropic properties and the elasticity relations (1.4) become valid. The components of the stiffness matrix for the k-th layer are, in rotated coordinates,

$$\begin{gathered} b_{11}{}^k=b_{11}=c_{11}\zeta^4+2(c_{12}+2c_{66})\xi^2\zeta^2+c_{22}\xi^4; \quad b_{22}{}^k=b_{22}=c_{11}\xi^4+2(c_{12}+ \\ +2c_{66})\xi^2\zeta^2+c_{22}\zeta^4; \quad b_{12}{}^k=b_{12}=[c_{11}+c_{22}-2(c_{12}+2c_{66})]\xi^2\zeta^2+c_{12}; \\ b_{66}{}^k=b_{66}=[c_{11}+c_{22}-2(c_{12}+2c_{66})]\xi^2\zeta^2+c_{66}; \\ b_{16}{}^k=(-1)^k b_{16}=(-1)^k[(c_{12}+2c_{66}-c_{11})\xi\zeta^3+(c_{22}-c_{12}-2c_{66})\xi^3\zeta]; \\ b_{26}{}^k=(-1)^k b_{26}=(-1)^k[(c_{12}+2c_{66}-c_{11})\xi^3\zeta+(c_{22}-c_{12}-2c_{66})\xi\zeta^3]; \\ \xi=\sin\gamma; \quad \zeta=\cos\gamma. \end{gathered} \tag{3.1}$$

The elasticity constants in relations (1.8) will be calculated from the expressions

$$\begin{gathered} a_{44}{}^k=a_{44}=G_{13}{}^{-1}\xi^2+G_{23}{}^{-1}\zeta^2; \quad a_{55}{}^k=a_{55}=G_{13}{}^{-1}\zeta^2+G_{23}{}^{-1}\xi^2; \\ a_{45}{}^k=(-1)^k a_{45}=(-1)^k(G_{23}{}^{-1}-G_{13}{}^{-1})\xi\zeta. \end{gathered} \tag{3.2}$$

Expressions (1.5) and (3.1) yield

$$\begin{gathered} (A_{11}, A_{12}, A_{22}, A_{66})=h(b_{11}, b_{12}, b_{22}, b_{66}); \\ (C_{11}, C_{12}, C_{22}, C_{66})=\frac{h^3}{12}(b_{11}, b_{12}, b_{22}, b_{66}); \quad (B_{16}, B_{26})=\frac{h^2}{4L}(b_{16}, b_{26}). \end{gathered} \tag{3.3}$$

The components of the stiffness matrix missing in expressions (3.3) are assumed to be equal to zero. We will not concretize the form of the function which characterizes the distribution of shearing stresses across the stack thickness. Let $f(z)=\frac{3}{2h}\left(1-\frac{4z^2}{h^2}\right)$. Then relations (1.8) and (3.2), together with this expression for f(z), yield $\tilde{q}_{44}=\frac{5h}{6a_{55}}$, $\tilde{q}_{55}=\frac{5h}{6a_{44}}$, and $\tilde{q}_{45}$ = 0 needed for calculation of the transverse forces.

It is obvious that the smoothing algorithm introduces some error in the determination of the state of stress and strain of anisotropic multilayer shells. For an estimate of this error we consider the circular toroidal shell whose geometry has been thoroughly analyzed here. The mechanical characteristics of this shell will be selected close to those of a factory-produced tire, namely: elasticity constants of the cord material $E_c = 10^4$ kgf/cm² and ν_c = 0.3, elasticity constants of the rubber E_r = 60 kgf/cm² and ν_r = 0.49, diameter of a cord filament d_c = 0.07 cm, thickness of an elementary corded-rubber layer h_0 = 0.12 cm, angle between a cord filament and the tire meridian at the equator γ_0 = 52°, frequency of filament around the equator i_0 = 9, number of corded-rubber layers N = 8 (L = 4), and internal pressure q = 5 kgf/cm² (Fig. 2). The expressions for the angle γ that a cord filament makes with the meridian and for the frequency i of cord filaments are [24]

$$\sin\gamma = \frac{r}{R_0}\sin\gamma_0; \quad i = i_0\frac{R_0\cos\gamma_0}{r\cos\gamma},$$

where r is the distance from the axis of rotation to the parallel of the shell on which γ and i are defined. As is customary, we assume a rigid joint at the rim.

The basic variables which determine the strain state of the shell are given in Table 2 as function of the central angle φ. They have been calculated by two different methods. First the geometrical parameters of the circular toroidal shell were defined by analytical expressions, then they were calculated according to the algorithm of smoothing splines. The input data needed for numerical calculations were taken from the example considered in the preceding (second) part of this article. Specifically, a grid with 41 nodes was used and an angle α_n = 120° was selected.

The data in Table 2 indicate that the ANSTIM program ensures an accuracy which is adequate for practical calculations and, therefore, can be used for numerical solution of more complex problems pertaining to strength of multilayer shells of revolution made of materials with highly anisotropic properties. The shape of the original shell surface does not have to be stipulated beforehand and can be entirely arbitrary.

LITERATURE CITED

1. É. I. Grigolyuk and I. T. Selezov, Nonclassical Theories of Vibration of Beams, Plates, and Shells. Progress in Science and Engineering. Mechanics of Deformable Solids [in Russian], Vol. 5, Moscow (1973).
2. V. E. Spiro, "A variant of geometrically nonlinear theory of anisotropic shells with transverse shear taken into account," Mekh. Polim., No. 5, 863-871 (1969).
3. V. N. Mal'gin, "Algorithms of solution of problems pertaining to strength and vibration of shells of revolution based on equations of the Timoshenko kind," Metody Reshen. Zadach Uprug. Plastichn., No. 7, 137-142 (1973).
4. É. V. Raman, "Nonlinear equations of statics for glass-plastic shells of revolution taking into account strain due to transverse shear," in: Trans. N. É. Bauman Moscow Higher Technical School [in Russian], No. 206, Moscow (1976), pp. 107-118.
5. N. S. Bakhvalov, Numerical Methods [in Russian], Moscow (1973).
6. S. B. Stechkin and Yu. N. Subbotin, Splines in Computer Mathematics [in Russian], Moscow (1976).
7. C. H. Reinsch, "Smoothing by spline functions," Numer. Math., 10, No. 4, 177-183 (1967).
8. A. K. Malmeister, V. P. Tamuzh, and G. A. Teters, Resistance of Rigid Polymer Materials [in Russian], 2nd ed., Riga (1972).
9. Ya. M. Grigorenko and A. T. Vasilenko, "Solution of problems pertaining to axisymmetric deformation of anisotropic multilayer shells of revolution," Prikl. Mekh., 7, No. 8, 3-8 (1971).
10. Ya. M. Grigorenko, Isotropic and Anisotropic Multilayer Variable-Stiffness Shells of Revolution [in Russian], Kiev (1973).
11. G. A. Cohen, "FASOR: A second generation shell-of-revolution code," Comput. Struct., 10, Nos. 1-2, 301-309 (1979).
12. V. V. Novozhilov, Fundamental Nonlinear Theory of Elasticity [in Russian], Leningrad (1948).
13. É. I. Grigolyuk and V. I. Mamai, "One variant of equations of the theory of finite displacements in deep shells," Prikl. Mekh., 10, No. 2, 3-13 (1974).
14. V. V. Novozhilov, Theory of Thin Shells [in Russian], Leningrad (1962).
15. P. M. Ogibalov and M. A. Koltunov, Shells and Plates [in Russian], Moscow (1969).
16. A. V. Karmishin, V. A. Lyaskovets, V. I. Myachenkov, and A. N. Frolov, Statics and Dynamics of Thin-Walled Shell Structures [in Russian], Moscow (1975).
17. C. H. Reinsch, "Smoothing by spline functions, Part 2," Numer. Math., 16, No. 5, 451-454 (1971).

18. V. L. Biderman and B. L. Bukhin, "Design of reticular zero-moment shells," in: Trans. Sixth All-Union Conf. on the Theory of Shells and Plates [in Russian], Moscow (1966), pp. 948-953.
19. H. K. Brewer, "Tire stress and deformation from composite theory," Tire. Sci. Technol., 1, No. 1, 47-76 (1973).
20. J. D. Walter, G. N. Avgeropoulos, M. L. Janssen, and G. R. Potts," Advances in tire composite theory," Tire. Sci. Technol., 1, No. 2, 210-250 (1973).
21. V. L. Biderman and M. M. Gershenzon, "Design of a pneumatic radial tire as an orthotropic triple-layer shell," Izv. Vyssh. Uchebn. Zaved., Mashinostr., No. 6, 83-87 (1979).
22. I. K. Nikolaev, "A mathematical model and a numerical method for the design of tires for nonaxisymmetric loads," in: Int. Conf. on Caoutchouc and Rubber [in Russian], Section C, No. 2, Kiev (1978).
23. Yu. K. Rasteryaev, É. N. Kvasha, and A. P. Prusakov, "Design of a large diagonal tire for internal pressure and local service loads," in: Int. Conf. on Caoutchouc and Rubber [in Russian], Section C, No. 2, Kiev (1978).
24. V. L. Biderman, R. L. Guslitser, S. P. Zakharov, B. V. Nenakhov, I. I. Seleznev, and S. M. Tsukerberg, Automobile Tires [in Russian], Moscow (1963).

STABILITY OF ORTHOTROPIC TRIPLE-LAYER SHELLS UNDER A COMPOUND LOAD

K. N. Lebedev

UDC 624.074:678.067

The stability problem is solved here for orthotropic triple-layer shells with a three-dimensional elastic filler and orthotropic carrier sheaths of an asymmetric structure under a compound load, and the results of an experimental stability study of such shells are presented.

In the derivation of the stability equations it has been assumed that the carrier sheaths are thin orthotropic shells for which the Kirchhoff—Love hypothesis holds true, their two principal axes oriented lengthwise and around the circumference, respectively. The filler is a three-dimensional elastic cylinder fastened to one carrier sheath on the outside surface and to one carrier sheath on the inside surface. Before a loss of stability occurs, the shell is in a zero-moment stress—strain state. With this arrangement the equations of stability for the carrier sheaths are

$$\frac{\partial T_{ix}}{\partial x}+\frac{\partial S_i}{\partial y}-(-1)^i q_{ix}=0;\ \frac{\partial S_i}{\partial x}+\frac{\partial T_{iy}}{\partial y}-(-1)^i q_{iy}=0;\ \frac{\partial^2 M_{ix}}{\partial x^2}+$$

$$+\frac{2\partial^2 H_i}{\partial x\partial y}+\frac{\partial M_{iy}}{\partial y^2}-\frac{T_{iy}}{R_i}+T_{ix}{}^0\frac{\partial^2 w_i}{\partial x^2}+T_{iy}{}^0\frac{\partial^2 w_i}{\partial y^2}+2S_i{}^0\frac{\partial^2 w_i}{\partial x\partial y}-(-1)^i q_{ir}=0. \tag{1}$$

Here x, y, r are, respectively, axial, circumferential, and radial coordinates; T_{ix}, T_{iy}, S_i, M_{ix}, M_{iy}, H_i, forces and moments on the i-th sheath (i = 1, outer sheath; i = 2, inner sheath); $T_{ix}{}^\circ$, $T_{iy}{}^\circ$, $S_i{}^\circ$, subcritical forces on the carrier sheaths; q_{ix}, q_{iy}, q_{ir}, contact forces between the carrier sheaths and the filler; u_i, v_i, w_i, displacements of the median surfaces of the carrier sheaths; and R_i, radius of the median surface of the i-th sheath.

Using a solution for the filler analogous to that given in another study [1] for the stability of shells with a filler under axial compression, with the components of displacements and stresses inside the filler in the solution to the three-dimensional problem in the theory of elasticity expressed through Bessel functions of the first kind and the second kind, we can obtain the resolvent system of stability equations from Eqs. (1) and the appropriate elasticity relations. Unlike in that other study [1], however, here the conditions of contact and bonding at the boundaries between the carrier sheaths and the filler will be expressed as

$$\sigma_r(R_i+a)=q_{ir};\ \tau_{rx}(R_i+a)=q_{ix};\ \tau_{ry}(R_i+a)=q_{iy}, \tag{2}$$

Translated from Mekhanika Kompozitnykh Materialov, No. 4, pp. 646-650, July-August, 1981. Original article submitted August 20, 1980.