Z. Wahrscheinlichkeitstheorie verw. Gebiete
51, 239–248 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Mixture of Independent Identically Distributed Random Variables Need Not Admit a Regular Conditional Probability Given the Exchangeable σ-Field*

David A. Freedman

Department of Statistics, University of California, Berkeley CA 94720, USA

This paper is a sequel to [1]. Let $I=[0,1]$ and $\mathscr{B}$ be the Borel σ-field in I. Let $S\subset I$, and $\mathscr{F}=S\cap\mathscr{B}$. Let S^* be the set of probabilities ϕ on $(S,\mathscr{F})$, equipped with the weak * σ-field $\mathscr{F}^*$ generated by $\{\phi\colon \phi(F)>t\}$ as F ranges over $\mathscr{F}$ and t over I. Define I^* likewise. Consider the infinite product space S^∞ equipped with the product σ-field $\mathscr{F}^\infty$. Let $\{\xi_n\}$ be coordinate process on S^∞:

$$\xi_n(x)=x_n \qquad \text{where } x=(x_1,x_2,\ldots)\in S^\infty.$$

If $\phi\in S^*$, the power probability ϕ^∞ on $(S^\infty,\mathscr{F}^\infty)$ makes the ξ_n independent, with common distribution ϕ. A probability P on $(S^\infty,\mathscr{F}^\infty)$ is said to be *presentable* if it is a mixture of power probabilities: for some probability μ on $(S^*,\mathscr{F}^*)$,

$$(1)\qquad P=\int_{S^*}\phi^\infty\,\mu(d\phi).$$

The mixing measure μ is unique: see (3.4) of [1].

A permutation π of the positive integer is *finite* if $\pi(n)=n$ for all but finitely many n. Each π induces a measurable mapping $\tilde{\pi}$ on S^∞ as follows:

$$\tilde{\pi}(x_1,x_2,\ldots)=(x_{\pi(1)},x_{\pi(2)},\ldots).$$

The *exchangeable σ-field* $\mathscr{E}_S$ in S^∞ is the collection of $A\in\mathscr{F}^\infty$ which are invariant under all $\tilde{\pi}$. The exchangeable σ-field $\mathscr{E}_I$ in I^∞ is defined the same way.

A probability P on S^∞ is said to be *exchangeable* if P is invariant under all $\tilde{\pi}$. One version of De Finetti's theorem is that for Borel sets S, all exchangeable P's are presentable. In [1], an example was given of an S and an exchangeable P on S^∞ which is not presentable. It was also noted [1, 4.3] that if P admits a regular conditional probability given the exchangeable σ-field $\mathscr{E}_S$, then P is presentable. The converse was left open. The object of this note is to show that the converse is false.

* This research was partially supported by National Science Foundation Grant MCS 77-01665

0044-3719/80/0051/0239/$02.00

(2) **Theorem.** *Let $(I, \mathscr{B})$ be the Borel unit interval. There is a subset S of I, with the relative Borel σ-field $\mathscr{F} = S \cap \mathscr{B}$, and a presentable probability P on $(S^\infty, \mathscr{F}^\infty)$ which does not admit a regular conditional probability given the exchangeable σ-field $\mathscr{E}_S$.*

The construction is a modification of the example presented in Sect. 2 of [1]. To review briefly, for $t \in I$ let t_j be the j^{th} digit in the binary expansion of t, so

$$t = \sum_{j=1}^{\infty} t_j/2^j, \quad t_j = 0 \text{ or } 1.$$

For $0 \leqq p \leqq 1$, let θ_p be the probability on $(I, \mathscr{B})$ which makes the t_j's independent, with common distribution

$$\theta_p\{t_j = 1\} = p, \qquad \theta_p\{t_j = 0\} = 1 = p.$$

Let $Q = \int \theta_p^\infty \, dp$, an exchangeable (and presentable) probability on $(I^\infty, \mathscr{B}^\infty)$. Let

$$Z(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} t_j$$

on the subset L of I where this limit exists. Let G be the set of $x \in I^\infty$ satisfying the following conditions:

(3) $x_j \in L$ and $Z(x_j) = Z(x_1)$ for all j

(4) $\frac{1}{n} \sum_{j=1}^{n} \delta_{x_j} \to \theta_{R(x)}$

where $R(x)$ is the common value for $Z(x_j)$, and δ_t is point mass at t, and the convergence is weak-star. Clearly, $G \in \mathscr{B}^\infty$ and $Q(G) = 1$. The next result constructs the state space S for Theorem (2).

(5) **Proposition.** *There is a subset S of the unit interval I having the property that for each $B \in \mathscr{B}^\infty$ with $Q(B) > 0$:*

(6) *there is a sequence x in $S^\infty \cap B \cap G$ such that*

$$S \cap \{t\colon t \in L \text{ and } Z(t) = R(x)\}$$

is countable

(7) *there is a sequence y in $S^\infty \cap B \cap G$ such that*

$$S \supset \{t\colon t \in L \text{ and } Z(t) = R(y)\}.$$

Proof. Let K be the set of ordinals of cardinality strictly less than c, the cardinality of the continuum. There is a $1-1$ mapping $\alpha \to B_\alpha$ of K onto the collection of B's in $\mathscr{B}^\infty$ with positive Q-measure. For each $\alpha \in K$, choose points x_α and y_α in $B_\alpha \cap G$ as follows. Fix $\beta \in K$, and suppose by induction that x_α and y_α have been chosen for all $\alpha < \beta$. Let T_β be the set of relative frequencies observed so far, namely $\bigcup_{\alpha < \beta} \{R(x_\alpha), R(y_\alpha)\}$. The cardinality of T_β is strictly less than c. In particular, $\tilde{T}_\beta = \{z\colon z \in G \text{ and } R(z) \in T_\beta\}$ has inner Q-measure 0 by (2.6) of [1]. So $B_\beta \cap G - \tilde{T}_\beta$ is non-empty.

Choose $x_\beta \in B_\beta \cap G - \tilde{T}_\beta$. Let $T_\beta^+ = T_\beta \cup R(x_\beta)$ and $\widetilde{T_\beta^+} = \{z\colon z \in G \text{ and } R(z) \in T_\beta^+\}$ and choose $y_\beta \in B_\beta \cap G - \widetilde{T_\beta^+}$. This completes the induction.

Now define S as follows: $x_\alpha \in I^\infty$ has coordinates $x_{\alpha j}$, and likewise for $y_{\alpha j}$;

$$S = \bigcup_{\alpha \in K} \Big[\bigcup_j \{x_{\alpha j}, y_{\alpha j}\}\Big] \cup \{t\colon t \in L \text{ and } Z(t) = R(y_\alpha)\}.$$

Fix $B \in \mathscr{B}^\infty$ with $Q(B) > 0$. Then $B = B_\beta$ for some $\beta \in K$. Clearly $x_\beta \in B_\beta \cap G$ by construction, and $x_\beta \in S^\infty$ because S contains all the coordinates of x_β. Likewise, $y_\beta \in S^\infty \cap B_\beta \cap G$. Property (6) follows because $x_\alpha \notin \tilde{T}_\alpha$ and $y_\alpha \notin \widetilde{T_\alpha^+}$ for all α, so each relative frequency is used at most once during the construction. To be explicit,

$$S \cap \{t\colon t \in L \text{ and } Z(t) = R(x_\beta)\} = \bigcup_{j=1}^{\infty} \{x_{\beta j}\}.$$

Property (7) is immediate. □

Before going on, it may be helpful to review the notations of *induced* and *traced* probabilities from [1, 2.10]. (As usual, outer measure is denoted by an asterisk.)

(8) *Review of Definitions.* Let (X, Σ) be an abstract measurable space, and X_0 a subset of X, not necessarily an element of Σ. Let $\Sigma_0 = X_0 \cap \Sigma$. Let μ_0 be a probability on (X_0, Σ_0). Then μ_0 induces a probability $\eta\,\mu_0$ on (X, Σ):

$$(\eta\,\mu_0)(A) = \mu_0(X_0 \cap A) \qquad \text{for all } A \in \Sigma.$$

Moreover, $(\eta\,\mu_0)^*(X_0) = 1$. Conversely, suppose μ is a probability on (X, Σ) and $\mu^*(X_0) = 1$. Then μ has a trace μ_0 on (X_0, Σ_0):

$$\mu_0(X_0 \cap A) = \mu(A) \qquad \text{for all } A \in \Sigma.$$

(9) *Definition.* As (6) or (7) implies, $Q^*(S^\infty) = 1$. Let P be the trace of Q on S^∞ in the sense given above. This defines the probability P for Theorem (2).

(10) **Lemma.** *Let (X, Σ, μ) be an abstract probability triple. Let $X_0 \subset X$ have $\mu^*(X_0) = 1$, and let μ_0 be the trace of μ on $(X_0, X_0 \cap \Sigma)$. Let $f \geqq 0$ be Σ-measurable on X. Then*

$$\int_{X_0} f\,d\mu_0 = \int_X f\,d\mu.$$

Proof. If f is an indicator function, this is true by definition, and extension is routine. □

(11) **Lemma.** *The P defined in* (9) *is presentable.*

Proof. Write ω for a generic sequence in S^∞ or I^∞. Clearly,

$$(12) \quad Q = \int_G \theta_{R(\omega)}^\infty\, Q(d\omega).$$

Let G^* be the set of $\omega \in S^\infty \cap G$ with $\theta_{R(\omega)}^*(S) = 1$. Then $Q^*(G^*) = 1$; indeed, suppose $B \in \mathscr{B}^\infty$ and $B \cap G^* = \phi$ but $Q(B) > 0$. By (7), there is a sequence y in $S^\infty \cap B \cap G$ such that

$$S \supset \{t\colon t \in L \text{ and } Z(t) = R(y)\}.$$

So $\theta^*_{R(y)}(S) = 1$, and $y \in B \cap G^*$. This contradiction proves $Q^*(G^*) = 1$.

Let $\hat{Q}$ be the trace of Q on G^*. By (10),

(13) $$Q = \int_{G^*} \theta^\infty_{R(\omega)} \hat{Q}(d\omega).$$

Trace this formula onto S^∞ to get

(14) $$P = \int_{S^*} \phi^\infty \nu(d\phi),$$

where ν is the $\hat{Q}$-distribution of $\omega \to \tau\,\theta_{R(\omega)}$ and τ traces $\theta_{R(\omega)}$ onto $(S, \mathscr{F})$. In more detail, if $\omega \in G^*$ then $\theta^*_{R(\omega)}(S) = 1$. By a theorem of von Neumann, $(\theta^\infty_{R(\omega)})^*(S^\infty) = 1$. And the trace of $\theta^\infty_{R(\omega)}$ on S^∞ is just $(\tau\,\theta_{R(\omega)})^\infty$. Hence

$$P = \int_{G^*} (\tau\,\theta_{R(\omega)})^\infty \hat{Q}(d\omega).$$

The rest of the argument for (14) is omitted, being routine. □

(15) **Lemma.** $\mathscr{E}_S = S^\infty \cap \mathscr{E}_I$.

Proof. Clearly, $S^\infty \cap \mathscr{E}_I \subset \mathscr{E}_S$. For the converse, let $F \in \mathscr{E}_S$. Then $F = S^\infty \cap B$ for some $B \in \mathscr{B}^\infty$, and

$$F = \tilde{\pi} F = S^\infty \cap \tilde{\pi} B = S^\infty \cap E$$

where

$$E = \bigcup_\pi \tilde{\pi} B \in \mathscr{E}_I. \quad \square$$

Proof of Theorem (2). Suppose by way of contradiction that $P(\omega, F)$ were a regular conditional P-probability on $(S^\infty, \mathscr{F}^\infty)$ given $\mathscr{E}_S$. Let $A \in \mathscr{B}$. It will be shown that

(16) There is a P-null set $N_A \in \mathscr{E}_S$ with

$$P(\omega, \{\xi_1 \in S \cap A\}) = \theta_{R(\omega)}(A) \qquad \text{for } \omega \in (S^\infty \cap G) - N_A.$$

Indeed, let E range over $\mathscr{E}_I$, so $S^\infty \cap E$ ranges over $\mathscr{E}_S$. Let $\zeta_1(x) = x$, for $x \in I^\infty$. Then

$$\begin{aligned}\int_{S^\infty \cap E} P(\omega, \{\xi_1 \in S \cap A\})\, P(d\omega) &= P(S^\infty \cap E \text{ and } \xi_1 \in S \cap A)\\ &= Q(E \text{ and } \zeta_1 \in A)\\ &= \int_E 1_G(\omega)\, \theta_{R(\omega)}(A)\, Q(d\omega)\\ &= \int_{S^\infty \cap E} 1_G(\omega)\, \theta_{R(\omega)}(A)\, P(d\omega)\end{aligned}$$

by (10), completing the proof of (16).

Let $\mathscr{B}_0$ be a countable field generating $\mathscr{B}$, and $N = \cup\{N_A\colon A \in \mathscr{B}_0\}$. Then $N \in \mathscr{E}_S \subset \mathscr{F}^\infty$ and $P(N) = 0$. In particular, $(S^\infty \cap G) - N = S^\infty \cap B \cap G$, where $B \in \mathscr{B}^\infty$ and $Q(B) = 1$. For $\omega \in S^\infty \cap B \cap G$, the monotone class argument proves

(17) $$P\{\omega, \{\xi_1 \in S \cap A\}) = \theta_{R(\omega)}(A) \qquad \text{for all } A \in \mathscr{B}.$$

Now a contradiction emerges: (6) yields an $\omega \in S^\infty \cap B \cap G$ for which

$$S \cap \{t: t \in L \text{ and } Z(t) = R(\omega)\}$$

is only countable, so $\theta^*_{R(\omega)}(S) = 0$. Take the inf of (17) over all Borel supersets A of S, getting $1 = 0$. □

The construction of this paper and [1] may be put into a slightly more general framework, as follows. Let H be the set of $\omega \in I^\infty$ for which $\frac{1}{n} \sum_{i=1}^{n} \delta_{\omega_i}$ converges weak-star; call the limit θ_ω. Let Q be an exchangeable probability on I^∞, so

(18) $Q(H) = 1$ and $Q = \int_H \theta_\omega^\infty Q(d\omega)$.

Let S be an arbitrary subset of I. Consider the following three conditions:

(19) $Q^*(S^\infty) = 1$

(20) $Q^*(H_S) = 1$, where H_S is the set of $\omega \in H$ with $\theta^*_\omega(S) = 1$.

(21) $Q^*(S^\infty) = 1$. Furthermore, there is a $B \in \mathscr{B}^\infty$ with $Q(B) = 1$ and $S^\infty \cap B \subset H_S$.

The following two results will be proved.

(22) **Theorem.** *Condition* (21) *implies* (20), *and* (20) *implies* (19).

(23) **Theorem.** (a) *Q can be traced onto an exchangeable probability P in S^∞* iff (19) *holds.*

(b) *The trace P is presentable* iff (20) *holds.*

(c) *The trace P admits a regular conditional probability given the exchangeable σ-field $\mathscr{E}_S$ if* (21) *holds.*

As this paper and [1] show, for non-standard S, condition (21) is genuinely stronger than (20), and (20) is genuinely stronger than (19).

Proof of Theorem (22). First, (21) implies (20). Indeed, suppose (21) and let $C \in \mathscr{B}^\infty$ with $C \supset H_S$: it must be proved that $Q(C) = 1$. Choose $B \in \mathscr{B}^\infty$ with $Q(B) = 1$ and $S^\infty \cap B \subset H_S$. Then $S^\infty \cap B \subset C$, so $B - C$ is disjoint from S^∞ and $Q(B - C) = 0$. In particular, $Q(C) \geqq Q(B) = 1$. Therefore, $Q(C) = 1$ and $Q^*(H_S) = 1$, deriving (20) from (21).

Next, (20) implies (19): even more, (20) implies

(24) $Q^*(S^\infty \cap H_S) = 1$.

To see this, fix $\omega \in H_S$ and let $\phi = \theta_\omega$. Then $\phi^*(S) = 1$, and $(\phi^\infty)^*(S^\infty) = 1$. Let

$$H_\phi = \{\omega: \omega \in H \text{ and } \theta_\omega = \phi\}.$$

Now $H_\phi \in \mathscr{B}^\infty$ and $\phi^\infty(H_\phi) = 1$ and $H_\phi \subset H_S$, so $(\phi^\infty)^*(S^\infty \cap H_S) = 1$. Let $C \in \mathscr{B}^\infty$ with $C \supset S^\infty \cap H_S$. Then $\theta^\infty_\omega(C) = 1$ for all $\omega \in H_S$, so

$$Q\{\omega: \omega \in H \text{ and } \theta^\infty_\omega(C) = 1\} = 1$$

and

$$Q(C)=\int_H \theta_\omega^\infty(C)\,Q(d\omega)=1$$

by (18). This proves (24). □

Proof of Theorem (23). Claim (a). This is clear.

Claim (b). The "if" part follows by the argument for (11). Conversely, suppose $P=\int_{S^*}\phi^\infty\,\nu(d\phi)$. Then $Q=\int_{S^*}(\eta\,\phi)^\infty\,\nu(d\phi)$ where $\eta\,\phi$ is the probability induced on $(I,\mathscr{B})$ by ϕ, in the sense of (8). So $(\eta\,\phi)^*(S)=1$. Let $\mu=\nu\eta^{-1}$. Then

(25) $$Q=\int_{I^*}\theta^\infty\,\mu(d\theta)$$

(26) $$\mu^*\{\theta\colon \theta^*(S)=1\}=1.$$

Compare (18) and (25): by the uniqueness part of the Hewitt-Savage theorem [1, 3.4], the Q-distribution of $\omega\to\theta_\omega$ is μ. Now condition (20) follows from (26), via Lemma (27) below: in the weak * σ-fields, the set of probabilities on $(I^\infty,\mathscr{B}^\infty)$ is a standard Borel space, as is the set I^* of probabilities on $(I,\mathscr{B})$. □

Claim (*c*). Suppose (21) holds. Recall that P is the trace of Q on S^∞. Then $P_*(S^\infty\cap H_S)=1$: that is, $\theta_\omega^*(S)=1$ for P-almost all $\omega\in S^\infty$. For the good ω's θ_ω has a trace ϕ_ω on $(S,\mathscr{F})$, and the requisite conditional probability is $\omega\to\phi_\omega^\infty$. If there is a regular conditional probability, then (21) holds by the argument used to prove Theorem (2) above. In more detail, if $P(\omega,F)$ is a regular conditional P-probability given $\mathscr{E}_S$, there is a $B\in\mathscr{B}^\infty$ with $Q(B)=1$, such that for $\omega\in S^\infty\cap B\cap H$,

$$P(\omega,\{\xi_1\in S\cap A\}=\theta_\omega(A)\qquad \text{for all } A\in\mathscr{B};$$

clearly, $\theta_\omega^*(S)=1$ for such ω. □

(27) **Lemma.** *Let (X,Σ) and (X',Σ') be measurable spaces. Let f be a measurable function from X to X' and μ a probability on Σ. Let $\nu=\mu f^{-1}$. Let A be an arbitrary subset of X'. Then $\mu^*(f^{-1}A)\leqq\nu^*(A)$. If (X,Σ) and (X',Σ') are standard Borel spaces, then $\mu^*(f^{-1}A)=\nu^*(A)$.*

Proof. The first assertion is easy. For the second, let $B\in\Sigma$ be disjoint from $f^{-1}A$ and have maximal μ-measure among all such sets. Verify that fB is disjoint from A, so $f^{-1}fB$ is larger than B but still disjoint from A. Now X and X' are standard Borel spaces, and f is a Borel function, so fB and $f^{-1}fB$ are analytic sets. Thus

$$\nu^*(A)\leqq 1-\nu(fB)\leqq 1-\mu(B)=\mu^*(f^{-1}A).\quad □$$

Some extensions of (27) may be of interest.

(28) **Lemma.** *Let (X,Σ) be an abstract measurable space, X_0 an arbitrary subset of X, and $\Sigma_0=X_0\cap\Sigma$. Let μ_0 be a probability on (X_0,Σ_0), and $\eta\,\mu_0$ the induced probability on (X,Σ), in the sense of* (8). *Let A be an arbitrary subset of X_0. Then $\mu_0^*(A)=(\eta\,\mu_0)^*(A)$.*

Proof. Let B range over Σ. Then $B\supset A$ iff $X_0\cap B\supset A$, and

$$\mu_0^*(A) = \inf\{\mu_0(X_0 \cap B): X_0 \cap B \supset A\}$$
$$= \inf\{(\eta\,\mu_0)(B): B \supset A\} = (\eta\,\mu_0)^*(A). \quad \square$$

(29) **Corollary.** *In* (27), *if* (X, Σ) *is a standard Borel space, and* Σ' *is separable, then* $\mu^*\{f \in A\} = (\mu f^{-1})^*(A)$.

Proof. Without loss of generality, suppose X' is a subset of I and $\Sigma' = X' \cap \mathscr{B}$, where $(I, \mathscr{B})$ is the Borel unit interval. Recall that $\nu = \mu f^{-1}$ is a probability on (X', Σ'). Let $\eta\,\nu$ be the probability induced on $\mathscr{B}$ by ν, in the same sense of (8). Let $\tilde{f}$ be that I-valued function on X which agrees with the X'-valued function $\tilde{f}$. Clearly, $\tilde{f}$ is Borel, and its range is a subset of X'. Also, $\mu \tilde{f}^{-1} = \eta\,\nu$. Now

$$\begin{aligned}
\mu^*(f \in A) &= \mu^*(\tilde{f} \in A) \\
&= (\mu \tilde{f}^{-1})^*(A) && \text{by (27)} \\
&= (\eta\,\nu)^*(A) \\
&= \nu^*(A) && \text{by (28).} \quad \square
\end{aligned}$$

(30) **Corollary.** *Let* (X, Σ) *be a standard Borel space. Let* X' *be a set, and* Σ' *a separable* σ*-field of subsets of* X'. *Consider the product space* $(X' \times X, \Sigma' \times \Sigma)$. *Let* π *project* $X' \times X$ *onto* X'. *Let* μ *be a probability on* $\Sigma' \times \Sigma$, *and* $\nu = \mu \pi^{-1}$. *Let* A *be an arbitrary subset of* X'. *Then* $\mu^*\{\pi \in A\} = \nu^*(A)$.

Proof. As in (29): take X' to be a subset of I, and consider the probabilities induced by μ and ν on $I \times X$ and I respectively. $\square$

(31) *Example.* In (27) and (29), the assumption that the domain of f be standard is needed. Let W be a subset of I having inner Lebesgue measure 0 and outer measure 1. Equip W with the relative Borel σ-field $\mathscr{F} = W \cap \mathscr{B}$. Let λ be the trace of Lebesgue measure on $(W, \mathscr{F})$, in the sense of (8). Let f embed W back into I, namely $f(x) = x$ for $x \in W$. Then λf^{-1} is Lebesgue measure. Let $A = I - W$. Then $\lambda^*\{f \in A\} = 0$ but $(\lambda f^{-1})^*(A) = 1$.

(32) *Example.* In (30), the assumption that the vertical edge X be standard is needed. Continuing (31), consider the product space $(I \times W, \mathscr{B} \times \mathscr{F})$. Define the probability μ on $\mathscr{B} \times \mathscr{F}$ as the λ-distribution of the map $x \to (x, x)$ from W into $I \times W$: this installs λ on the diagonal $D = \{(x, y): x \in I \text{ and } y \in W \text{ and } x = y\}$. Verify that $D \in \mathscr{B} \times \mathscr{F}$ and $\mu(D) = 1$. Recall that $A = I - W$, so $(A \times W) \cap D = \emptyset$ and $\mu^*(A \times W) = 0$. Now $\mu \pi^{-1}$ is Lebesgue measure: $\mu^*\{\pi \in A\} = 0$ but $(\mu \pi^{-1})^*(A) = 1$. $\square$

One more remark on the construction for Proposition (5): the probability Q there is a mixture of powers of continuous probabilities. Discrete probabilities will not do. To state this more sharply, let H^d be the set of $\omega \in H$, as defined before (18), where θ_ω is discrete. Then H^d is Borel, by (2.13) of [2].

(33) **Proposition.** *Let* Q *be an exchangeable probability on* $(I^\infty, \mathscr{B}^\infty)$, *with* $Q(H^d) = 1$. *Let* S *be a subset of* I. *If* Q *can be traced onto* S^∞, *in the sense of* (8), *the trace admits a regular conditional probability given* $\mathscr{E}_S$, *the exchangeable* σ*-field in* S^∞.

Proof. We assume condition (19), and derive (21). Let $Z_{ij}(\omega) = 1$ if $\omega_i = \omega_j$, and $Z_{ij}(\omega) = 0$ otherwise. Let

$$Z_i(\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} Z_{ij}(\omega)$$

on the set H_i where the limit exists. Let $N_i(\omega)$ be the least j with $\omega_j=\omega_i$, and let $V(\omega)$ be the set of distinct values among $N_1(\omega), N_2(\omega), \ldots$. Let H^g be the set of ω in H^d such that

$$(34) \quad \omega\in H_i \quad \text{and} \quad Z_i(\omega)>0 \qquad \text{for all } i$$

and

$$(35) \quad \sum_{j\in V(\omega)} Z_j(\omega)=1.$$

Condition (34) is that any value which appears among $\omega_1, \omega_2, \ldots$ does so with positive limiting relative frequency. Condition (35) is that the sum of these relative frequencies is 1.

As is easily seen, H^g is Borel. If θ is discrete, then $\theta^\infty(H^g)=1$ by the strong law. So, if Q is exchangeable and $Q(H^d)=1$, then $Q(H^g)=1$:

$$Q(H^g)=\int \theta_\omega^\infty(H^g)\, Q(d\omega)=\int_{H^d} \theta_\omega^\infty(H^g)\, Q(d\omega).$$

If $\omega\in H^g$, then $\theta_\omega\{\omega_1,\omega_2,\ldots\}=1$: indeed $\frac{1}{n}\sum_{i=1}^{n}\delta_{\omega_i}$ converges weak * to θ_ω, and converges in variation norm to the probability asigning mass $Z_j(\omega)$ to $j\in V(\omega)$. As a result, if $\omega\in S^\infty\cap H^g$ then $\theta_\omega(S)=1$. This derives (21) from (19), with H^g for B. Theorem (23) completes the proof. □

A final remark on [1] may be in order. There is a subset S of $[0,1]$ and an exchangeable probability in S^∞ which is not presentable. This brings into question the nature of the extreme exchangeable probabilities in S^∞. However, let $(S,\mathscr{F})$ be an abstract measurable space, and $\mathscr{E}_S$ the exchangeable σ-field in S^∞. The following result is known [3], but the proof may be new.

(36) **Proposition.** *Let P be an exchangeable probability on $(S^\infty,\mathscr{F}^\infty)$. The following three conditions are equivalent:*

(i) *P is extreme,*

(ii) *P is $0-1$ on $\mathscr{E}_S$,*

(iii) *P is ϕ^∞ for some probability ϕ on $(S,\mathscr{F})$.*

Condition (i) implies (ii). If $A\in\mathscr{E}_S$ and $0<P(A)<1$, then P cannot be extreme, because

$$P=P(A)\,P(\cdot\,|A)+P(B)\,P(\cdot\,|B)$$

where B is the complement of A in S^∞.

Condition (ii) implies (iii). Let $\xi_1,\xi_2,\ldots$ be the coordinate process on S^∞. Let f be a measurable function from $(S,\mathscr{F})$ to the Borel unit interval. As is well known,

$$(37) \quad T_N=\frac{1}{N}\sum_{n=1}^{N} f(\xi_n)\to E\{f(\xi_m)\,|\,\mathscr{E}_S\} \qquad \text{a.s.}$$

where the limit does not depend on m. This may be derived from the martingale covergence theorem, for

$$T_N = E\{f(\xi_m) | T_N, T_{N+1}, \ldots\}.$$

Here is another, more interesting, proof of (37). The main idea is due to de Finetti. By a direct calculation which exploits the exchangeability, for $N < M < \infty$,

$$E\{(T_N - T_M)^2\} = \left(\frac{1}{N} - \frac{1}{M}\right) \{\mathrm{Var}[f(\xi_1)] - \mathrm{Cov}[f(\xi_1), f(\xi_2)]\}.$$

In particular, $\{T_N\}$ is fundamental in probability, and so converges in probability to some limit T. This limit T is clearly $\mathscr{E}_S$-measurable. Next,

$$(38) \quad E\{[f(\xi_m) - T] \cdot [f(\xi_2) - T]^3\} = 0 \qquad \text{for } m \neq 2.$$

Indeed, the left side of (38) does not depend on m, by exchangeability. Averaging over $m = 1, 3, 4, \ldots, N$, the left side of (38) equals

$$\frac{N}{N-1} E\{[T_N - T] \cdot [f(\xi_2) - T]^3\} - \frac{1}{N-1} E\{[f(\xi_2) - T]^4\}.$$

This converges to 0, because $T_N \to T$ and $0 \leqq T_N \leqq 1$, proving (38). Likewise,

$$(39\text{a}) \quad E\{[f(\xi_1) - T] \cdot [f(\xi_2) - T] \cdot [f(\xi_3) - T] \cdot [f(\xi_4) - T]\} = 0$$

$$(39\text{b}) \quad E\{[f(\xi_1) - T] \cdot [f(\xi_2) - T] \cdot [f(\xi_3) - T]^2\} = 0$$

$$(39\text{c}) \quad E\{[f(\xi_1) - T] \cdot [f(\xi_2) - T]^3\} = 0.$$

Hence

$$\begin{aligned} E[(T_N - T)^4] &= \frac{1}{N^3} E[(f(\xi_1) - T)^4] \\ &\quad + \binom{4}{2} \binom{N}{2} \frac{1}{N^4} E\{[f(\xi_1) - T]^2 \cdot [f(\xi_2) - T]^2\} \\ &= 0 \left(\frac{1}{N^2}\right) \end{aligned}$$

is summable in N. As a result, $T_N \to T$ almost surely.

To see that $T = E\{f(\xi_m) | \mathscr{E}_S\}$, let $A \in \mathscr{E}_S$. Then $\int_A f(\xi_m)\, dP$ does not depend on m; as such, this integral equals

$$\int_A T_N\, dP \to \int_A T\, dP.$$

This completes the second proof of (37).

Now it is easy to derive (iii) from (ii). Indeed, let A_i and $A \in \mathscr{F}$ and $B = \{\xi_1 \in A_1, \ldots, \xi_k \in A_k\}$. Then,

$$
\begin{aligned}
P(B \text{ and } \xi_{k+1}\in A) &= P(B \text{ and } \xi_n\in A) \qquad \text{for } n>k\\
&= \int_B \frac{1}{N}\sum_{n=k+1}^{k+N} 1_A(\xi_n)\,dP\\
&\to \int_B P(\xi_{k+1}\in A\,|\,\mathscr{E}_S)\,dP\\
&= P(B)\,P(\xi_{k+1}\in A)
\end{aligned}
$$

because P is $0-1$ on $\mathscr{E}_S$.

Condition (iii) implies (i). Suppose Q is exchangeable and absolutely continuous with respect to ϕ^∞. Let A_1 and $A_2 \in \mathscr{F}$. Now

$$
(40)\qquad \frac{1}{N}\sum_{n=1}^{N} 1_{A_1}(\xi_n)\,1_{A_2}(\xi_{n+1}) \to \phi(A_1)\,\phi(A_2)
$$

ϕ^∞-almost surely, by the strong law applied separately to the even n's and to the odd n's. Next, (40) holds Q-almost surely, because $Q \ll \phi^\infty$. Integrate (40) with respect to Q:

$$
Q(\xi_1\in A_1 \text{ and } \xi_2\in A_2) = \phi(A_1)\,\phi(A_2).
$$

By similar argument,

$$
Q(\xi_1\in A_1 \text{ and } \xi_2\in A_2 \text{ and } \dots \text{ and } \xi_k\in A_k) = \phi(A_1)\,\phi(A_2)\dots\phi(A_k),
$$

so $Q=\phi^\infty$. To sum up,

(41) If Q is exchangeable, and absolutely continuous with respect to ϕ^∞, then $Q=\phi^\infty$.

Now, to prove ϕ^∞ is extreme, suppose $\phi^\infty = \frac{1}{2}Q + \frac{1}{2}Q'$, where Q and Q' are exchangeable. Clearly, Q and Q' are absolutely continuous with respect to ϕ^∞. Hence, $Q=Q'=\phi^\infty$ by (41). □

References

1. Dubins, L.E., Freedman, D.A.: Exchangeable processes need not be mixtures of independent, identically distributed variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 1979
2. Dubins, L.E., Freedman, D.A.: Measurable sets of measures, Pac. J. Math. **14**, 1211–1222 (1964)
3. Hewitt, E., Savage, L.J.: Symmetric measures on Cartesian products, Trans. Amer. Math. Soc., **80**, 470–501 (1955)

Received June 20, 1979