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Point Processes of Cylinders, Particles and Flats

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Abstract. Point processes X of cylinders, compact sets (particles), or flats in $\mathbf{R}^d$ are mathematical models for fields of sets as they occur, e.g., in practical problems of image analysis and stereology. For the estimation of geometric quantities of such fields, mean value formulas for X are important. By a systematic approach, integral geometric formulas for curvature measures are transformed into density formulas for geometric point processes. In particular, a number of results which are known for stationary and isotropic Poisson processes of convex sets are generalized to nonisotropic processes, to non-Poissonian processes, and to processes of nonconvex sets. The integral geometric background (including recent results from translative integral geometry), the fundamentals of geometric point processes, and the resulting density formulas are presented in detail. Generalizations of the theory and applications in image analysis and stereology are mentioned shortly.

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1. Introduction

In many practical situations problems of the following type occur. There is a collection of sets in $\mathbf{R}^2$ or $\mathbf{R}^3$, from which a transformed image is observed, e.g., a section, a projection, or the part within a 'sampling window'. Typically, the sets are compact particles or flats (lines, planes) or cylinders. In practice they may be pores in a porous medium, tubules in a tissue, fibres in a fabric, etc. The problem is to estimate geometric quantities of the entire collection based on information from the observed image. This is a classical stereological situation to which the statistical theory, developed on the basis of integral geometry by Davy and Miles (see Davy (1978), Miles (1978), or Weil (1983a), for surveys), may be applied. However, if the field $\mathcal{F}$ of sets is significantly extended (e.g., compared to the bounded sampling window) and if the interest is mainly in mean values (mean surface area of particles per unit volume, mean thickness of cylinders per unit volume, mean number of lines per unit area, etc.), $\mathcal{F}$ may be also considered as the outcome $\mathcal{F} = X(\omega)$ of a random process X, a geometric point process. In particular, if X can be assumed to have certain invariance properties (stationarity, isotropy) of its distribution, this point of view has a number of advantages. First of all, sections and sampling windows need not be chosen randomly in this case. Moreover, special point process models allow a statistical analysis which is quite complicated in the deterministic case. Finally, various geometric point processes can be generated by computer programs and, hence, many numerical results can be obtained by simulation.

The basic point process model used in the literature is that of a stationary and isotropic Poisson process of convex sets (particles, cylinders, flats). Formulas for such point processes are collected in Matheron (1975) and Davy (1978). More recently it has been shown that many results can be obtained under weaker assumptions, for example, for stationary and nonisotropic processes, for non-Poissonian processes, and for processes of nonconvex sets. Such results are found here and there in the literature and are obtained by quite different methods under varying assumptions. Matheron, in his book (Matheron (1975)), already emphasized the role of the curvature measures of a set as a basic notion in stochastic geometry. This opinion was supported by results in Weil (1983b), (1984), Zähle (1986). It turns out that integral geometric formulas for curvature measures can be transformed easily into density formulas for geometric point processes from which many (but, of course, not all) stereological results for mean values come out as special cases.

In the following this method is exploited systematically in d-dimensional space $\mathbf{R}^d$ and the resulting formulas are collected. Most of them are known but some are new and the others are shown to be true under quite general assumptions. In order to allow a unified approach to all the results the underlying class of sets is of importance. We base our considerations upon the 'convex ring' $\mathcal{R}_d$, a choice which is essential in view of the integral formulas for curvature measures which are used. Since compact sets can be approximated by sets from the convex ring, formulas given in the literature for more general sets (e.g., fibres, surfaces, etc.) may be obtained immediately from corresponding results on $\mathcal{R}_d$. This is sufficient for applications but it should be mentioned that the approximation of one set class by another such that geometric functionals converge is, in general, a difficult mathematical problem.

2. Integral Formulas for Curvature Measures

We need the following classes of sets in $\mathbf{R}^d$: $\mathcal{K}_d$ the class of convex bodies (compact convex subsets of $\mathbf{R}^d$), $\mathcal{R}_d$ the convex ring (finite unions of convex bodies), $\mathcal{L}_q^d$ the set of q-dimensional linear subspaces of $\mathbf{R}^d$, and $\mathcal{E}_q^d$ the set of q-flats (q-dimensional affine subspaces) in $\mathbf{R}^d$. For a subspace $L \in \mathcal{L}_q^d$ and a set K in the orthogonal space $L^\perp$, the vector sum $K+L$ is called a *cylinder* with basis K. Let $\mathcal{Z}_q(K)$ be the set of all cylinders (properly) congruent to $K+L$. The Lebesgue measure in $\mathbf{R}^d$ is denoted by λ_d. Correspondingly, λ_E is the Lebesgue measure in $E \in \mathcal{E}_q^d$ and, for a face F of a polytope or a cylinder, λ_F is the appropriate Lebesgue measure restricted to F. For L, $L' \in \mathcal{L}_q^d$, $[L, L']$ is the volume (of appropriate dimension) of the parallelepiped spanned by the following vectors. First choose an orthonormal basis in $L \cap L'$, then extend it to an orthonormal basis in L and an orthonormal basis in L'.

For faces F, F' of convex polytopes K, K' (or cylinders with polytopal basis) we define the angle $\gamma(F, F', K, K')$ as the external angle of $K \cap (K'+x)$ at the face $F \cap (F'+x)$, where $x \in \mathbf{R}^d$ is such that the relative interior of F meets the relative interior of $F'+x$. This definition does not depend on the choice of x as long as $F \cap (F'+x)$ has dimension at most $\max\{0, \dim F + \dim F' - d\}$ for all x. If this is the case for all faces, we say that K and K' are in *general relative position*.

For $K \in \mathcal{K}_d$ the *curvature measures* $\Psi_0(K, \cdot), \ldots, \Psi_d(K, \cdot)$ can be introduced by the local Steiner formula

$$\lambda_d(A_\epsilon(K, \beta)) = \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} \Psi_j(K, \beta). \tag{2.1}$$

Here, $\beta \subset \mathbf{R}^d$ is a Borel set, κ_k is the volume of the k-dimensional unit ball, $\epsilon > 0$, and $A_\epsilon(K, \beta)$ is a local parallel set (the set of all $x \in \mathbf{R}^d$ such that the metric projection $\text{proj}_K(x)$ of x onto K obeys $\text{proj}_K(x) \in \beta$ and $\|x\text{-proj}_K(x)\| \leq \epsilon$). An important property of the curvature measures is that they depend additively on K. An additive extension to sets $K \in \mathcal{R}_d$ was given by Schneider (1980a). For $K = \bigcup_{i=1}^{n} K_i$, $K_i \in \mathcal{K}_d$, this extension $\Psi_j(K, \cdot)$ fulfills

$$\begin{aligned} \Psi_j(K, \cdot) = \sum_{i=1}^{n} \Psi_j(K_i, \cdot) - \sum_{1 \leq i_1 < i_2 \leq n} \Psi_j(K_{i_1} \cap K_{i_2}, \cdot) + \\ - \cdots + (-1)^{n+1} \Psi_j(K_1 \cap \cdots \cap K_n, \cdot). \end{aligned} \tag{2.2}$$

Equation (2.2) cannot be used as a definition of $\Psi_j(K, \cdot)$ since the right side may depend on the choice of the sets $K_1, \ldots, K_n$. An alternative approach was given by Schneider (1980a) who showed that there is a generalization of (2.1) to sets K in $\mathcal{R}_d$. For properties of the curvature measures, we refer to the surveys Schneider (1979) and Weil (1983a). We mention only that for j-sets $K \in \mathcal{R}_d$, i.e., sets which are the union of convex bodies $K_1, \ldots, K_n$ of dimension at most j, the curvature measure $\Psi_j(K, \cdot)$ equals the j-dimensional Hausdorff measure on K. Since the curvature measures are locally defined, they extend immediately to unbounded sets, which are locally finite countable unions of convex bodies. In particular, this holds for cylinders Z. More precisely, for a cylinder Z, a convex body K', and a Borel set β in the interior int K' of K', we have

$$\Psi_j(Z, \beta) = \Psi_j(Z \cap K', \beta).$$

For stereological applications, the total measures $V_j(K) = \Psi_j(K, \mathbf{R}^d)$ are important. $V_j(K)$ is proportional to the $(d-j)$th quermass integral of K, in particular $V_d(K)$ is the volume, $V_{d-1}(K)$ is half the surface area, $V_{d-2}(K)$ is proportional to the integral mean curvature, ..., $V_1(K)$ is proportional to the (additively extended) mean width, and $V_0(K)$ is the Euler–Poincaré characteristic of $K \in \mathcal{R}_d$. It is convenient to define $V_j(Z)$ for a cylinder $Z = K + L$ not as the total curvature measure but as $V_{j-q}(K)$, $j = q, \dots, d$.

The basic integral geometric result for curvature measures is the following translation formula for cylinders (Schneider and Weil (1986)). Let $L \in \mathcal{L}_q^d$, $K' \in \mathcal{R}_d$ with $K' \subset L^\perp$, $K \in \mathcal{R}_d$, and Borel sets $\beta \subset \mathbf{R}^d$, $\beta' \subset L^\perp$ be given. Then, for $j \in \{0, \dots, d\}$,

$$\begin{aligned}\int_{L^\perp} &\Psi_j(K \cap (K' + L + x), \beta \cap (\beta' + L + x))\, d\lambda_{L^\perp}(x) \\ &= \Psi_j(K, \beta)\Psi_{d-q}(K', \beta') + \sum_{k=j+1}^{d-1} \varphi_k^{(j)}(K, K' + L, \beta \times \beta') + \\ &\quad + \Psi_d(K, \beta)\Psi_{j-q}(K', \beta')\end{aligned} \tag{2.3}$$

(curvature measures with index $m < 0$ are defined to be zero). $\varphi_k^{(j)}(K, K' + L, \cdot)$ is a (signed) Borel measure on $\mathbf{R}^d \times \mathbf{R}^d$ which depends in an additive and measurable way on K, K' (and L). For $k > d + j - q$, $\varphi_k^{(j)}(K, K' + L, \cdot)$ vanishes, hence the summation in (2.3) really extends only to $d + j - q$, if $j < q$. The measures $\varphi_k^{(j)}(K, K' + L, \cdot)$ with $k \leq d + j - q$, are concentrated on $\operatorname{bd} K \times \operatorname{bd} K'$ (here bd denotes the boundary). For $k \leq d + j - q$, $\varphi_k^{(j)}(K, K' + L, \cdot)$ is explicitly known if K, K' are convex polytopes and $K, K' + L$ are in general relative position. Then we have

$$\begin{aligned}&\varphi_k^{(j)}(K, K' + L, \cdot) \\ &= \sum_{F \in \mathcal{F}_k(K)} \sum_{F' \in \mathcal{F}_{d+j-q-k}(K')} \gamma(F, F' + L, K, K' + L) \times \\ &\quad \times [L(F), L(F') + L] \lambda_F \otimes \lambda_{F'}.\end{aligned} \tag{2.4}$$

Here, $\mathcal{F}_m(P)$ is the set of m-dimensional faces of the polytope P and $L(F) \in \mathcal{L}_m^d$ is the subspace parallel to $F \in \mathcal{F}_m(P)$. Two special cases of (2.3) are of interest. First, if $L = \{0\}$, we get

$$\begin{aligned}\int_{\mathbf{R}^d} &\Psi_j(K \cap (K' + x), \beta \cap (\beta' + x))\, d\lambda_d(x) \\ &= \Psi_j(K, \beta)\Psi_d(K', \beta') + \\ &\quad + \sum_{k=j+1}^{d-1} \varphi_k^{(j)}(K, K', \beta \times \beta') + \Psi_d(K, \beta)\Psi_j(K', \beta').\end{aligned} \tag{2.5}$$

Next, if $K' = \{0\}$, the measure $\varphi_k^{(j)}(K, K' + L, \cdot)$ is concentrated on $\operatorname{bd} K \times \{0\}$, hence it may be viewed as a measure on $\mathbf{R}^d$ (concentrated on bd K). Since $\varphi_k^{(j)}(K, K' + L, \cdot)$ depends homogeneously of degree $d + j - q - k$ on K' it must vanish as long as $d + j - q - k \neq 0$. Hence, from (2.3) it follows that

$$\begin{aligned}\int_{L^\perp} &\Psi_j(K \cap (L + x), \beta \cap (L + x))\, d\lambda_{L^\perp}(x) \\ &= \varphi_{d+j-q}^{(j)}(K, L, \beta).\end{aligned} \tag{2.6}$$

An important property of the measure $\varphi_k^{(j)}(K, K'+L, \cdot)$ concerns its rotation integral. If SO_d denotes the rotation group with invariant measure ν, $\nu(SO_d)=1$, then

$$\int_{SO_d} \varphi_k^{(j)}(K, \theta(K'+L), \beta \times \theta\beta')\,d\nu(\theta) = \alpha_{djk}\Psi_k(K,\beta)\Psi_{d+j-q-k}(K',\beta'). \tag{2.7}$$

Combining (2.7) with (2.3), we get the kinematic formula for cylinders (Schneider 1980b)

$$\int_{SO_d}\int_{L^\perp} \Psi_j(K\cap\theta(K'+L+x), \beta\cap\theta(\beta'+L+x))\,d\lambda_{L^\perp}(x)\,d\nu(\theta) = \sum_{k=j}^{d} \alpha_{djk}\Psi_k(K,\beta)\Psi_{d+j-q-k}(K',\beta'). \tag{2.8}$$

Again as special cases, (2.8) contains the *principal kinematic formula* and the *Crofton formula* for curvature measures. Instead of using double integrals we consider here the group G_d of rigid motions with invariant measure μ and the space $\mathscr{E}_q^d$ with invariant measure μ_q. Then

$$\int_{G_d} \Psi_j(K\cap gK', \beta\cap g\beta')\,d\mu(g) = \sum_{k=j}^{d} \alpha_{djk}\Psi_k(K,\beta)\Psi_{d+j-k}(K',\beta') \tag{2.9}$$

and

$$\int_{\mathscr{E}_q^d} \Psi_j(K\cap E, \beta\cap E)\,d\mu_q(E) = \alpha_{djq}\Psi_{d+j-q}(K,\beta). \tag{2.10}$$

The coefficients α_{djk} occurring in (2.7) to (2.10) are given by

$$\alpha_{djk} = \frac{\binom{k}{j}\kappa_k\kappa_{d+j-k}}{\binom{d}{k-j}\kappa_j\kappa_d}.$$

A variant of (2.3) and (2.8) concerns the case of projected thick sections. For a cylinder $K'+L$ denote by $A\mid L$ the projection of the set $A\subset\mathbf{R}^d$ onto the direction space L of $K'+L$. Then, we have for $K, K'\in\mathscr{K}_d$

$$\int_{L^\perp} \Psi_j([K\cap(K'+L+x)]\mid L, [\beta\cap(\beta'+L+x)]\mid L)\,d\lambda_{L^\perp}(x) = \varphi_{d+j-q}^{(j)}(K+(-K'), L, \beta+(-\beta')), \tag{2.11}$$

where

$$-K' = \{-x \mid x\in K'\}.$$

The analog of (2.8) would be

$$\int_{SO_d}\int_{L^\perp} \Psi_j([K\cap\theta(K'+L+x)]\,|\,\theta L,[\beta\cap\theta(\beta'+L+x)]\,|\,\theta L)\,d\lambda_{L^\perp}(x)\,d\nu(\theta) = \sum_{k=j}^{d+j-q} \gamma_{djkq}\Psi_k(K,\beta)\Psi_{d+j-q-k}(K',\beta') \tag{2.12}$$

with

$$\gamma_{djkq} = \frac{\binom{q}{j}\kappa_q\kappa_k\kappa_{d-k}}{\binom{d}{d-k}\kappa_{q-j}\kappa_d\kappa_j}.$$

So far, (2.12) is known only for the quermassintegrals, i.e., for $\beta = \mathbf{R}^d$, $\beta' = L^\perp$ (see Schneider (1981)). The two formulas can be extended to sets $K\in\mathcal{R}_d$ which are unions of convex bodies $K_1,\dots,K_n$, provided the different parts $K_1,\dots,K_n$ of K can be distinguished in the projection and the integrands are modified appropriately. For practical applications, the following case is important: K is a 1-set, i.e., $K_1,\dots,K_n$ are line segments, no two of which have more than one point in common. Then, (2.12) holds for the quermassintegral V_1 without further modification, if $d\geqslant 3$.

Since

$$\Psi_j(K,\mathbf{R}^d) = V_j(K),\quad j=0,\dots,d,\ K\in\mathcal{R}_d,$$

the integral formulas (2.3) to (2.12) contain, as special cases, integral formulas for quermassintegrals. It is therefore of interest to obtain more explicit expressions for the total measures $\Phi_k^{(j)}(K,Z) = \varphi_k^{(j)}(K,Z,\mathbf{R}^d\times\mathbf{R}^d)$ (resp. $\Phi_k^{(j)}(K,L) = \varphi_k^{(j)}(K,L,\mathbf{R}^d)$). We have (Goodey and Weil (1986))

$$\Phi_{d+j-q}^{(j)}(K,K'+L) = \frac{\binom{d}{q-j}}{\kappa_{q-j}} V(\underbrace{K,\dots,K}_{d+j-q},\underbrace{B_L,\dots,B_L}_{q-j}), \tag{2.13}$$

where $V(K_1,\dots,K_d)$ denotes the mixed volume of $K_1,\dots,K_d$ and B_L is the q-dimensional unit ball in L. Also,

$$\Phi_k^{(0)}(K,K'+L) = \frac{\binom{d}{q}\binom{d-q}{k}}{\kappa_q} V(\underbrace{K,\dots,K}_{k},\underbrace{-K',\dots,-K'}_{d-q-k},\underbrace{B_L,\dots,B_L}_{q}), \tag{2.14}$$

$k=0,\dots,d-q$, and

$$\Phi_k^{(j)}(K, K'+L) = \frac{\binom{d}{q}\binom{d-q}{k-j}\binom{d}{j}\kappa_d}{\kappa_q\kappa_{d-j}\kappa_j} \times \int_{\mathscr{E}_{d-j}^d} V(\underbrace{K\cap E,\ldots,K\cap E}_{k-j}, \underbrace{-K',\ldots,-K'}_{d+j-q-k}, \underbrace{B_L,\ldots,B_L}_{q})\, d\mu_{d-j}(E), \tag{2.15}$$

$k=j+1,\ldots,\min(d-1, d+j-q)$. Because of the use of mixed volumes, these results first hold for convex bodies K, K', but they can be extended to sets K, $K'\in\mathscr{R}^d$ by additivity. The extension is possible since the mixed volumes in (2.13), (2.14), and (2.15) depend additively and continuously on the sets K, $K'\in\mathscr{K}_d$ (see the article of McMullen and Schneider (1983), for more details).

Of special interest is the case where K is an r-dimensional convex body, $0\leq r\leq d-1$. Then using (2.15) we see that $\Phi_k^{(j)}(K, K'+L)=0$ if $k>r$. Thus, if we assume $0<r-j\leq d-q$, the last summand in (2.3) (for $\beta\times\beta'=\mathbf{R}^d\times L^\perp$) is $\Phi_r^{(j)}(K, K'+L)$. Here, formula (2.15) can be simplified (Goodey and Weil (1986)):

$$\Phi_r^{(j)}(K, K'+L) = \frac{\binom{d}{q}\binom{d-q}{r-j}}{\kappa_{r-j}\kappa_q} V_r(K)\, V(\underbrace{K',\ldots,K'}_{d+j-r-q}, \underbrace{B_L,\ldots,B_L}_{q}, \underbrace{B_M,\ldots,B_M}_{r-j}) \tag{2.16}$$

($M\in\mathscr{L}_r^d$ is the subspace parallel to the affine hull of K).

3. Geometric Point Processes

Let $\mathscr{Z}_q$ be the set of all cylinders in $\mathbf{R}^d$ with q-dimensional direction space and basis in $\mathscr{R}_d$. For some fixed $L\in\mathscr{L}_q^d$, $\mathscr{Z}_q$ is the union of all sets $\mathscr{Z}_q(K)$, $K\subset L^\perp$, $K\in\mathscr{R}_d$. A *point process* X on $\mathscr{Z}_q$ is given by a probability measure P_X, the distribution of X, on the set $\mathscr{M}(\mathscr{Z}_q)$ of locally finite counting measures on $\mathscr{Z}_q$ supplied with its usual σ-algebra. Since $\mathscr{Z}_q$ is a measurable subset of the set $\mathscr{F}$ of all closed sets in $\mathbf{R}^d$, this definition of X fits into the general theory of point processes on $\mathscr{F}$ (see Matheron (1975) for details). Contrary to the definition of point processes in general spaces (see e.g., Neveu (1977)), we call a measure φ on $\mathscr{Z}_q$ *locally finite*, if

$$\varphi(\{Z\in\mathscr{Z}_q \mid \operatorname{conv} Z\cap K\neq\emptyset\})<\infty$$

for all $K\in\mathscr{R}_d$. Alternatively, we consider a point process X as a measurable mapping from some abstract probability space $(\Omega, \mathscr{A}, P)$ into $\mathscr{M}(\mathscr{Z}_q)$. We therefore write $X(\omega)$ for a realization of X. Since we only work with simple point processes, we may interpret a counting measure η on $\mathscr{Z}_q$ also as a collection of cylinders. This allows us to write $Z \in X(\omega)$ or $Z \in X$.

The point process X on $\mathscr{Z}_q$ is *stationary* if P_X is translation invariant, it is *isotropic* if P_X is invariant w.r.t. rotations. The *intensity measure* Θ of X is defined by

$$\Theta(\mathscr{B}) = \mathbb{E}X(\mathscr{B})$$

for any Borel set $\mathscr{B} \subset \mathscr{Z}_q$. We assume throughout that Θ is locally finite. If X is stationary (isotropic), Θ has the same invariance properties but the converse is obviously false. X is called *weakly stationary* (*weakly isotropic*) if Θ is stationary (isotropic).

The following decomposition is important. Let s: $\mathscr{R}_d \to \mathbf{R}^d$ be a measurable mapping which associates to each set $K \in \mathscr{R}_d$ a center $s(K)$ in a motion covariant way. Familiar choices of such centers are the midpoint of the circumsphere or the Steiner point (the usual centroid does not exist for all sets in $\mathscr{R}_d$).

Let $\mathscr{Z}_q^0$ be the set of all cylinders $Z = K + L$ such that $L \in \mathscr{L}_q^d$, $K \in \mathscr{R}_d$, $K \subset L^\perp$, and $s(K) = 0$, and let

$$\tilde{\mathscr{Z}}_q = \{(x, Z) \mid Z \in \mathscr{Z}_q^0, Z = K + L, x \in L^\perp\}.$$

Then i: $(x, Z) \mapsto Z + x$ is an isomorphism of $\tilde{\mathscr{Z}}_q$ onto $\mathscr{Z}_q$. Assume now that X is weakly stationary. Then there exists a number $\gamma \geq 0$ and a probability measure P_0 on $\mathscr{Z}_q^0$ such that

$$i \circ \Theta(A \times C) = \gamma \int_C \lambda_{L(Z)^\perp}(A)\, \mathrm{d}P_0(Z), \tag{3.1}$$

for Borel sets $A \subset \mathbf{R}^d$, $C \subset \mathscr{Z}_q^0$. Here, $L(Z)$ is the direction space of the cylinder Z. We call γ the *intensity* and P_0 the *shape distribution of* the point process X. Moreover, X is weakly isotropic, if and only if P_0 is rotation invariant. Frequently, we use the abbreviation $P_0(f)$ for the integral $\int_{\mathscr{Z}_q^0} f(Z)\, \mathrm{d}P_0(Z)$. If X is a process of (convex) particles, P_0 is concentrated on $\mathscr{R}_d^0$ or $\mathscr{K}_d^0$, the set of particles K in $\mathscr{R}_d$ resp. $\mathscr{K}_d$ with $s(K) = 0$. If X is a process of q-flats, P_0 is concentrated on $\mathscr{L}_q^d$.

An important tool for the following is the Campbell theorem. In view of (3.1), it states that for every measurable and Θ-integrable function f on $\mathscr{Z}_q$

$$\mathbb{E} \sum_{Z \in X} f(Z) = \gamma \int_{\mathscr{Z}_q^0} \int_{L^\perp} f(K + L + x)\, \mathrm{d}\lambda_{L^\perp}(x)\, \mathrm{d}P_0(K + L). \tag{3.2}$$

We first use this result to give an interpretation of γ. For $K \in \mathscr{K}_d$ let

$$\mathscr{Z}_q(K) = \{Z \in \mathscr{Z}_q \mid Z \cap K \neq \emptyset\}$$

and let $1_{\mathscr{Z}_q(K)}(\cdot)$ be the indicator function of $\mathscr{Z}_q(K)$. Let B denote the unit ball in $\mathbf{R}^d$.

THEOREM 3.1. *We have*

$$\gamma = \lim_{r\to\infty} \frac{1}{r^{d-q}\kappa_{d-q}} \mathsf{E} \sum_{Z\in X} 1_{\mathscr{Z}_q(rB)}(Z).$$

Proof. From (3.2) we get

$$\begin{aligned}
&\mathsf{E} \sum_{Z\in X} 1_{\mathscr{Z}_q(rB)}(Z) \\
&\quad = \gamma \int_{\mathscr{Z}_q^0} \int_{L^\perp} 1_{\mathscr{Z}_q(rB)}(K+L+x)\, \mathrm{d}\lambda_{L^\perp}(x)\, \mathrm{d}P_0(K+L) \\
&\quad = \gamma \int_{\mathscr{Z}_q^0} \lambda_{L^\perp}[(rB\cap L^\perp)+K]\, \mathrm{d}P_0(K+L).
\end{aligned}$$

In the same way, we get from the Steiner formula (2.1)

$$\begin{aligned}
&\Theta(\{Z\in \mathscr{Z}_q \mid \operatorname{conv} Z\cap rB\neq \emptyset\}) \\
&\quad = \gamma \int_{\mathscr{Z}_q^0} \lambda_{L^\perp}[(rB\cap L^\perp)+\operatorname{conv} K]\, \mathrm{d}P_0(K+L) \\
&\quad = \sum_{j=0}^{d-q} r^{d-q-j}\kappa_{d-q-j} \int_{\mathscr{Z}_q^0} V_j(\operatorname{conv} K)\, \mathrm{d}P_0(K+L).
\end{aligned}$$

Since

$$\begin{aligned}
&\left| \frac{1}{r^{d-q}\kappa_{d-q}} \lambda_{L^\perp}[(rB\cap L^\perp)+K] \right| \\
&\quad \leqslant \sum_{j=0}^{d-q} r^{-j} \frac{\kappa_{d-q-j}}{\kappa_{d-q}} V_j(\operatorname{conv} K)
\end{aligned}$$

and since Θ is locally finite, the integrand in the following integral is uniformly integrable and we get

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{r^{d-q}\kappa_{d-q}} \mathsf{E} \sum_{Z\in X} 1_{\mathscr{Z}_q(rB)}(Z) \\
&\quad = \gamma \int_{\mathscr{Z}_q^0} \lim_{r\to\infty} \frac{1}{r^{d-q}\kappa_{d-q}} \lambda_{L^\perp}[(rB\cap L^\perp)+K]\, \mathrm{d}P_0(K+L) \\
&\quad = \gamma.
\end{aligned}$$

If in Theorem 3.1 the indicator function $1_{\mathscr{Z}_q(rB)}$ is replaced by $1_{\mathscr{Z}_q(rB)}\cdot f$, where f is a translation invariant function on $\mathscr{Z}_q$ such that

$$\int_{\mathscr{Z}_q^0} V_i(\operatorname{conv} Z)|f(Z)|\, \mathrm{d}P_0(Z) < \infty$$

for $i = q, \dots, d$, then the same arguments hold true and we get

$$\lim_{r\to\infty} \frac{1}{r^{d-q}\kappa_{d-q}} \mathbb{E} \sum_{Z\in X} 1_{\mathcal{Z}_q(rB)}(Z) \cdot f(Z)$$
$$= \gamma \cdot P_0(f).$$

We are especially interested in the case $f = V_j$.

THEOREM 3.2. *Suppose* $\int_{\mathcal{Z}_q^0} V_i(\mathrm{conv}\, Z) |V_j(Z)| \, \mathrm{d}P_0(Z) < \infty$ *for* i, $j = q, \dots, d$. *Then*

$$D_j(X) := \lim_{r\to\infty} \frac{1}{r^{d-q}\kappa_{d-q}} \mathbb{E} \sum_{Z\in X} 1_{\mathcal{Z}_q(rB)}(Z) V_j(Z)$$

exists and fulfills

$$D_j(X) = \gamma P_0(V_j), \quad j = q, \dots, d.$$

We call $D_j(X)$ the *jth quermass density* of the point process X. If the cylinders Z of the process are all simply connected, then $V_q(Z) = 1$ and, hence, $D_q(X) = \gamma$.

Results similar to Theorems 3.1 and 3.2 hold if the unit ball B is replaced by an arbitrary convex body K with dimension at least $d - q$, but the normalizing constant κ_{d-q} then looks more complicated. It has a simple form, namely $V_{d-q}(K)$, if X is weakly isotropic. Two special cases are of interest. If the cylinders are a.s. q-flats, i.e., if P_0 is concentrated on those $Z = K + L \in \mathcal{Z}_q^0$ with $K = \{0\}$, and if X is weakly isotropic, then γ gives the mean number of flats of the process X per unit $(d - q)$-dimensional volume. A better mathematical basis of this interpretation of γ will be given later. Second, if $q = 0$, then X is a process of particles. In this case we have

$$\gamma = \lim_{r\to\infty} \frac{1}{V_d(rK_0)} \mathbb{E} \sum_{K\in X} 1_{\mathcal{R}_d(rK_0)}(K)$$

and

$$D_j(X) = \gamma P_0(V_j) = \lim_{r\to\infty} \frac{1}{V_d(rK_0)} \mathbb{E} \sum_{K\in X} 1_{\mathcal{R}_d(rK_0)}(K) V_j(K)$$

without requiring that X is isotropic. Here $K_0 \in \mathcal{K}_d$ is arbitrary (but with inner points) and $\mathcal{R}_d(rK_0) = \{K \in \mathcal{R}_d \mid K \cap rK_0 \neq \emptyset\}$. The proof is similar to that of Theorem 3.1 and Theorem 3.2 but, instead of the Steiner formula, mixed volumes have to be used.

4. Formulas for Quermass Densities

With the help of (3.2) it is now easy to apply integral formulas for curvature measures to point processes. Let us start with (2.3). Our assumptions are the following. We consider a point process X of cylinders (i.e., a point process on $\mathscr{Z}_q$ with $q \in \{0, \ldots, d-1\}$) which is weakly stationary (and fulfills the appropriate integrability conditions of Section 3) and a set $K_0 \in \mathscr{R}_d$ which serves as a sampling window. Moreover, we fix a Borel set β_0 (associated with K_0) and we assume that to each cylinder $Z = K + L$ a Borel set $\beta(Z) = \beta(K) + L$ is associated in a measurable and translation covariant way, i.e., we have

$$\beta(Z + x) = \beta(Z) + x$$

for all $x \in \mathbf{R}^d$. Then from (3.2) and (2.3) we get

$$\begin{aligned}
&\mathsf{E} \sum_{Z \in X} \Psi_j(K_0 \cap Z, \beta_0 \cap \beta(Z)) \\
&\quad = \gamma \int_{\mathscr{Z}_q^0} \int_{L^\perp} \Psi_j(K_0 \cap (K + L + x), \beta_0 \cap (\beta(K) + L + x))\, \mathrm{d}\lambda_{L^\perp}(x)\, \mathrm{d}P_0(K + L) \\
&\quad = \gamma \Big[\Psi_j(K_0, \beta_0) \int_{\mathscr{Z}_q^0} \Psi_{d-q}(K, \beta(K))\, \mathrm{d}P_0(K + L) + \qquad (4.1) \\
&\qquad + \sum_{k=j+1}^{d-1} \int_{\mathscr{Z}_q^0} \varphi_k^{(j)}(K_0, Z, \beta_0 \times \beta(Z))\, \mathrm{d}P_0(Z) + \\
&\qquad + \Psi_d(K_0, \beta_0) \int_{\mathscr{Z}_q^0} \Psi_{j-q}(K, \beta(K))\, \mathrm{d}P_0(K + L) \Big],
\end{aligned}$$

for $j = 0, \ldots, d$.

Because of our notation, the last summand in (4.1) vanishes if $j < q$ and, in this case, the summation is from $j+1$ to $d+j-q$.

Let us examine some special cases of formula (4.1). First, if $\beta_0 = \beta(Z) = \mathbf{R}^d$, then we get

$$\begin{aligned}
&\mathsf{E} \sum_{Z \in X} V_j(K_0 \cap Z) \\
&\quad = \gamma V_j(K_0) P_0(V_d) + \qquad (4.2) \\
&\qquad + \gamma \sum_{k=j+1}^{d-1} P_0(\Phi_k^{(j)}(K_0, \cdot)) + \gamma V_d(K_0) P_0(V_j).
\end{aligned}$$

The right-hand side can be expressed by means of mixed volumes using (2.14) and (2.15). In view of the homogeneity properties, (4.2) implies the following interpretation of $D_j(X)$.

THEOREM 4.1. *We have*

$$D_j(X) = \lim_{r \to \infty} \frac{1}{V_d(rK_0)} \mathsf{E} \sum_{Z \in X} V_j(rK_0 \cap Z)$$

for all $K_0 \in \mathscr{R}_d$ with inner points, $j = q, \ldots, d$.

To give another interpretation of $D_j(X)$ let $\beta_0 \subset \operatorname{int} K_0$ and $\beta(Z) = \mathbf{R}^d$. Then

$\Psi_j(K_0, \beta_0) = 0$, for $j < d$, and $\varphi_k^{(j)}(K_0, Z, \beta_0 \times \mathbf{R}^d) = 0$, for $k < d$. Hence, (4.1) implies

$$\mathbb{E} \sum_{Z \in X} \Psi_j(K_0 \cap Z, \beta_0) = \gamma \Psi_d(K_0, \beta_0) P_0(V_j). \tag{4.3}$$

Since the curvature measures are locally defined and since we may choose to each bounded Borel set β_0 a set $K_0 \in \mathcal{R}_d$ with $\beta_0 \subset \operatorname{int} K_0$, this implies the following result.

THEOREM 4.2. *We have*

$$\mathbb{E} \sum_{Z \in X} \Psi_j(Z, \cdot) = D_j(X) \cdot \lambda_d,$$

for $j = q, \ldots, d$.

Whilst the equation

$$D_j(X) = \gamma P_0(V_j), \quad j = q, \ldots, d,$$

presents the most direct way to define the quermass density $D_j(X)$ for the point process X, the formulas in Theorems 3.2, 4.1, and 4.2 describe other possible approaches, namely approaches of 'ergodic' type and the random measure approach. As the results show, the different methods lead to the same quantity, namely $D_j(X)$. Similar considerations have been made for random sets and point processes of sets from other set classes (Weil, 1984; Weil and Wieacker, 1984; Zähle, 1986). Using Lemma 6 in Weil and Wieacker (1984) one also gets

$$D_j(X) = \mathbb{E} \sum_{Z \in X} [V_j(C_0 \cap Z) - V_j(\delta^+ C_0 \cap Z)]$$

$j = q, \ldots, d$, where C_0 is a unit cube and $\delta^+ C_0$ its 'upper right' boundary (see Weil and Wieacker (1984) for details).

If the point process X is ergodic, then the formulas in Theorems 3.2 and 4.1 hold for almost all realizations $X(\omega)$ (or in L^1) without the expectation sign. In particular, any Poisson process X is ergodic (see Nguyen and Zessin (1979) and Wieacker (1982) for more details).

As a consequence of Theorem 4.2 (or (4.3)) we get the following interesting formula:

$$\mathbb{E} \sum_{Z \in X} \Psi_j(Z, \operatorname{int} K_0) = V_d(K_0) D_j(X), \quad j = q, \ldots, d. \tag{4.4}$$

As other special cases of (4.1) we put $\beta_0 = \mathbf{R}^d$ and $\beta(Z) = \operatorname{int} Z$. Then

$$\mathbb{E} \sum_{Z \in X} \Psi_j(K_0, \operatorname{int} Z) = V_j(K_0) D_d(X). \tag{4.5}$$

For $\beta_0 = \text{bd } K_0$ and $\beta(Z) = \text{bd } Z$, we get

$$\mathbb{E} \sum_{Z \in X} \Psi_j(K_0 \cap Z, \text{bd } K_0 \cap \text{bd } Z) = \gamma \sum_{k=j+1}^{d-1} P_0(\Phi_k^{(j)}(K_0, \cdot)) \tag{4.6}$$

which can be expressed by mixed volumes again.

For weakly isotropic X, these formulas can be simplified further in view of the integral formula (2.8). In this case, (4.1) becomes

$$\mathbb{E} \sum_{Z \in X} \Psi_j(K_0 \cap Z, \beta_0 \cap \beta(Z)) = \gamma \sum_{k=j}^{d} \alpha_{djk} \Psi_k(K_0, \beta_0) \int_{\mathscr{X}_q^{0}} \Psi_{d+j-q-k}(K, \beta(K)) \, dP_0(K+L), \tag{4.7}$$

$j = 0, \ldots, d$.

The corresponding changes in (4.2) and (4.6) are obvious.

For a process X of particles (i.e., $q = 0$), (4.2) can be written as

$$\begin{aligned} \mathbb{E} \sum_{K \in X} V_j(K_0 \cap K) &= \gamma \Bigg[V_j(K_0) P_0(V_d) + \sum_{k=j+1}^{d-1} \frac{\binom{d}{k-j}\binom{d}{j}\kappa_d}{\kappa_{d-j}\kappa_j} \times \\ &\times \int_{\mathscr{E}_{d-j}^d} P_0(V(\underbrace{-K_0 \cap E, \ldots, -K_0 \cap E}_{k-j}, \underbrace{\cdot, \ldots, \cdot}_{d+j-k})) \, d\mu_{d-j}(E) + \\ &+ V_d(K_0) P_0(V_j) \Bigg]. \end{aligned} \tag{4.8}$$

For weakly isotropic X, this formula becomes

$$\mathbb{E} \sum_{K \in X} V_j(K_0 \cap K) = \gamma \sum_{k=j}^{d} \alpha_{djk} V_k(K_0) P_0(V_{d+j-k}),$$

a result which was first obtained by Fava and Santaló (1978), (1979).

For another consequence of (4.2) let X be a process of flats. If $j = q$, we get a sharper form of Theorem 4.1.

THEOREM 4.3. *For a process X of q-flats we have*

$$\gamma = \frac{1}{V_d(K_0)} \mathbb{E} \sum_{L \in X} V_q(K_0 \cap L).$$

Thus, the intensity of X can be also interpreted as the mean q-dimensional volume of the q-flats in X per unit volume of $\mathbf{R}^d$. If $j \neq q$, there is only one term on the right side of (4.2) which does not vanish, namely the summand $P_0(\Phi^{(j)}_{d+j-q}(K_0,\cdot))$. It can be expressed more explicitly with the help of (2.13).

THEOREM 4.4. *Let $K_0 \in \mathcal{R}_d$ and X be a point process of q-flats. Then*

$$\mathsf{E}\sum_{L\in X} V_j(K_0\cap L) = \gamma \frac{\binom{d}{q-j}}{\kappa_{q-j}} P_0(V(\underbrace{K_0,\ldots,K_0}_{d+j-q},\underbrace{B_\bullet,\ldots,B_\bullet}_{q-j})).$$

In particular, if the affine hull of K_0 is $(d+j-q)$-dimensional (and $L_0\in\mathcal{L}^d_{d+j-q}$ parallel to this affine hull), then

$$V(\underbrace{K_0,\ldots,K_0}_{d+j-q},\underbrace{B_L,\ldots,B_L}_{q-j}) = V_{d+j-q}(K_0)\frac{\kappa_{q-j}}{\binom{d}{q-j}}[L_0,L]$$

and hence

$$\mathsf{E}\sum_{L\in X} V_j(K_0\cap L) = \gamma\, V_{d+j-q}(K_0)\int_{\mathcal{L}^d_q}[L_0,L]\,\mathrm{d}P_0(L). \tag{4.9}$$

If X is weakly isotropic, we use (4.7) instead of (4.1) and get a generalization of Theorem 4.3.

THEOREM 4.5. *Let X be a weakly isotropic point process of q-flats, let $K_0\in\mathcal{R}_d$, and $q\in\{0,\ldots,d-1\}$. Then*

$$\gamma = \frac{1}{\alpha_{djq}V_{d+j-q}(K_0)}\sum_{L\in X} V_j(K_0\cap L)$$

for $j=0,\ldots,q$.

In particular, this result includes q different interpretations of γ as mean value per unit content if we take a $(d+j-q)$-dimensional set K_0.

5. Point Processes Induced on Lower Dimensional Flats

Let X be a point process of cylinders with the properties mentioned at the beginning of the last section (weakly stationary, etc.) and let E_r be a fixed r-flat, $r\in\{1,\ldots,d-1\}$. We consider the intersection process $X\cap E_r$ which consists of the sets $Z\cap E_r$, $Z\in X$. If $Z=K+L$ and if L and E_r are in general relative position, then $Z\cap E_r$ is either empty or a cylinder in $\mathcal{Z}_p$ with $p=\max\{0,q+r-d\}$. Let $\mathcal{Z}^0_{q,E_r}$ be the set of all cylinders $Z=K+L\in\mathcal{Z}^0_q$, for which L and E_r are not in general relative position. Then

$$\mu_r(\{E_r\mid P_0(\mathcal{Z}^0_{q,E_r})>0\})=0,$$

i.e., for μ_r-almost all E_r, almost all realizations $X(\omega)$ of X have the property that L and E_r are in general relative position for all $K+L \in X(\omega)$. Hence, for almost all E_r the intersection process $X \cap E_r$ is a process of cylinders in $\mathscr{Z}_p$. It may, however, be possible, even in this case that $Z \cap E_r = \emptyset$ for all $Z \in X$. This event occurs either with probability zero or with probability one. While the first possibility is, of course, of no significance, the second occurs, if and only if X is a process of q-flats and $q+r<d$. We may then include this 'empty' process $X \cap E_r$ into our considerations as a point process of cylinders with intensity zero.

After these explanations we can assume that $X \cap E_r$ is a point process of cylinders in $\mathscr{Z}_p$ (lying in E_r). It is obvious that $X \cap E_r$ has the same invariance properties as X. In particular, for weakly stationary X, $X \cap E_r$ is also weakly stationary. Therefore, its intensity measure is determined by the shape distribution $P_0^{E_r}$ and the intensity γ_{E_r}. Our goal is to express γ_{E_r}, the mean values $P_0^{E_r}(V_j)$, and the quermass densities $D_j(X \cap E_r)$ by means of γ, $P_0(V_k)$, and $D_k(X)$. For this purpose, let $K_0 \subset E_r$ be r-dimensional, $r \geq j$, $\beta(Z) = \mathbf{R}^d$, and β_0 in the relative interior of K_0. Then

$$\Psi_j(K_0 \cap Z, \beta_0) = \Psi_j(Z \cap E_r, \beta_0).$$

Also,

$$\varphi_k^{(j)}(K_0, Z, \beta_0 \times \mathbf{R}^d) = 0,$$

for $k \neq r$, and (for $Z = K+L$ and $r-j \leq d-q$)

$$\varphi_r^{(j)}(K_0, Z, \beta_0 \times \mathbf{R}^d) = \frac{\binom{d}{q}\binom{d-q}{r-j}}{\kappa_{r-j}\kappa_q} V(\underbrace{K, \ldots, K}_{d+j-r-q}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-j}, \underbrace{B_L, \ldots, B_L}_{q}) \cdot \lambda_{E_r}(\beta_0).$$

This follows for polytopes K, K_0 (in general relative position) from (2.4) and (2.16), for $K, K_0 \in \mathscr{K}_d$ by approximation, and for $K, K_0 \in \mathscr{R}_d$ by additivity. Hence, we conclude from (4.1)

$$\mathsf{E} \sum_{Z \in X} \Psi_j(Z \cap E_r, \cdot) = \frac{\binom{d}{q}\binom{d-q}{r-j}}{\kappa_{r-j}\kappa_q} \times$$
$$\times \gamma\left(\int_{\mathscr{Z}_q^0} V(\underbrace{K, \ldots, K}_{d+j-q-r}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-j}, \underbrace{B_L, \ldots, B_L}_{q}) \, \mathrm{d}P_0(K+L)\right) \cdot \lambda_{E_r}$$

for $j = p, \ldots, r$.

On the other hand,

$$\mathbb{E} \sum_{Z \in X} \Psi_j(Z \cap E_r, \cdot) = \mathbb{E} \sum_{Z \in X \cap E_r} \Psi_j(Z, \cdot) = D_j(X \cap E_r) \cdot \lambda_{E_r}$$

by Theorem 4.2. Thus, we have proved the following formula for the quermass densities of the induced point process $X \cap E_r$.

THEOREM 5.1. *We have*

$$D_j(X \cap E_r) = \frac{\binom{d}{q}\binom{d-q}{r-j}}{\kappa_{r-j}\kappa_q} \times \gamma \int_{\mathscr{X}_q^0} V(\underbrace{K, \ldots, K}_{d+j-q-r}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-j}, \underbrace{B_L, \ldots, B_L}_{q}) \, \mathrm{d}P_0(K+L),$$

for $j = p, \ldots, r$.

If the cylinders of the process X are all convex, then $D_p(X \cap E_r) = \gamma_{E_r}$ for all E_r, hence, Theorem 5.1 gives an expression for γ_{E_r},

$$\gamma_{E_r} = \begin{cases} \dfrac{\binom{d}{q}\binom{d-q}{r}}{\kappa_r\kappa_q}\gamma \times \displaystyle\int_{\mathscr{X}_q^0} V(\underbrace{K, \ldots, K}_{d-q-r}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r}, \underbrace{B_L, \ldots, B_L}_{q}) \, \mathrm{d}P_0(K+L), & q+r<d, \\ \gamma \displaystyle\int_{\mathscr{X}_q^0} [E_r, L] \, \mathrm{d}P_0(K+L), & q+r \geqslant d. \end{cases} \tag{5.1}$$

Here, we have used

$$V(\underbrace{B_{E_r}, \ldots, B_{E_r}}_{d-q}, \underbrace{B_L, \ldots, B_L}_{q}) = \frac{\kappa_{d-q}\kappa_q}{\binom{d}{q}}[E_r, L].$$

Also, in this case, Theorem 3.2 implies

$$P_0^{E_r}(V_j) = \frac{\binom{d}{q}\binom{d-q}{r-j}}{\kappa_{r-j}\kappa_q} \frac{\gamma}{\gamma_{E_r}} \times \int_{\mathscr{X}_q^0} V(\underbrace{K, \ldots, K}_{d+j-q-r}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-j}, \underbrace{B_L, \ldots, B_L}_{q}) \, \mathrm{d}P_0(K+L). \tag{5.2}$$

Matheron (1975) has corresponding formulas for Poisson processes of convex particles. In order to show the connection, let X be a process of convex particles. Since

$$V(\underbrace{K, \dots, K}_{d-r}, \underbrace{B_{E_r}, \dots, B_{E_r}}_{r})$$

is (up to a constant) the $(d-r)$-content of the projection $K \mid E_r^\perp$ of K onto $E_r^\perp$, we get from (5.1)

$$\gamma_{E_r} = \binom{d}{r} \cdot \gamma P_0(V_{d-r}(\cdot \mid E_r^\perp)). \tag{5.3}$$

For $j \geq 1$, the following projection formula can be deduced, e.g., from (9.7) in Schneider and Weil (1983):

$$\int_{\mathscr{L}_{r-j}^{E_r}} V_{d+j-r}(K \mid M^\perp) \, d\nu_{r-j}^{E_r}(M)$$

$$= \frac{\binom{d}{r-j}\kappa_j}{\binom{r}{j}\kappa_{d+j-r}} V(\underbrace{K, \dots, K}_{d+j-r}, \underbrace{B_{E_r}, \dots, B_{E_r}}_{r-j}). \tag{5.4}$$

Here $\nu_{r-j}^{E_r}$ is the normalized invariant measure on $\mathscr{L}_{r-j}^{E_r}$, the space of $(r-j)$-dimensional subspaces in E_r. Therefore, (5.2) implies

$$P_0^{E_r}(V_j) = \frac{\binom{r}{j}\kappa_{d+j-r}}{\kappa_{r-j}\kappa_j} \frac{\gamma}{\gamma_{E_r}} \int_{\mathscr{L}_{r-j}^{E_r}} P_0(V_{d+j-r}(\cdot \mid M^\perp)) \, d\nu_{r-j}^{E_r}(M), \tag{5.5}$$

$j = 1, \dots, r$.

Equations (5.3) and (5.5) are Matheron's formulas; as we have seen, they are valid without Poisson assumptions.

If X is weakly isotropic, then we use (2.8) instead of (2.4) and get

$$D_j(X \cap E_r) = \alpha_{djr} \gamma P_0(V_{d+j-r}), \tag{5.6}$$

$j = p, \dots, r$. For convex cylinders, (5.6) implies

$$\gamma_{E_r} = \begin{cases} \alpha_{d0r}\gamma P_0(V_{d-r}), & q + r < d \\ \alpha_{d,q+r-d,r}, & q + r \geq d, \end{cases} \tag{5.7}$$

and

$$P_0^{E_r}(V_j) = \alpha_{djr} \frac{\gamma}{\gamma_{E_r}} P_0(V_{d+j-r}). \tag{5.8}$$

For Poisson processes of particles ($q = 0$), (5.7) and (5.8) were first proved by

Matheron (1975); his result was extended to non-Poissonian processes by Stoyan (1979), (1982).

If X is a process of flats, Theorem 5.1 implies that

$$D_j(X \cap E_r) = 0,$$

for all j if $q + r < d$. In particular, $\gamma_{E_r} = D_0(X \cap E_r) = 0$, hence $X \cap E_r$ is the 'empty process', as we have mentioned earlier. If $q + r \geq d$, then

$$D_j(X \cap E_r) = 0$$

for $j = q + r - d + 1, \ldots, r$. The following formula for $D_{q+r-d}(X \cap E_r)$ is just the second part of (5.1).

THEOREM 5.2. *Let X be a point process of q-flats with $q + r \geq d$. Then*

$$\gamma_{E_r} = \gamma \int_{\mathscr{L}_q^d} [E_r, L] \, \mathrm{d}P_0(L).$$

Equation (5.7) gives the corresponding result for weakly isotropic X. Again, for Poisson processes of flats, similar formulas are obtained in Matheron (1975).

Finally, let us assume that X is a process of j-sets $K \in \mathcal{R}_d$, $j \in \{0, \ldots, d-1\}$. Then we can strengthen the formula in Theorem 5.1. For a j-set $K \in \mathcal{R}_d$, there are sets $K_1, \ldots, K_n \in \mathcal{R}_d$ of dimension at most j with $K = \bigcup_{i=1}^n K_n$ and

$$\Psi_j(K, \cdot) = \sum_{i=1}^n \Psi_j(K_i, \cdot),$$

Let

$$\eta_K = \sum_{i=1}^n V_j(K_i) \epsilon_{L(K_i)} \tag{5.9}$$

where $\epsilon_{L(K_i)}$ is the Dirac measure on $L(K_i) \in \mathscr{L}_j^d$, a subspace parallel to the affine hull of K_i. It is easily seen that η_K depends only on K and not on the special representation $K = \bigcup_{i=1}^n K_i$. η_K is a measure on $\mathscr{L}_j^d$ with $\eta_K(\mathscr{L}_j^d) = V_j(K)$, hence

$$\tilde{P}_0 = \frac{1}{P_0(V_j)} P_0(\eta.)$$

is a probability measure on $\mathscr{L}_j^d$. (Here we have to assume $P_0(V_j) > 0$, but the case $P_0(V_j) = 0$ is of no interest for the following. We also leave out the details for the measurability of $K \mapsto \eta_K$, which follows from the definition of X by counting measures.)

We call $\tilde{P}_0$ the ***directional distribution*** of X.

THEOREM 5.3. *For a process X of j-sets $K \in \mathcal{R}_d$ we have*

$$\begin{aligned} &D_{j+r-d}(X \cap E_r) \\ &\quad = D_j(X) \int_{\mathscr{L}_j^d} [L, E_r] \, \mathrm{d}\tilde{P}_0(L), \quad r = d - j, \ldots, d. \end{aligned}$$

Proof. From Theorem 5.1, we know that

$$D_{j+r-d}(X \cap E_r) = \frac{\binom{d}{j}}{\kappa_{d-j}} \gamma \int_{\mathscr{R}_d^0} V(\underbrace{K, \ldots, K}_{j}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{d-j}) \, \mathrm{d}P_0(K).$$

Using the representation $K = \bigcup_{i=1}^n K_i$ underlying (5.9), we get

$$\begin{aligned} V(\underbrace{K, \ldots, K}_{j}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{d-j}) &= \sum_{i=1}^{n} V(\underbrace{K_i, \ldots, K_i}_{j}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{d-j}) \\ &= \frac{\kappa_{d-j}}{\binom{d}{j}} \sum_{i=1}^{n} V_j(K_i)[L(K_i), E_r] \\ &= \frac{\kappa_{d-j}}{\binom{d}{j}} \int_{\mathscr{L}_j^d} [L, E_r] \, \mathrm{d}\eta_K(L), \end{aligned}$$

hence

$$\begin{aligned} D_{j+r-d}(X \cap E_r) &= \gamma \int_{\mathscr{L}_j^d} [L, E_r] \, \mathrm{d}(P_0(\eta.)(L)) \\ &= D_j(X) \int_{\mathscr{L}_j^d} [L, E_r] \, \mathrm{d}\tilde{P}_0(L). \end{aligned}$$

For a process X of j-sets we may consider the union set Y,

$$Y(\omega) = \bigcup_{K \in X(\omega)} K,$$

which is a weakly stationary, random closed set in $\mathbf{R}^d$. If the particles of X do not overlap (i.e., if their intersections are $(j-1)$-sets), then the quermass densities $D_j(X)$ and $D_{j+r-d}(X \cap E_r)$ depend only on Y.

More precisely, from Theorem 4.2 it follows that $D_j(X) = D_j(Y)$ and

$$D_{j+r-d}(X \cap E_r) = D_{j+r-d}(Y \cap E_r)$$

(see Weil (1983b) for details on quermass densities of random sets). Also, the directional distribution $\tilde{P}_0$ of X depends only on Y. To see this, the following formula can be derived similarly to (4.3). Let $\mathscr{A}$ be a Borel set in $\mathscr{L}_j^d$ and, for a j-set $K \in \mathscr{R}_d$, let $\beta(K, \mathscr{A})$ be the closure of all midpoints of j-dimensional balls in K which are parallel to subspaces in $\mathscr{A}$. As one can easily show, $K \mapsto \beta(K, \mathscr{A})$ is measurable and translation covariant. Let K_0 be a ball with $V_d(K_0) = 1$. Then

$$\tilde{P}_0(\mathscr{A}) = \frac{1}{D_j(Y)} \mathbb{E}\Psi_j(K_0 \cap Y, \beta(K_0 \cap Y, \mathscr{A})).$$

For stationary fibre processes ($j = 1$) in $\mathbf{R}^2$ or $\mathbf{R}^3$ and for stationary surface processes ($j = 2$) in $\mathbf{R}^3$, Theorem 5.3 was obtained by Mecke and Stoyan (1980a), Mecke and Nagel (1980), and Pohlmann, Mecke, and Stoyan (1981) (see also Stoyan, Mecke and Pohlmann (1980) for a special nonisotropic fibre process in $\mathbf{R}^2$). Here fibre processes and surface processes are point processes on more general set classes but, as we have mentioned in the introduction, our model can serve as an approximation. As a more general result, which contains Theorem 5.2 and Theorem 5.3 as special cases, Zähle (1982) proved intersection formulas for point processes of Hausdorff rectifiable closed sets. The correspondence of $\tilde{P}_0$ with the distribution of tangents or normals in the above-mentioned papers can be seen from (5.9). We finally remark that $\tilde{P}_0$ can be interpreted as the distribution of the tangent space in a 'typical point' of the random set Y.

In order to obtain a second point process in E_r we consider the case of projected thick sections. Let the convex cylinder $Z_0 = K_0 + E_r$ be a thickening of the flat E_r by a set $K_0 \in \mathscr{K}_d$, $K_0 \subset E_r^{\perp}$. If X is a weakly stationary point process of convex particles, we denote by $(X \cap Z_0) \mid E_r$ the collection of projected intersections $(K \cap Z_0) \mid E_r$, $K \in X$. $(X \cap Z_0) \mid E_r$ is also a weakly stationary process of convex particles (in E_r). We denote its intensity by γ_{Z_0} and its shape distribution by $P_0^{Z_0}$. In order to get formulas for γ_{Z_0}, $P_0^{Z_0}(V_j)$, and $D_j((X \cap Z_0) \mid E_r)$, let β be a Borel set in the relative interior of a convex body $K' \subset E_r$. As in Section 4, we get from the Campbell theorem (3.2)

$$\begin{aligned}
&\mathbb{E} \sum_{K \in X} \Psi_j((K \cap Z_0) \mid E_r, \beta) \\
&= \gamma \int_{\mathscr{K}_d^0} \int_{\mathbf{R}^d} \Psi_j(((K + x) \cap Z_0) \mid E_r, \beta) \, \mathrm{d}\lambda_d(x) \, \mathrm{d}P_0(K) \qquad (5.10) \\
&= \gamma \int_{\mathscr{K}_d^0} \int_{E_r} \int_{E_r^{\perp}} \Psi_j(((K + x + y) \cap Z_0) \mid E_r, \beta) \, \mathrm{d}\lambda_{E_r^{\perp}}(y) \, \mathrm{d}\lambda_{E_r}(x) \, \mathrm{d}P_0(K).
\end{aligned}$$

Using (2.11), we can simplify the inner integral

$$\begin{aligned}
&\int_{E_r^{\perp}} \Psi_j(((K + x + y) \cap Z_0) \mid E_r, \beta) \, \mathrm{d}\lambda_{E_r^{\perp}}(y) \\
&= \int_{E_r^{\perp}} \Psi_j(((K + x) \cap (Z_0 + y)) \mid E_r, \beta) \, \mathrm{d}\lambda_{E_r^{\perp}}(y) \qquad (5.11) \\
&= \varphi_{d+j-r}^{(j)}(K + (-K_0) + x, E_r, \beta + E_r^{\perp}).
\end{aligned}$$

If K, K_0 are polytopes in general relative position, (2.4) implies

$$\begin{aligned}
&\int_{E_r} \varphi^{(j)}_{d+j-r}(K+(-K_0)+x, E_r, \beta+E_r^{\perp})\,\mathrm{d}\lambda_{E_r}(x)\\
&\quad = \int_{E_r} \sum_{F\in\mathcal{F}_{d+j-r}(K+(-K_0)+x)} \gamma(F, E_r, K+(-K_0)+x, E_r)\times\\
&\qquad \times[L(F), E_r]\lambda_F(\beta+E_r^{\perp})\,\mathrm{d}\lambda_{E_r}(x) \qquad (5.12)\\
&\quad = \sum_{F\in\mathcal{F}_{d+j-r}(K+(-K_0))} \gamma(F, E_r, K+(-K_0), E_r)[L(F), E_r]\times\\
&\qquad \times\left(\int_{E_r} \lambda_F(\beta+x+E_r^{\perp})\,\mathrm{d}\lambda_{E_r}(x)\right).
\end{aligned}$$

By (2.3)

$$\begin{aligned}
&\int_{E_r} \lambda_F(\beta+x+E_r^{\perp})\,\mathrm{d}\lambda_{E_r}(x)\\
&\quad = \int_{E_r} \Psi_{d+j-r}(F\cap(K'+E_r^{\perp}+x), \beta+E_r^{\perp}+x)\,\mathrm{d}\lambda_{E_r}(x) \qquad (5.13)\\
&\quad = V_{d+j-r}(F)\lambda_{E_r}(\beta).
\end{aligned}$$

Combining (5.12) and (5.13) and using (2.13) and the multilinearity of mixed volumes, we get

$$\begin{aligned}
&\int_{E_r} \varphi^{(j)}_{d+j-r}(K+(-K_0)+x, E_r, \beta+E_r^{\perp})\,\mathrm{d}\lambda_{E_r}(x)\\
&= \Phi^{(j)}_{d+j-r}(K+(-K_0), E_r)\lambda_{E_r}(\beta)\\
&= \frac{\binom{d}{r-j}}{\kappa_{r-j}} \sum_{i=j}^{d+j-r} \binom{d+j-r}{i} V(\underbrace{K,\dots,K}_{i}, \underbrace{-K_0,\dots,-K_0}_{d+j-r-i}, \underbrace{B_{E_r},\dots,B_{E_r}}_{r-j})\cdot\lambda_{E_r}(\beta),
\end{aligned}$$

first for polytopes in general relative position, then for arbitrary K, K_0 by approximation.

In view of (5.10), (5.11), and Theorem 4.2, this result gives a formula for $D_j((X\cap Z_0)\mid E_r)$.

THEOREM 5.4. *We have*

$$\begin{aligned}
&D_j((X\cap Z_0)\mid E_r)\\
&\quad = \frac{\binom{d}{r-j}}{\kappa_{r-j}}\gamma \sum_{i=j}^{d+j-r} \binom{d+j-r}{i}\times\\
&\qquad \times\int_{\mathcal{K}_d^0} V(\underbrace{K,\dots,K}_{i}, \underbrace{-K_0,\dots,-K_0}_{d+j-r-i}, \underbrace{B_{E_r},\dots,B_{E_r}}_{r-j})\,\mathrm{d}P_0(K),
\end{aligned}$$

for $j=0,\dots,r$.

For weakly isotropic X,

$$\begin{aligned}
&\int_{\mathcal{K}_d^0} V(\underbrace{K, \ldots, K}_{i}, \underbrace{-K_0, \ldots, -K_0}_{d+j-r-i}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-j}) \,\mathrm{d}P_0(K) \\
&= \int_{SO_d} \int_{\mathcal{K}_d^0} V(\underbrace{\theta K, \ldots, \theta K}_{i}, \underbrace{-K_0, \ldots, -K_0}_{d+j-r-i}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-j}) \,\mathrm{d}P_0(K) \,\mathrm{d}\nu(\theta) \\
&= \int_{\mathcal{K}_d^0} \int_{SO_d} V(\underbrace{K, \ldots, K}_{i}, \underbrace{-\theta K_0, \ldots, -\theta K_0}_{d+j-r-i}, \underbrace{\theta B_{E_r}, \ldots, \theta B_{E_r}}_{r-j}) \,\mathrm{d}\nu(\theta) \,\mathrm{d}P_0(K) \\
&= \frac{\kappa_{r-j}}{\binom{d}{r-j}} \gamma_{djir} P_0(V_i) V_{d+j-r-i}(K_0),
\end{aligned}$$

where we have used a rotation formula in Schneider (1981). Thus, Theorem 5.4 implies

$$D_j((X \cap Z_0) \mid E_r) = \gamma \sum_{i=j}^{d+j-r} \gamma_{djir} P_0(V_i) V_{d+j-r-i}(K_0), \tag{5.14}$$

$j = 0, \ldots, r$.

Since $\gamma_{Z_0} = D_0((X \cap Z_0) \mid E_r)$, formulas for γ_{Z_0} and $P_0^{Z_0}(V_j)$ follow from Theorem 5.4 and (5.14). For circular cylinders Z_0, (5.14) is due to Davy (1976). As we have mentioned, (2.11) and (2.12) can be generalized to sets in $\mathcal{R}_d$ with appropriate modifications. We only consider the case of a fibre process X, i.e., a process of 1-sets. Here

$$\begin{aligned}
&D_1((X \cap Z_0) \mid E_r) \\
&= \frac{r\binom{d}{r}}{\kappa_{r-1}} \gamma \int_{\mathcal{R}_d^0} V(K, \underbrace{-K_0, \ldots, -K_0}_{d-r}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-1}) \,\mathrm{d}P_0(K),
\end{aligned}$$

which can be simplified using the directional distribution $\tilde{P}_0$

$$\begin{aligned}
&D_1((X \cap Z_0) \mid E_r) \\
&= \frac{r\binom{d}{r}}{\kappa_1 \kappa_{r-1}} D_1(X) \int_{\mathcal{L}_1^d} V(B_L, \underbrace{K_0, \ldots, K_0}_{d-r}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r-1}) \,\mathrm{d}\tilde{P}_0(L).
\end{aligned} \tag{5.15}$$

If X is weakly isotropic, then

$$D_1((X \cap Z_0) \mid E_r) = \frac{r\kappa_r \kappa_{d-1}}{d\kappa_{r-1}\kappa_d} D_1(X) V_{d-r}(K_0). \tag{5.16}$$

If the fibres of the process X have no segments in common, i.e., if the intersections of different fibres are 0-sets, then for μ_r-almost all E_r and for $d \geq 3$, $D_1((X \cap Z_0) \mid E_r)$ depends only on the union set of $(X \cap Z_0) \mid E_r$.

For $D_0((X \cap Z_0) \mid E_r)$ or γ_{Z_0} no simple formulas exist, in general. If X is a weakly stationary process of simply connected fibres each fibre $K \in X$ intersects Z_0 in finitely many (simply) connected parts. In order to determine γ_{Z_0}, not only the number of these parts must be recognized from their projections onto E_r but also it must be clear which parts in Z_0 come from the same original fibre $K \in X$. Quite often this is impossible and it is much more realistic to consider the different connected parts of $K \cap Z_0$, $K \in X$, as the fibres of a new fibre process (in Z_0) which are then projected onto E_r. Let the resulting fibre process in E_r be denoted by $\tilde{X}_{Z_0}$ and its intensity by $\tilde{\gamma}_{Z_0}$. For $\tilde{\gamma}_{Z_0}$, a simple formula can be given. By construction of $\tilde{X}_{Z_0}$ and from Theorem 3.1 we have

$$\tilde{\gamma}_{Z_0} = \lim_{t\to\infty} \frac{1}{V_r(tB_{E_r})} \mathsf{E} \sum_{K\in X} V_0(K \cap (K_0 + tB_{E_r})).$$

Now we can use (4.8)

$$\begin{aligned} \tilde{\gamma}_{Z_0} &= \lim_{t\to\infty} \frac{1}{V_r(tB_{E_r})} \gamma [V_d(K_0 + tB_{E_r}) + \\ &\quad + dP_0(V(-K_0 + tB_{E_r}, \ldots, -K_0 + tB_{E_r}, \cdot))] \\ &= \lim_{t\to\infty} \frac{1}{V_r(tB_{E_r})} \gamma [V_{d-r}(K_0)\, V_r(tB_{E_r}) + \\ &\quad + dP_0(V(\underbrace{-K_0, \ldots, -K_0}_{d-r}, \underbrace{tB_{E_r}, \ldots, tB_{E_r}}_{r-1}, \cdot)) + \\ &\quad + dP_0(V(\underbrace{-K_0, \ldots, -K_0}_{d-r-1}, \underbrace{tB_{E_r}, \ldots, tB_{E_r}}_{r}, \cdot))], \end{aligned}$$

hence

$$\tilde{\gamma}_{Z_0} = \gamma \Big[V_{d-r}(K_0) + \frac{d}{\kappa_r} P_0(V(\underbrace{K_0, \ldots, K_0}_{d-r-1}, \underbrace{B_{E_r}, \ldots, B_{E_r}}_{r}, \cdot)) \Big]. \tag{5.17}$$

Analogously to (5.15), a simplification of (5.17) is possible using $\tilde{P}_0$. For weakly isotropic X, (5.17) becomes

$$\tilde{\gamma}_{Z_0} = \gamma \Big[V_{d-r}(K_0) + \frac{2\kappa_{d-1}}{d\kappa_d} V_{d-r-1}(K_0) P_0(V_1) \Big]. \tag{5.18}$$

For $d = 3$ and $r = 2$, (5.15) and (5.16) were proved by Nagel (1983); for arbitrary d and $r = d - 1$, (5.16) and (5.18) are due to Zähle (1984).

6. Discussion and Comments

Before we study a few applications of the results obtained so far, we want to discuss some generalizations and interrelationships between the formulas. Also, some comments on the literature are in order here.

We have studied point processes X of cylinders not only because they are interesting models for several natural phenomena, but also as a unifying notion for both, point processes of particles and point processes of flats. An important step in our considerations was the decomposition (3.1) of the intensity measure, which was based on the isomorphism i. i^{-1} transforms X into a point process $\tilde{X}$ on $\tilde{\mathscr{Z}}_q \subset \mathbf{R}^d \times \mathscr{Z}_q^0$. $\tilde{X}$ can be interpreted as a marked point process in $\mathbf{R}^d$, given by an underlying point process $\bar{X}$ in $\mathbf{R}^d$ (which is the process of centers $s(K)$, $K+L \in X$) and with mark space $\mathscr{Z}_q^0$. Some authors prefer to work with these marked point processes instead of X. For processes of particles, the two concepts are indeed equivalent (as long as the mark space contains only the shapes $\mathscr{K}_d^0$ and no additional information). For cylinder processes our approach is preferable since then the mappings $X \to \tilde{X}$ and $X \to \bar{X}$ do not preserve invariance properties, i.e., for stationary X, $\tilde{X}$ and $\bar{X}$ need not be stationary. For a stationary process of particles, γ is the intensity of the underlying ordinary (and stationary) point process $\bar{X}$ and P_0 is the mark distribution. Of course, $\bar{X}$ may have multiple points even if X is simple.

As an even more general notion, point processes X of closed sets may be considered (see Matheron (1975)). Especially, if the sets $C \in X$ are in the class $\mathscr{S}_d$ of locally finite, countable unions of convex bodies, then the curvature measures $\Psi_j(C, \cdot)$, $j = 0, \ldots, d$, exist as (signed) Radon measures. However, a decomposition of the type (3.1) is more difficult in this case and, therefore, methods and results of the last sections do not immediately generalize to such processes X. For weakly stationary X, the quermass densities $D_j(X)$ can be defined by

$$\mathbb{E} \sum_{C \in X} \Psi_j(C, \cdot) = D_j(X) \cdot \lambda_d, j = 0, \ldots, d. \tag{6.1}$$

Here, of course, some conditions on X are necessary (similar to the conditions in Theorem 3.2) which guarantee that $\mathbb{E} \sum_{C \in X} \Psi_j(C, \cdot)$ is a locally finite (signed) Radon measure on $\mathbf{R}^d$. (6.1) then follows from the stationarity of Θ and the Campbell theorem. If X is, moreover, weakly isotropic, a modification of the argument in Section 3 (see Zähle (1986) for a similar proof) leads to the following analog of (4.7)

$$\mathbb{E} \sum_{C \in X} \Psi_j(K_0 \cap C, \beta_0) = \sum_{k=j}^{d} \alpha_{djk} \Psi_k(K_0, \beta_0) \cdot D_{d+j-k}(X), \tag{6.2}$$

$j = 0, \ldots, d.$

In the same way, (5.6) and (5.14) can be generalized to processes X on $\mathscr{S}_d$:

$$D_j(X \cap E_r) = \alpha_{djr} D_{d+j-r}(X), \qquad j = 0, \ldots, r, \tag{6.3}$$

$$D_j((X \cap Z_0) \mid E_r) = \sum_{i=j}^{d+j-r} \gamma_{djir} D_i(X)\, V_{d+j-r-i}(K_0), \qquad j = 0, \ldots, r. \tag{6.4}$$

Since cylinder processes X are special processes on $\mathscr{S}_d$, (6.4) implies that (5.14) is true for processes of cylinders, too.

There is an obvious connection between point processes on $\mathscr{Z}_d$ (or $\mathscr{S}_d$) and random $\mathcal{S}_d$-sets, since for each point process X we can consider its union set Y. If the curvature measures obey

$$\Psi_j(Y, \cdot) = \sum_{C \in X} \Psi_j(C, \cdot)$$

(this is the case if the sets $C \in X$ have at most $(j-1)$-dimensional intersections), then

$$D_j(Y) = D_j(X),$$

hence, we have similar formulas for the jth quermass density of X and Y. Such formulas for random $\mathcal{S}_d$-sets are obtained in Davy (1978), Weil (1983a), (1984), Weil and Wieacker (1984). For more general random sets, see Mecke (1981a) and Zähle (1982), (1986). A result in the opposite direction was obtained by Weil and Wieacker (1984) who show that any random $\mathcal{S}_d$-set Y is the union set of some point process X on $\mathcal{K}_d$. The construction can be modified so that X has the same invariance properties as Y (Weil and Wieacker (1986)). The connection between X and Y has led some authors to use the term 'process' in some cases for Y, too. For instance, fibre processes are sometimes random aggregates of fibres, where the individual fibres cannot be distinguished.

For a process X of q-flats, the intensity γ coincides with $D_q(X)$, the mean q-content of X per unit volume (Theorem 4.3). Thus, the intensity γ depends only on the union set Y of X. For a process X of q-sets (without overlapping), the intensity γ and $D_q(X)$ are different quantities. While γ depends strongly on the number of different particles of X, $D_q(X)$ is the same as $D_q(Y)$ (where Y is again the union set). For this reason, some authors call $D_q(X)$ the intensity of X, especially if the process is only given by its union set Y.

From the intersection formulas which we have given, it is quite easy to obtain formulas for quermass densities of intersections and superpositions of two (or more) independent processes. The invariance conditions must be imposed only on one of the processes. In particular, this means that there are generalizations of the formulas of Section 5, where the r-flat E_r is replaced by an r-set K_0 or a (not necessarily stationary) point process X_0 of r-sets. Formulas of this kind have been investigated in Mecke (1981b) and Zähle (1986).

As we mentioned, the quermass densities of the process X and the union set Y are the same only in special cases. General formulas for the connection between $D_j(X)$ and $D_j(Y)$ are known only for Poisson processes (see Davy, 1976, 1978; Wieacker, 1982; Weil, 1983a; Kellerer, 1984; Weil and Wieacker, 1984; Zähle, 1986 for more details).

We have exploited the basic formula (3.2) only for the curvature measures since we aimed to apply the integral formulas of Section 2. It is obvious that the procedure underlying Section 4 can be performed if the curvature measures are multiplied by a 'weighting factor' $g = g(Z)$ which depends on $Z \in \mathcal{Z}_q$ in a translation invariant way. For example, (4.1) then reads

$$\mathbb{E} \sum_{Z\in X} g(Z)\Psi_j(K_0\cap Z, \beta_0\cap \beta(Z))$$
$$= \gamma\Big[\Psi_j(K_0,\beta_0)\int_{\mathscr{X}_q^0} g(K+L)\Psi_{d-q}(K,\beta(K))\,dP_0(K+L)+$$
$$+\sum_{k=j+1}^{d-1}\int_{\mathscr{X}_q^0} g(Z)\varphi_k^{(j)}(K_0,Z,\beta_0\times\beta(Z))\,dP_0(Z)+ \tag{6.5}$$
$$+\Psi_d(K_0,\beta_0)\int_{\mathscr{X}_q^0} g(K+L)\,\Psi_{j-q}(K,\beta(K))\,dP_0(K+L)\Big],$$

$j = 0, \ldots, d.$

Analogously, the other formulas in Sections 3 and 4 can be modified. Of course, the integrability conditions on X must be changed according to g, too. The variants which follow from this concept are too numerous to mention them all. We will use some special weighting factors in the following section. We give only two further results of this kind. If X is a process of convex cylinders, $K_0\in \mathscr{K}_d$, and $g = V_j$, then using (4.2) we have

$$\mathbb{E}\sum_{Z\in X} V_j(Z)1_{\mathscr{X}_q(K_0)}(Z)$$
$$= \gamma\Big[P_0(V_j\cdot V_d)+\sum_{k=1}^{d-q} P_0(\Phi_k^{(0)}(K_0,\cdot)\cdot V_j)\Big]. \tag{6.6}$$

If K_0 is a point x_0, (6.6) holds for arbitrary cylinder processes X and becomes

$$\mathbb{E}\sum_{\substack{Z\in X,\\ x_0\in Z}} V_j(Z) = \gamma P_0(V_j\cdot V_d).$$

If X is a process of q-flats, $E_r\in \mathscr{E}_r^d$, $\mathscr{B}\subset \mathscr{L}_q^d$ a Borel set, $\tilde{\mathscr{B}} = \{L+x \mid L\in\mathscr{B}, x\in L^\perp\}$, and

$$g(L) = \begin{cases} [L, E_r]^{-1} & \text{if } L\in\tilde{\mathscr{B}} \\ 0 & \text{if } L\notin\tilde{\mathscr{B}}\end{cases}$$

then

$$\mathbb{E}\sum_{\substack{L\in X,\\ L\in\tilde{\mathscr{B}}}} \frac{1}{[L,E_r]}\Psi_{q+r-d}(L\cap E_r,\cdot)$$
$$= \gamma P_0(\mathscr{B})\lambda_{E_r}, \tag{6.7}$$

by Theorem 5.2 and the argument before Theorem 5.1. Similarly, for a process X of convex q-sets,

$$\mathbb{E}_{\substack{K\in X\\ L(K)\in\mathscr{B}}} \frac{1}{[L(K),E_r]}\Psi_{q+r-d}(K\cap E_r,\cdot)$$
$$= D_q(X)\tilde{P}_0(\mathscr{B})\lambda_{E_r}. \tag{6.8}$$

Since each process X of q-sets can be decomposed into a process X' of convex q-sets with the same density $D_q(X) = D_q(X')$, (6.8) holds for arbitrary (weakly stationary) processes of q-sets with the appropriate definition of $L(K)$. Special mention should be made of the case $q = d - 1$. Here, the result applies to the process of boundaries of a particle process X, provided bd $K \in \mathcal{R}_d$ for $K \in X$ (e.g., boundaries of convex polytope processes). For dimensions 2 and 3, (6.7) and (6.8) are due to Ambartzumian (1977), (1982), Mecke and Stoyan (1980a), Pohlmann, Mecke and Stoyan (1981). Again, there are similar formulas with E_r replaced by an r-set K_0 or a process X_0 of r-sets (e.g., Ohser (1981)).

Formula (6.5) and its consequences have a close connection to another famous problem in stereology. If the shape distribution P_0 is concentrated on a class $\tilde{\mathcal{Z}} \subset \mathcal{Z}_q^0$, the elements of which are characterized by one real parameter $h(Z)$ (e.g., diameter of balls), or if we are interested only in one real parameter $h(Z)$ for $Z \in \mathcal{Z}_q^0$ and its distribution P_h (which is the image of P_0 under h), then we may choose h as weight g in (6.5). Consequently, we obtain results analogous to those of Section 5 in which the distribution of h for those $Z \in X$ with $Z \cap E_r \neq \emptyset$ is related to P_h. Notice that these formulas involve $h(Z)$, for $Z \cap E_r \neq \emptyset$, $Z \in X$, and not $h(Z \cap E_r)$. This is a variant of the classical Wicksell problem; formulas for processes of balls, discs, etc. are investigated by Mecke and Stoyan (1980), Pohlmann, Mecke and Stoyan (1981).

Finally, we mention a special class of processes X of particles, the random mosaics. X is a random mosaic, if the union set of X is almost surely $\mathbb{R}^d$ and if the intersection $K \cap K'$ of different particles $K, K' \in X(\omega)$ is a $(d-1)$-set (for almost all realizations of X). The particles then constitute the cells of the mosaic (random mosaics with cylindrical cells can be defined in a similar way, but are of less interest). The classical case are mosaics with convex cells (i.e., necessarily convex polytopes as cells), random curved mosaics have been treated by Weiss and Zähle (1986). The j-dimensional faces of the cells of a random mosaic X with convex cells form another point process $X^{(j)}$, $j = 0, \ldots, d-1$, with the same invariance properties as X. The main interest is in relations between the quermass densities of the processes $X^{(0)}, \ldots, X^{(d-1)}$ and X. Formulas of this kind are collected in Ambartzumian (1974), Cowan (1980), Mecke (1980), (1984), Radecke (1980), Weiss and Zähle (1986).

7. Applications

The formulas which we have presented deal with the quantities γ and $D_j(X)$, $j = 0, \ldots, d$, and the distribution P_0 (resp. $\tilde{P}_0$) of a given geometrical point process X, as well as corresponding notions of transformed images of X. Therefore, they can be used for the estimation of γ, $D_j(X)$, P_0; in particular, they show which natural estimators are unbiased and which are not. In principal, each of the formulas can be exploited in this way. In the following, we will give some examples which are of special practical interest.

As a first example we consider a typical problem in image analysis related to edge effects in sampling windows. Let X be a stationary process of simply connected particles in $\mathcal{R}_2$. The process X is observed in a sampling window $K_0 \in \mathcal{R}_2$ (with area 1). The mean number of particles γ is to be estimated. Natural estimators of γ are g_1 = 'number of particles intersecting K_0' and g_2 = 'number of particles $K \in X$ with $K \subset K_0$'. Of course, g_1 is an overestimation of γ, and g_2 an underestimation of γ. More precisely, if X is isotropic and if the particles of X and K_0 are convex the bias of g_1 is given by

$$\gamma[2P_0(V_1) \cdot V_1(K_0) + P_0(V_2)]$$

(in view of (4.8)). If in g_1 each particle K is counted with weighting factor

$$[1 + 2V_1(K_0)V_1(K) + V_2(K)]^{-1},$$

the new estimator $\tilde{g}_1$ is unbiased. Analogously, g_2 can be changed into an estimator $\tilde{g}_2$ if each particle $K \in X$, $K \subset K_0$, is weighted by a factor which depends in a more complicated way on the geometry of K and K_0 (see Miles (1974) and Weil (1982) for details). If all particles are small enough w.r.t. K_0, $\tilde{g}_2$ is unbiased. The disadvantages of these estimators are the following. $\tilde{g}_2$ is no longer unbiased if there are particles $K \in X$ which do not fit into K_0. Moreover, if the particles of X and K_0 do not have simple shapes, the calculation of the weighting factor is quite complicated. The weighting factor in $\tilde{g}_1$ can only be determined if $V_1(K)$ and $V_2(K)$, $K \in X$, are known. This requires either knowledge about the shapes in X or the possibility to observe the part $K \backslash K_0$ of the particles $K \in X$ with $K \cap K_0 \neq \emptyset$.

Since γ is also the intensity of the underlying point process X of centers $s(K)$, $K \in X$, an obviously unbiased estimator of γ is g_3 = 'number of particles $K \in X$ with $s(K) \in K_0$'. This method of 'associated points' can be generalized by associating more than one point with each particle and by counting particles $K \in X$ with weights α, $0 \leq \alpha \leq 1$, according to the number of associated points of K which are in K_0. For more details, see Jensen and Sundberg (1985), (1986). A variant of the method of associated points is the tangent count of DeHoff (1978). If the particles do not have simple shapes, the associated points method usually also has to use information outside the sampling window. Another disadvantage is that not all particles which are observed in K_0 are represented in the estimator.

An unbiased estimator which uses the full information of $X \cap K_0$ can be based on the isotropic version of the system of linear equations (4.8), for $j = 0, 1, 2$. Solving this system yields unbiased estimators of $D_0(X)$, $D_1(X)$, $D_2(X)$ which are linear combinations of

$$\sum_{K \in X} V_0(K_0 \cap K), \quad \sum_{K \in X} V_1(K_0 \cap K) \quad \text{and} \quad \sum_{K \in X} V_2(K_0 \cap K).$$

Of course, here one has to assume that either X is isotropic or K_0 is circular.

This estimator was proposed in the book of Santaló (1976), pp. 282–286; it was studied in more detail by Schwandtke, Ohser and Stoyan (1986). Another estimator of γ which uses the full information is the sum of the total Gaussian curvatures in int K_0,

$$g_4 = \sum_{K \in X} \Psi_0(K, \text{int } K_0).$$

In view of (4.4), g_4 is unbiased. This estimator works for nonisotropic X and for arbitrary shapes $K \in X$ without requiring that the different parts of K in K_0 are recognizable. Since in the plane $\Psi_0(K, \text{ int } K_0)$ is determined by the angles between the normals of bd K in the points of bd $K \cap$ bd K_0, the estimator is also mathematically simple. If normals are difficult to determine in practice, DeHoff's tangent count with different directions of tangents can be used as a discrete approximation. Finally, if $K_0 = C_0$ a simple unbiased estimator for γ is given by

$$\sum_{K \in X} [V_0(C_0 \cap K) - V_0(\delta^+ C_0 \cap K)]$$

(see the discussion after Theorem 4.2). For more details about edge effects in particle counting, see the survey of Gundersen (1978) and the recent paper of Schwandtke, Ohser and Stoyan (1986).

Concerning the variances of these estimators, simulations of special point process models with simple shapes have been performed recently by Kellerer (1985), Jensen and Sundberg (1986), Schwandtke, Ohser and Stoyan (1986).

As a second example, we remark that (5.6) contains, for $d = 2$ or 3 and $r = 0$, 1, 2, the set of 'fundamental formulas of stereology'. For simplicity, we consider only the case of a process X of simply connected particles. In the notation which is used in the stereological literature, the volume density $D_d(X)$ is written as V_V, A_A, L_L, P_P, for dimensions $d = 3, 2, 1, 0$, respectively. The surface area density $2D_{d-1}(X)$ is written as S_V, L_A, P_L, according to the dimensions 3, 2, and 1. Moreover, $(2\pi/(d-1))\, D_{d-2}(X)$ which is the density of the integral mean curvature is denoted by K_V if $d = 3$, and the number density (or intensity) $D_0(X) = \gamma$ is denoted by N_V and N_A, for $d = 3$ and 2. The same symbols are used for the intersection processes $X \cap E_r$, $r = 0, 1, \dots, d-1$. The set of fundamental formulas then reads

$$V_V = A_A = L_L = P_P, \qquad S_V = \frac{4}{\pi} L_A = 2P_L, \qquad K_V = 2\pi N_A. \tag{7.1}$$

For lower dimensional particles (sheets or fibres) the same symbols are used, partially with a different (but obvious) meaning; e.g., for fibre processes in $\mathbf{R}^2$, L_A stands for $D_1(X)$ (and not for $2D_1(X)$). Also other symbols with obvious interpretations may be used (A_V, L_V, P_A, etc.), see Stoyan and Mecke (1983) for the modifications. If the point process X is not weakly isotropic, the formulas

(7.1) still hold for expectations if the sectioning plane is chosen with a random direction (independently of X).

If the intersection processes $X \cap E_r$ are used to get some information about the anisotropy of X then of course one has to work with flats E_r of different nonrandom directions. In particular, this is the case if information about the directional distribution $\tilde{P}_0$ of fields of fibres or sheets is wanted. Since the direction of a plane in $\mathbf{R}^3$ as well as the direction of lines in $\mathbf{R}^2$ or $\mathbf{R}^3$ is determined by a unit vector, the directional distribution $\tilde{P}_0$ for $d = 3$ or 2 can be viewed as an even probability measure on the unit sphere Ω in $\mathbf{R}^3$ or $\mathbf{R}^2$. The quermass densities $D_{j+r-d}(X \cap E_r)$, for $d = 3$ and $r = 1$, 2, or $d = 2$ and $r = 1$, when E_r is supposed to be variable, are functions on $\mathscr{L}_r^d$, hence, they can also be represented as functions on Ω. With the notation used earlier, Theorem 5.3 then gives the formulas

$$L_A(x) = S_V \int_\Omega |\sin \alpha(u, x)| \, \mathrm{d}\tilde{P}_0(u) \tag{7.2}$$

and

$$P_L(x) = S_V \int_\Omega |\cos \alpha(u, x)| \, \mathrm{d}\tilde{P}_0(u) \tag{7.3}$$

for a process X of sheets in $\mathbf{R}^3$,

$$N_A(x) = L_V \int_\Omega |\cos \alpha(u, x)| \, \mathrm{d}\tilde{P}_0(u) \tag{7.4}$$

for a process X of fibres in $\mathbf{R}^3$, and

$$P_L(x) = L_A \int_\Omega |\sin \alpha(u, x)| \, \mathrm{d}\tilde{P}_0(u) \tag{7.5}$$

for a process X of fibres in $\mathbf{R}^2$. Here, $\alpha(u, x)$ is the angle between u and x and x is the direction in which the intersection is taken. By a well-known uniqueness result (see the survey of Schneider and Weil (1983)), $\tilde{P}_0$ is uniquely determined by any of the functions on the left sides of (7.2) up to (7.5). The inversion is simple in the case of (7.5), since here $P_L(\cdot)$ is the support function of a centrally symmetric convex body $K_0 \subset \mathbf{R}^2$ and $\tilde{P}_0$ is just the normalized 'length measure' of K_0 (the image of $\Psi_1(K_0, \cdot)$ under the spherical image map from bd K_0 onto Ω). For smooth K_0, $\tilde{P}_0$ can be determined analytically; for a polygon K_0, $\tilde{P}_0$ can be given directly. This also indicates a simple procedure for the estimation of $\tilde{P}_0$ if $P_L(x)$ is only known for finitely many directions x. The inversion of (7.2), (7.3), and (7.4) can also be done analytically (under appropriate smoothness assumptions) but the estimation of $\tilde{P}_0$ from finitely many directions is a much more complicated problem. As we already mentioned in the last section, the situation is much easier if it is possible to observe in the plane (line) of intersection E_r the angles of the sheets or fibres with E_r. Then, $\tilde{P}_0$ can be estimated directly from

(6.8). Another method is possible if the fibres are replaced by cylindrical tubes (Stoyan (1984), (1985b)).

Finally, we want to discuss the situation of projected thick sections. The most interesting case for applications is that of a circular cylinder $Z_0 = E_1 + tB_{E_1^\perp}$ or a thickened plane $Z_0 = E_2 + tB_{E_2^\perp}$ in $\mathbf{R}^3$, $t > 0$.

If we first consider a weakly isotropic field of convex particles, then the quermass density on the left side of (5.14) is a function of t alone and we get for the two situations mentioned above

$$\begin{aligned} P_L(t) &= \tfrac{1}{2} S_V + tK_V + 2\pi t^2 N_V, \\ L_L(t) &= V_V + \frac{\pi}{4} tS_V + \tfrac{1}{2} t^2 K_V, \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} N_A(t) &= \frac{1}{2\pi} K_V + 2tN_V, \\ L_A(t) &= \frac{\pi}{4} S_V + tK_V, \\ A_A(t) &= V_V + \tfrac{1}{2} tS_V. \end{aligned} \tag{7.7}$$

Of course, for $t = 0$ the formulas (7.6) and (7.7) reduce to (7.1). If thick sections with at least two different thicknesses are available, Equations (7.6) or (7.7) can be solved for N_V, K_V, S_V, and V_V and, hence, unbiased estimators for these quantities result. In particular, this allows an estimation of the intensity N_V (as was first observed by Matheron (1976)) which is not possible by sections of zero thickness. If more variation of the thickness is possible, regression methods can be applied (see Voss and Stoyan (1985)). By combination of (7.6) or (7.7) with (7.1) simple approximation formulas for V_V (or S_V) can be obtained (see Stoyan (1985a) for details).

If X is a fibre process, then (5.16) (or (7.6) and (7.7)) gives

$$L_L(t) = \frac{\pi}{2} t^2 L_V \quad \text{and} \quad L_A(t) = \frac{\pi}{2} tL_V.$$

If we write $\tilde{N}_A$, respectively $\frac{1}{2}\tilde{P}_L$, for the intensity $\tilde{\gamma}_{Z_0}$, (5.18) gives

$$\tilde{P}_L(t) = \pi tL_V + 2\pi t^2 N_V \quad \text{and} \quad \tilde{N}_A(t) = \tfrac{1}{2} L_V + 2tN_V.$$

For nonisotropic processes X, the formulas are more intricate. We mention only the results for fibre processes. Of course, now the densities for the projected thick sections depend on t and on the direction x of the cylinder Z_0, $x \in \Omega$. From (5.15) we then get

$$L_L(t, x) = \pi t^2 L_V \int_\Omega |\cos \alpha(u, x)| \, \mathrm{d}\tilde{P}_0(u)$$

and

$$L_A(t, x) = 2tL_V \int_\Omega |\sin \alpha(u, x)| \, d\tilde{P}_0(u),$$

(5.17) implies

$$\tilde{P}_L(t, x) = 2\pi t^2 N_V + 4tL_V \int_\Omega |\sin \alpha(u, x)| \, d\tilde{P}_0(u),$$

$$\tilde{N}_A(t, x) = 2tN_V + L_V \int_\Omega |\cos \alpha(u, x)| \, d\tilde{P}_0(u).$$

8. Final Remarks

The main purpose of this article was to transform general integral formulas for curvature measures into density formulas for geometric point processes. To this end, the integral geometric results, the fundamentals of geometric point processes, and the resulting general density formulas have been presented with the corresponding background and complete proofs, if necessary. The generalizations in Section 6 and the applications in Section 7 are treated in a much more cursory fashion, they have been included to show the variety and significance of the theory as a unifying approach to the many existing results in the literature. In some cases, the situations have to be studied in more details, and this will be done elsewhere. In others, the interested reader should be able to fill in the gaps.

More information about geometric point processes (in particular, higher-order properties which we have omitted completely), and applications in image analysis and stereology can be obtained from the book of Stoyan and Mecke (1983) (see also the forthcoming English version by Stoyan, Kendall, and Mecke). There one also finds additional literature, since our list of references is concentrated on those articles which are directly connected with the material presented here.

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