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CONSTRUCTIVISM, MATHEMATICS AND MATHEMATICS EDUCATION

ABSTRACT. Learning theories such as behaviourism, Piagetian theories and cognitive psychology, have been dominant influences in education this century. This article discusses and supports the recent claim that Constructivism is an alternative paradigm, that has rich and significant consequences for mathematics education. In the United States there is a growing body of published research that claims to demonstrate the distinct nature of the implications of this view. There are, however, many critics who maintain that this is not the case, and that the research is within the current paradigm of cognitive psychology. The nature and tone of the dispute certainly at times appears to describe a paradigm shift in the Kuhnian model. In an attempt to analyse the meaning of Constructivism as a learning theory, and its implications for mathematics education, the use of the term by the intuitionist philosophers of mathematics is compared and contrasted. In particular, it is proposed that Constructivism in learning theory does not bring with it the same ontological commitment as the Intuitionists' use of the term, and that it is in fact a relativist thesis. Some of the potential consequences for the teaching of mathematics of a relativist view of mathematical knowledge are discussed here.

Constructivism has been described (e.g. Kilpatrick, 1987) as consisting of two hypotheses:

(1) Knowledge is actively constructed by the cognizing subject, not passively received from the environment.

(2) Coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.

The first of these is becoming generally accepted, certainly by mathematics educators, and is seen to be a useful and productive hypothesis when thinking about listening to children and their mathematical learning. The second is more controversial and perhaps worrying, since it appears to lead us immediately into problems on two levels: firstly, whether it is ever possible to understand what anyone else is saying or meaning, that is, problems of private languages, and secondly, what kind of meaning can thus be given to what we all accept as known, that is, the nature of knowledge in general and of mathematical knowledge in particular. One might suggest that we content ourselves with hypothesis (1), call ourselves 'weak' constructivists, (or the more pejorative term 'trivial'), and leave debate of hypothesis (2) to philosophers, and conferences, with the implication that it is not really relevant to the business at hand, the teaching of mathematics. (Those accepting both hypothesis have been called 'radical

Educational Studies in Mathematics **20**: 211–223, 1989.

constructivists'). This is rather unsatisfactory, though, since the connections between hypothesis (1) and (2) seem to be quite strong. After all, in mathematics, and in philosophy, we are accustomed to pursuing the consequences of an hypothesis, despite their sometimes disturbing nature.

In fact it is crucial that the second hypothesis is considered here, precisely because of the significance of the nature of mathematical knowledge for epistemology and philosophy in general. From Plato, through Descartes, Leibnitz, and Kant to modern philosophers, mathematical knowledge has served two essential functions at least: first, the ultimate test of the adequacy of the philosophical ideas proposed is whether they can include and explain mathematical truths; second, the apparently timeless, certain, *a priori*, tautological nature of mathematical propositions form the paradigm of knowledge. If one can establish the validity of propositions about 'justice', 'good', or 'freedom' with the kind of certainty that mathematical propositions appear to exhibit, the classical problems of philosophy can be solved. Thus it is often the case that philosophers begin with mathematical knowledge, and constantly refer to mathematical concepts, at the heart of their ideas. Plato increasingly used mathematical forms to characterise his theory, perhaps because there seem to be so many different forms of, for instance 'table'; Leibnitz's success with his notation for the calculus led to the proposal that such a notation should be developed for all reason, and Kant characterised Space and Time as transcendental categories. Developments in science in the last three centuries have reinforced the role of mathematics as the last bastion of certainty. If, however, one can present a case for the fallibility and relativity of mathematical knowledge, and of such concepts as 'proof' and 'truth', this has fundamental implications for philosophy. Bloor (1976) was so successful in proposing his relativist thesis which focussed on mathematics and logic, that debate about his ideas have drawn in many of the major British philosophers today (e.g. Hollis and Lukes, 1982).

One way out, discussed and criticised in depth by Stove (1982), is to continue as before, but to surround words such as proof, truth, etc. with inverted commas, as 'proof' and 'truth'. In that way, we can slightly weaken our claims in mathematics, but continue to use familiar words. It is as if we are saying "This is true, within the confines of present notions of truth". Or it can be taken to mean that the term is completely devalued. Stove demonstrates how Popper, Lakatos, Kuhn and Feyerabend use inverted commas, and puts forward the thesis that all four philosophers are in fact irrationalists, as seen by their use of this punctuation. I will return to this issue below, but a comment of Bloor's (1982) is relevant here, although he gives it in a different context:

> Are believers in a flat earth the only ones amongst us with the right to operate with the distinction between 'up' and 'down'? (p. 321)

Both 'weak' and 'radical' constructivism are concerned with learning theory, and find their modern roots in Piaget's genetic epistemology. However Piaget was attempting to resolve the ancient problems of how one comes to know anything, and the relationship between the individual and an objective world, and was proposing an alternative to empiricism or platonism. Thus the questions that were amongst those asked and discussed at the Eleventh International Conference on the Psychology of Mathematics Education, what does constructivism imply for mathematics, and does it have any implications for mathematics education, arise directly.

Much concern and disquiet has been expressed in recent years with the rigidity, appropriateness and applicability of the Piagetian stages of development. I suggest that part of Piaget's motive in constructing his elaborate theories of mental schema was his desire to establish objectivity, since his epistemological alternative to empiricism and platonism places the roots of knowledge in the individual, and thus borders on private thoughts and language. I will attempt to show that radical constructivism returns to genetic epistemology, but takes the full consequences of Piaget's philosophy without feeling the need to establish objectivity *in this sense.*

In this article I will examine what radical constructivism might mean in mathematics, and propose some implications for mathematics education. In the former discussion I will suggest that the present use of the word 'constructivism' has some similarities with its historical use in connection with Intuitionism, but is fundamentally different. I will then go on to propose that radical constructivism is a relativist epistemology, which does not leave us unable to say anything to each other, but actually endows philosophical discussions with content and controversy. In the latter discussion, I will not avoid the issue of the connections between theory and practice, in the sense of the question "How can different epistemologies lead to distinct and testable hypotheses for the teaching of mathematics?"

CONSTRUCTIVISM IN MATHEMATICS

i. *Constructivism in Intuitionism*

In the philosophy of mathematics, constructivism is usually used in connection with Intuitionism. Indeed Kline (1980, p. 241) uses the terms interchangeably. I will call this constructivism C_1. Whilst the origins of intuitionist ideas in mathematics can be traced back at least to Kronecker (1823–1891) and perhaps to Descartes, one looks to Brouwer, whose doctoral dissertation 'On the Foundations of Mathematics' was written in 1907, for the essence of this epistemological position. A clear description of the use of this notion of C_1 is given by Weyl (1963):

An existential statement, such as 'there exists an even number', is not considered a proposition in the proper sense that asserts a fact. An 'infinite logical summation' such as is called for by a statement of this kind (1 is even or 2 is even or 3 is even or . . . ad infinitum) is evidently incapable of execution. '2 is an even number', this is a real proposition (provided 'even' has been defined recursively . . .); 'there exists an even number' is nothing but a propositional abstract derived from that proposition. If I consider an insight a valuable treasure, then the propositional abstract is merely a document indicating the presence of a treasure without disclosing its location. Its only value may lie in the fact that it causes me to look for the treasure. It is a worthless piece of paper as long as it is not endorsed by a real proposition such as '2 is an even number'. Whenever nothing but the possibility of a construction is being asserted, we have no meaningful proposition; only by virtue of an effective construction, an executed proof, does an existential statement acquire meaning. In any of the numerous existential theorems in mathematics, what is valuable in each case is not the theorem as such but the construction carried out in its proof; without it the theorem is an empty shadow. (p. 51)

Clearly, there are already differences between this notion of constructivism, C_1, and the one which is discussed today in mathematics education, which I shall call C_2. The epistemology of the Intuitionists focusses on what mathematics *ought* to be, which methods, statements, proofs etc. are acceptable, in the construction of mathematics, in order to achieve the certainty that was sought by these philosophers of mathematics. Genetic epistemology, the origins of the present notion C_2, focuses on the activity of construction as the process by which the individual learns, and by which knowledge is created. Mathematics as such, is taken for granted by Piaget, his concern was not with valid or invalid mathematical statements, but with how the individual gains that knowledge. There are some similarities also. Heyting (1956, in Benacerraf and Putnam, 1964) writes, in a manner that sounds much like the speech of constructivists today:

. . . a mathematical theorem expresses a purely empirical fact, namely the success of a certain construction. "$2+2=3+1$" must be read as an abbreviation for the statement: "I have effected the mental constructions indicated by "$2+2$" and by "$3+1$" and I have found that they lead to the same result". (p. 61)

The role of language is one of the distinguishing features between C_1 and C_2. For the Intuitionists, language is secondary to thought, and only serves to communicate that thought to others. Thought, which is right intuition, is the essence, and language is an imperfect device for communication. I will discuss more fully below, what essential function language might serve in an elaboration of C_2, but briefly a concept is identified by its use, it gains its meaning from the shared social interpretation which is its use, and hence language, which itself is socially negotiated, and finds its meaning only in its use, is integrally connected with the notion of a concept.

Other major differences are revealed in an examination of the ontology of the Intuitionists. As with C_2, the Intuitionists do not rely on a notion of transcendental existence of mathematical entities. They maintain that there may be such objects, that exist independent of acts of human thought, but their existence "is guaranteed only insofar as they can be determined by thought .... Faith in transcendental existence, unsupported by concepts, must be rejected as a means of mathematical proof" (Heyting in Benacerraf and Putnam 1964, p. 42). Similarly, faith in the existence of a real objective world without establishing some means of connecting with it, must be rejected as providing us with certainty. However, it is just this **certainty** that distinguishes C_2 from C_1. The programme of the Intuitionists was the Euclidean one of resolving the problems in the foundations of mathematics, and establishing the certainty of mathematical knowledge. They maintained that one must reject claims to any transcendental existence, but they needed some certain foundations from which to build the structure of mathematics with constructive proofs. Their method, in fact, was to weaken but not reject Kantian intuitionism, which relied on transcendental categories of time and space, and which had been fundamentally threatened by the invention of non-Euclidean geometries, by "abandoning Kant's apriority of space, but adhering the more resolutely to the apriority of time" (Bouwer 1912 in Benacerraf and Putnam 1964, p. 69). They then appeal to intuition of the integers as the 'basal intuition of mathematics'. Kronecker is said to have declared, in an after-dinner speech, "God made the integers; all the rest is the work of man." It is interesting to note that Brouwer, Heyting and their supporters were known as radical Intuitionists, whereas others were prepared to accept, for instance, the real numbers rather than just the integers, as God-given and the rest as human construction. Whether these latter were known as 'weak' intuitionists, or 'trivial' intuitionists, I do not know.

To summarise, C_1 has similarities to C_2 in that both reject transcendental knowledge of the real world as providing proof, and are content to leave its existence an open question. Both assert the centrality of the notion of construction in concept-formation, and indicate the little value of pure existence theorems, whether in mathematical statements, or concepts. The fundamental difference is that Intuitionism is an epistemology of mathematical knowledge, concerned with a programme to establish the certainty of mathematics, based on the apriority of time and subsequently of the integers, to be grasped by intuition. C_2 is, in my interpretation, a more complete and consistent view of coming to know, of knowledge and of mathematics.

ii. *Constructivism C_2 and Mathematics*

For the Intuitionists, as for the Logicists and the Formalists at the time, certainty was the aim. They were trying to achieve a revised Euclidean programme for mathematics, which of course had implications for knowledge in general, since mathematics was seen as the last bastion of the absolutists. C_2 has no such teleology, it is not a philosophy built on trying to achieve certain goals. It suggests that we examine the consequences, honestly, of hypothesis (2) above, that coming to know does not discover an independent, pre-existing world outside the mind of the knower.

What, then, does 'coming to know' discover? What is meant by 'coming to know'? If the second constructivist hypothesis implies that there is no world outside the mind of the knower, an implication that many seem to assume, then we are certainly all doomed to solepsism. However, I suggest that this is not the case. The second hypothesis recognises experiences, it does not cast doubt on the idea that we all interact in some way with people and the world around us. When we ascribe meaning and significance, when we interpret and attempt to explain, when we propose theories based on those experiences, we are organizing our experiential world. The hypothesis implies that we are not 'discovering' the way the world works, in the sense that America was there and inhabited and then 'discovered' by Europeans, or whoever. In fact we cannot talk about the way the world actually, certainly and in a timeless way, works, simply because we cannot have knowledge that what we are describing is just what is. There is no Archimedian position from which to view our concepts and theories. But this has always been the case, and it has not halted people from developing theories, discussing explanatory power, or comparing evidence. Far from making one powerless, I suggest that research from a radical constructivist position, is empowering. If there are no grounds for the claim that a particular theory is ultimately the right and true one, then one is constantly engaged in comparing criteria of progress, truth, refutability etc., whilst comparing theories and evidence. This enriches the process of research.

If it is accepted that the knowledge we have of the way the world works is not forced upon us by that world (empiricism), nor do we have this knowledge innately (platonism), then what we know becomes conjecture, theory and hypothesis. It may be well-established conjecture, it may even be just about inconceivable that things could be otherwise, but this still does not provide certainty, and all our deductions and reasoning must adjust accordingly. Further, the theories, concepts and constructs are culturally and temporally relative. One need only think of the Greek notion of the centre of the earth being the centre of the universe, or modern creationist interpretations of the development of life, to illustrate this.

Thus, I suggest that what has to be abandoned with the rejection of the belief that we are discovering an independent pre-existing world outside the mind of the knower, is only that the knowledge we have can in any way be described as certain and ultimately true. What we lose is certainty and absoluteness, we do not lose the whole purpose in searching. There is still the possibility of making judgements, of using terms such as true and false, up and down, better or worse, more or less fruitful etc., and I will clarify in what sense one can use these terms below. Certainty is perhaps only a psychological necessity, or an emotional necessity, not a logical one in any sense. Loss of certainty means that different theories and conjectures are comparable, examinable, and equally valid, until one establishes some acceptable criteria of 'better'. Certainty has a tendency to lead one to say "That's it, no more discussion, we have the answer". Fallibilism, a view which accepts the potential refutation of all theories, and counter-examples to all concepts, allows one to ask how does one know that this answer is better than that one, what might constitute a notion of 'better', might they not both be possible, as with Euclidean and non-Euclidean geometries, or arithmetics with or without the Continuum Hypothesis.

I have attempted to demonstrate the radical constructivism, as I interpret it, is a relativist epistemology. I have suggested that it does not leave us unable to use the terms 'truth' and 'falsity', for instance, or make the whole process of investigation worthless. It is perhaps an interesting empirical question whether scientists, or in our case mathematicians, find relativism an empowering philosophy within which to work, or whether they find a form of platonism more fruitful, (this idea was first proposed by Reuben Hersh). One of the ways in which we may justify preferring one theory or explanation over another, is the comparative fruitfulness of those theories. Of course these notions are themselves problematic and have different interpretations in different paradigms, but this is not the place to discuss these issues. I have engaged in that debate elsewhere (Lerman, 1986). In suggesting that this question, whether mathematicians find one theory more empowering than another, is an interesting one for research, I wish to indicate that the relativist thesis, to be consistent, must be reflexive, that is, to adopt a critical and fallibilist position on relativism itself. One of the criticisms, misguided I believe, that Kilpatrick made of the radical constructivists (Kilpatrick, 1987) was that they were not reflexive in their advocacy of what I have described as relativism.

The relativist position poses difficult questions, particularly in relation to mathematics, amongst which are the following:

(a) How can one account for the apparent successes of mathematics? After all, buildings generally stay up, satellites reach their destinations, arithmetic, even with impredicative definitions, seems to work.

(b) Feyerabend claims that 'anything goes' (e.g. 1978), including astrology, creationism, etc. Is there a way of preferring?

(c) Can there be any communication, or are all languages and concepts private?

First, as a general comment, it is my view that one consequence of abandoning certainty and accepting fallibilism is that questions such as these are generated, and this is in itself a support for the relativist programme.

One can suggest that (a) gives clear evidence of the mathematical theories being true, in the sense of correspondence with the real world. It is certainly one of the wonders of mathematics, and one of the best teachers of these phenomena in recent years has been Professor Morris Kline. Thus it is instructive to read the final chapter in his *Mathematics – The Loss of Certainty* (Kline, 1980), where he surveys the comments of the great mathematicians and scientists on this enterprise. He, and they, confirm the thesis of his book, that one can only express wonderment, and a sense of the power of this human invention, which many do in almost mystical and religious terms, but this still does not make it possible, or indeed necessary to claim any absolute truth for the theories. After all there are revolutions, even in mathematical thought. On the other hand, Feyerabend's claim that 'anything goes' results from his, in my view justified, rejection of any absolute criteria for preferring one theory from another. We can give up absolute criteria, however, and still claim the mathematics progresses, or that one theory is to be preferred to another, even if only in hindsight, by accepting the relativism of such criteria, in relation to the scientific and cultural community in which they arise. To return to Stove's critique of Popper, Kuhn, Feyerabend and Lakatos as irrationalists, one can accept his arguments, that they are inconsistent in their theories, and that in the end they are led to irrationalism, in the sense that for instance Popper holds on to a progression towards truth, and Lakatos to the rational reconstruction of scientific history, despite their rejection of certainty. Stove goes on to reject all their ideas, because they are irrationalists, and to develop an alternative, empiricist epistemology, that provides the certainty that Stove requires. However, pointing out their inherent irrationalism can also lead to a strengthening of the relativist thesis, that forms such a large part of Popper's and Lakatos's work, and yet is opposed by their total programmes. C_2, as a relativist thesis, does not imply that we cannot use cognitive terminology, merely that we have to refer terms such as truth, and proof, and better, to a particular philosophical frame of reference.

Finally, on the question of ontology, or 'what is', again certain knowledge of an objectively-existing real world cannot be achieved. This is not to deny that the real world exists, only that even if it does, we can have no way of having "knowledge" of it, if we demand that this knowledge has to be certain and absolute. As discussed above, neither the empiricist answer nor the platonist one are adequate. One can pursue this ultimately to Descartes' "Cogito, ergo sum", but only if one is searching for absolute knowledge. Objectivity rests in the public nature of language, of concepts, of theories and hence of knowledge. These can change, as they are social constructions, publicly negotiated concepts, but relative to a particular culture, in a time and a place, they function as objective knowledge, without ascribing to them a transcendental existence. Bloor's highly illuminating discussion of objectivity takes the issue to the heart of mathematics, what we mean by the existence of mathematical objects (Bloor, 1977). Indeed he takes the views of Frege, one of the major figures in presenting the neo-platonist image of mathematics, to illustrate the social nature of mathematical knowledge. Frege uses the equator, the axis of the earth and the centre of mass of the solar system as examples of objective but non-physical entities. Bloor points out how these are just social constructions, invented by people to function as structuring and ordering concepts. In the Ptolemaic system, the centre of the earth served the same role, as the centre of the universe which consisted of concentric circles around the circular earth. Bloor comments that Frege would be as horrified by 'sociologism' as he was by 'psychologism', as he termed Mill's empiricist philosophy, the latter being the idea that concepts gain meaning in the individual mind, and the former in the social mind, as it were. Nevertheless Frege's description of mathematical objectivity appears to fit the characterisation of mathematics as a social construction, changeable and negotiated. As Wittgenstein (1967) puts it:

> If humans were not in general agreed about the colour of things, if undetermined cases were not exceptional, then our concepts of colour could not exist.' No: – our concept would not exist. (para 351)

This is not the place to pursue these arguments further, around (a) and (b), but I have given some indication of the direction such discussions can take, and have given a more complete account of these issues elsewhere (1986).

On 'Private Languages'

In this present discussion, however, it is most important to try to answer (c), since without a means of communication, teaching is certainly a wasted and futile effort, and this highlights some of the worries people express when considering C_2. I will attempt to do this with reference to the concept 'understanding', as this is a major concern for mathematics educators, and hence will lead to the final section on the implications for mathematics education.

Discussions about children's 'understanding', how we examine and identify 'understanding', and ultimately what 'understanding' means, form a central core of research in mathematics education. We talk about whether a student doing such and such would demonstrate understanding of the mathematics taught, as if there is some inner phenomenon, called 'understanding', which may be a 'correct understanding' or an 'incorrect understanding', and particular behaviour on the student's part would identify which one for us. Here is the essence of the difficulty, since if all understandings are private and individual constructions, no student behaviour will allow me to do anything other than make my own private construction about what the student 'understands' of my 'understanding' of the concept or idea in question. The difficulty, however, may exist only in an absolutist epistemology. If 'addition' has a transcendental existence as a concept, then the student either has that concept, or not, there are no partial stages. The job of the teacher is then to discover whether the student has the 'correct understanding' in its totality, otherwise it would not constitute 'understanding'.

If one abandons the absolutist epistemology, the discussion changes. Consider the familiar philosophical example, what is meant by saying that a child has learned and understands the concept 'hat'. This comes about by pointing out instances of 'hat', objects that have the use implied by that term. When the child points to a hat and says "Hat", we confirm that this is correct. When the child points to a tea-cosy and says "Hat" we have to explain that this object has a different use, and is not a hat. It is in the use, according to the public, objective notion of 'hat', that we can apply the word 'understand', and it has no application without this public connection. (Of course someone, initially called eccentric, and later perhaps a person who sets a new trend, may put the tea-cosy on their head, and call it a hat, and our concept will have to undergo a public change.) Wittgenstein says the following about understanding (Wittgenstein, 1974):

> Do I understand the word 'perhaps'? – And how do I judge whether I do? Well, something like this: I know how it's used, I can explain its use to somebody, say by describing it in made-up cases. I can describe the occasions of its use, its position in sentences, the intonation it has in speech. – Of course this only means that 'I understand the word "perhaps"' comes to the same as: 'I know how it is used etc.'; not that I try to call to mind its entire application in order to answer the question whether I understand the word. (p. 64)

To summarise, the shift from behaviourism to cognitive psychology focussed attention on teaching for understanding, but the problems of how to carry this out, and how to identify that 'it' had happened, remained as ongoing and major ones for mathematics education. It is suggested here that central to the difficulty is our notion of 'understanding', tied as it is to the idea of certain and absolute concepts. According to this view, the process of coming to understand a concept is one that takes place in the mind of an individual, and the final step of achieving that full understanding of a timeless, universal notion is a very private, almost mystical one. It is certainly beyond the power of any outsider, such as a teacher, to know that the process has taken place in full.

The difficulty of private languages thus arises with absolutist epistemologies, and not with relativism and the present use of constructivism C_2, contrary to the usual discussions. Accepting hypothesis (1) alone, "that knowledge is actively constructed by the cognizing subject, not passively received from the environment", does not ease any of our difficulties, and may in fact aggravate them. On its own, this hypothesis rejects coming to know through empirical means, but leaves us with a view of knowledge that cannot be actively constructed by the individual, since knowledge is objective in an absolutist sense. We need to be able to continue with our belief that if we create the right environment, in the classroom, in our teaching, learning and understanding will take place. Accepting hypothesis (2), however, forces us to re-examine what is meant by knowledge, and locates objectivity in the social domain, not the transcendental. Concepts are public, as their meaning is their use, and so too is understanding.

In the first section, the differences between the Intuitionists use of the term constructivism and the present use were compared, in order to clarify some of the issues in this paradigm shift in mathematics education, if that is what is taking place. In this section, I have attempted to show that epistemology, mathematical knowledge, and learning theories are interdependent areas of study. Piaget's genetic epistemology was a philosophical theory, resulting from his rejection of platonism and empiricism, and it placed the question of the nature of knowledge in the study of its acquisition. Radical constructivism C_2 is a re-examination of these ideas, and proposes that the programme has distinct consequences for mathematics education. I have attempted to support this view, and illustrate it by the discussion of the notion 'understanding'. In the final section, the issue of consequences of a theory for practice is discussed.

CONSTRUCTIVISM C_2 IN MATHEMATICS EDUCATION

In discussing the nature of scientific revolutions, Popper proposed that there are 'crucial experiments' that finally reveal which of two competing theories is the correct one. Both Lakatos and Kuhn pointed out that this was a naive idea, and that, for instance, had the Michelson-Morley experiment not supported Einsteinian mechanics, it would only have been declared that the equipment was unsatisfactory, or the experiment had been carried out incorrectly. Kuhn's analysis of scientific revolutions examined the issue of paradigms, the hard-core of theories, and the nature of the conflicts between scientific communities at the stage or paradigm shifts. I suggest that it would therefore be naive of us, in mathematics education, to expect to pick on a 'crucial experiment' to establish or refute C_2.

In order to maintain, however, that C_2 is an alternative and competing paradigm, it must at least be shown that there are potentially rich theories and ideas that are distinct to this view. In this article I have attempted to show, by drawing on the second hypothesis, one can identify a notion of understanding that is distinct from that of cognitive psychology, and is particularly accessible from the point of view of mathematics education, by its very nature.

Elsewhere I have discussed other possible influences of alternative perspectives of the nature of mathematical knowledge on aspects of mathematics education. I have suggested that a relativist view of mathematical knowledge, and I have attempted to show that C_2 is a relativist view, has implications for teaching styles (Lerman, 1983, 1986), for the way the curriculum is developing in relation to problem-solving and investigations (Lerman, 1987), and also draws issues of social values and politics into the mathematics classroom (Lerman 1988). These are theoretical developments of the consequences of C_2, and theory formation is an integral part of the development of research programmes. Indeed without such theory formation, proposing consequences of hypothesis (2) in particular, radical constructivism is perhaps vulnerable to the strong criticisms of e.g. Kilpatrick (1987), and to attempts to subsume the innovative research in the constructivist paradigm under that of earlier ideas.

Finally, there remains the question of how theory and practice relate.

This is an ancient question, but again one that, I suggest, is a consequence of epistemologies with an absolutist teleology. The ideas developed here, taking inspiration from Wittgenstein, propose that theories and concepts are rooted in practice, and obtain their meaning from use. They gain their objectivity in their public nature, in that theories written down become public property, subject to dispute, negotiation and adaptation. Their objectivity does not lie in their being the ultimate truths. Thus, there is in general a problem of the relationship between theory and practice, but not for Constructivism C_2.

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