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Large strain analysis of rubber-like materials based on a perturbed Lagrangian variational principle*

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Abstract. A mixed finite element method is presented for the large strain analysis of rubber-like materials, which are considered to be nearly incompressible. Two types of constitutive relations are included: generalized Rivlin and Ogden's models. The finite element equations are derived on the basis of a perturbed Lagrangian variational principle from which both the displacement and pressure fields are independently approximated by appropriate shape functions. A physically meaningful pressure parameter is introduced in the expression of complementary energy. In the paper, a special effort is made to split the deformation energy into two distinct parts: isochoric and hydrostatic parts. By doing this, a quadratic convergence rate of nonlinear iterative solution is achieved, particularly for problems deformed in the large strain range. The finite element equations are specialized for a two-dimensional 9-node Lagrange element with three-term pressure parameters. Five examples are given to demonstrate the application of the proposed numerical algorithm.

1 Introduction

Rubber and rubber-like materials (such as elastomers) have many industrial applications, e.g. automobile tires, load bearing pads, engine mounds, and numerous other useful products. Unfortunately, the mechanical responses of rubber, especially when subjected to complex loads, are not well understood. This is perhaps due to a combination of several reasons: (i) stress–strain behavior of rubber is highly nonlinear, (ii) the material usually experiences large strains, in the order of several hundred percents, and (iii) very little volumetric changes under deformations. Thus, in the solution of boundary value problems for rubber-like materials, one must deal with a set of highly nonlinear partial differential equations involving finite elasticity theory. Analytical solutions of such nonlinear problems, except those with simple geometries (Green and Zerna 1966; Rivlin 1951), are generally difficult to obtain. Therefore, the analysts usually resort into numerical techniques, e.g. the finite element method.

Other than the difficulties associated with the solution of nonlinear problems resulting from material and geometric nonlinearities, the greatest challenge in solving rubber elasticity problems via finite element method is the proper treatment of incompressibility condition. If the displacement based finite elements are employed in a structural analysis, the well-known "locking phenomenon" has often been observed (Nagtegaal et al. 1974; Oden and Carey 1984; Hughes 1987). Other related numerical difficulties include, for example, ill-conditioning of stiffness matrix and appearance of spurious pressure modes.

A literature view of finite element applications to incompressible materials in general can be found by Gadala (1986) and Hughes (1987), and to rubber-like materials in particular from the papers by Sussman and Bathe (1987) and Cescotto and Fonder (1979). Therefore, there is no need for the present paper to conduct a comprehensive review of literature. Nevertheless, in order to place our study in the proper perspective, a brief account on various methodologies closely related to the present work is listed. It is pointed out however that our reference list is by no means inclusive on the subject.

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Perhaps one of the most effective approaches to incorporating the incompressibility constraint in the finite element analysis is to use a mixed formulation. In this approach, a Lagrange multiplier is utilized to impose the incompressible condition in the potential energy theorem. One of the earliest investigators using this method is due to Herrmann (1965) for linear elastic, isotropic incompressible materials. Extension to orthotropic materials was made by Taylor et al. (1968) and Key (1969). Application of the mixed method to nonlinear response of rubbers including large strain analysis was made by a number of authors, e.g. Oden and Sato (1967), Oden and Key (1970, 1971), Agyris et al. (1974), Thompson (1975), Scharnhorst and Pian (1978), Berkovier (1978, 1981), Murakawa and Atluri (1979), Oden and Kikuchi (1982), Häggbland and Sundberg (1983) and Jankovich et al. (1985). In the variational formulation, a Lagrange multiplier, which is equivalent to hydrostatic pressure of the material, is introduced. By invoking finite element approximation, this parameter is interpolated independently in a way that the incompressibility condition is satisfied in the mean. Parallel to the mixed formulation, the conventional displacement element together with uniform or selective reduced integrations have also proven effective, e.g. by Oden and Kikuchi (1982), Fried (1974), Naylor (1974), Malkus (1976), Hughes (1977), and Markus and Hughes (1978). Tong (1969) employed a hybrid stress element to solve two-dimensional problems, where both the displacement and stress fields are independently assumed. Similar formulations were given by Scharnhorst and Pian (1978) and Murakawa and Atluri (1979) for nonlinear analysis.

More recently, in a paper by Atluri and Reissner (1989) a general frame work was developed to incorporate volume constraints into suitable multi-field variational theorems for the geometrically nonlinear analysis of both compressible and incompressible matrials. The generality of the proposed variational theorems is evident by their ability to include, up to seven independently discretized fields, e.g. displacements, volumetric and deviatoric stresses and strains, displacement gradient tensors, etc. in some of the functionals given by Atluri and Reissner (1989). However, from the standpoint of finite element applications at the present time, a reduced form of the generalized theorem is certainly more attractive. Indeed, several reduced forms, e.g. two-, four- and five-field theorems, are also included in the aforementioned paper. In particular, the formulation given in the present paper falls in the special form of two-field variational functional given by Atluri and Reissner (1989).

An important ingredient in the general functional forms alluded in the above concerns is the use of appropriate measures for the volumetric and deviatoric components of strain and/or "generalized" stress tensors. Such decomposition of volumetric and deviatoric responses, in the context of large strain analysis, has its mathematical origin in the *multiplicative splitting* of the deformation gradient tensor into a volume-preserving (isochoric or purely deviatoric) part and dilational part. This process is obviously in contrast to the "familiar" additive decomposition employed for the case of small deformation analysis. In fact, the origin of the notion in splitting the strain and stress tensors can be traced back to as early as 1961 in the work by Flory. The same decomposition process was subsequently used by Sidoroff (1974) in the context of large-strain viscoelasticity, and also more recently by Simo et al. (1985, 1988) for large-strain plasticity, Ogden (1972) and Sussman and Bathe (1987) for large-strain hyperelasticity.

For perfectly incompressible materials, the use of Lagrange multiplier results in non-positive definite coefficient matrix, which presents numerical difficulty to some equations solvers (such as Cholesky square root decomposition). As an alternative, a perturbed Lagrange Multiplier method was introduced by several authors to relax the incompressibility constraint, e.g., Bercovier et al. (1978, 1981), Häggbland and Sundberg (1983), Jankovich et al. (1985), Zdunek and Bercovier (1986), and Sussman and Bathe (1987). Further, a penalty method was proposed by Oden and Kikuchi (1982); also utilized by Bercovier (1978), in which the Lagrange multiplier is assumed to be proportional to the constraint equation. Usually, the penalty method is applied in conjunction with reduced and selective integration technique (Hughes 1987).

Indeed, numerous publications on the finite element analysis of incompressible materials have appeared in the literature. However, most of the reported work is limited to linear elastic analysis. The experience on nonlinear analysis of such materials, particularly including the effect of very large strains, still appears to be lacking. It is therefore the objective of this paper to report our research findings along this direction. In our study, we choose a mixed method based on the perturbed Lagrange formulation of a two-field variational principle, which is a special case of the generalized theorems by Atluri and Reissner (1989), as was mentioned previously. Similar to the early works by Sussman and Bathe (1987), Atluri and Reissner (1989), Simo et al. (1985, 1988), we again emphasize the importance of splitting the strain energy density function of the elastic material into two distinct parts: volumetric and isochoric (purely distortional) parts. By doing this, it enhances the convergence property of the nonlinear solution algorithm used, particularly for rubber deformed in the large strain regime. We note that the variational formulation presented in this paper is applicable to both compressible and nearly-incompressible materials. Two types of constitutive relations are considered: (1) Rivlin's 5-parameter model (Treloar 1975), and (2) Ogden's model (Ogden 1972, 1976). It is noted that any other models can be incorporated into the present work by following the same formulation procedure. The finite element equations are derived for a 9-node Lagrange two-dimensional element which is known to satisfy the Brezzi and Babuska conditions (Babuska et al. 1977; Brezzi 1974). Several numerical examples are included to demonstrate the utility of the proposed formulation.

2 Basic equations

In this section, some of the basic equations, i.e. definitions of strain energy density function, and its incremental relations necessary for finite element derivations, are summarized. In our derivations, we shall adopt the total Lagrangian formulation for description of deformations of an elastic solid and the reference frame is limited to a rectangular Cartesian system. We consider a continuum at time $t = 0$, occupying a domain V^0, with a bounding surface S^0. Let P_0 be a generic point of the continuum, which is identified by a position vector $\mathbf{x}^0$ with the Cartesian components x_i^0. At a time t, the point P_0, as a result of deformation, is moved to a new position P with a displacement vector $\mathbf{u}$. Correspondingly, the position vector of P is designated by $\mathbf{x}^t$ with the Cartesian components: $x_i^t = x_i^0 + u_i$. With the aforementioned deformation state, we define the following notations which will be used in our development: E_{ij} is the Green–Lagrange strain tensor; C_{ij}, Green deformation tensor; W, strain energy density function; S_{ij}, Piola–Kirchhoff stress tensor; σ_{ij}, Cauchy stress tensor.

In the later derivations, a hydrostatic pressure p will be referred. Instead of using a pseudo pressure term such as the one in the text by Green and Zerna (1966), a physically meaningful quantity directly related to the Cauchy stress is chosen in the present formulation and it is defined by

$$p = \frac{1}{3}\sigma_{ii} = \frac{1}{3J} C_{kl} \frac{\partial W}{\partial E_{ij}}. \tag{1}$$

Rubber is generally considered to be hyperelastic material with no change (or very little change) of volume upon deformation. Thus, it is also classified as incompressible or nearly-incompressible material (Treloar 1975). In finite element analysis, the requirement of incompressibility condition imposes a severe constraint to the discretized Galerkin equations, thus inducing locking to the finite element model. Since the incompressibility constraint is directly resulted from volumetric strain (or its stress counterpart, hydrostatic pressure), it is more advantageous to split the deformation energy (or strain energy) into two distinct parts: isochoric due to distortion and dilatation due to volume change; and these two parts can be interpolated differently in finite element approximations. In our splitting of strain energy function, it is accomplished by introducing modified strain invariants in the expression of deviatoric energy. This is slightly different, although accomplishing the same purpose, from the splitting of deformation gradients by Atluri and Reissner (1989) or deformation tensor due to Simo et al. (1985) by a multiplicative decomposition process.

Anticipating that a variational formulation will be used for both the compressible and nearly incompressible isotropic, hyperelastic materials, the strain energy density W is generally written as a function of three strain invariants

$$W = W(I_1, I_2, I_3). \tag{2}$$

Where I_2, I_2 and I_3 are the first, second and third invariants, respectively, and their definitions can be found in the text by Green and Zerna (1966). It is noted that the strain energy function of a hyperelastic material does not have to be written in the form of Eq. (2). In fact, any three independent invariants could be used to express the mathematical form of W. We therefore introduce the following convenient form

$$W = W(J, I_1^*, I_2^*). \tag{3}$$

Where J is a dilational ratio, i.e. a ratio between deformed volume and undeformed volume of the material, given by

$$J = \frac{dV^t}{dV^0} = \det\left(\frac{\partial x_j^t}{\partial x_i^0}\right) = I_3^{1/2} \tag{4}$$

where V^t = deformed volume at time t, V^0 = undeformed volume. Note here that J is related to the displacement field through the definition of I_3. In Eq. (3), I_1^*, I_2^* are the modified first and second strain invariants or modified principal stretches; their definitions are determined by satisfying two requirements: (1) the two quantities are mutually independent, and (2) they have no contribution to the hydrostatic pressure. The exact definitions of I_1^* and I_2^* may vary for a specific constitutive model and their expressions will be given in the next section.

In addition to the definition of J given in Eq. (4), we wish to establish a relationship between p and J since they are the conjugates to each other. This is done as follows.

Substituting Eq. (3) into (1), we arrive at

$$p = \frac{1}{3J} C_{kl} \frac{\partial W}{\partial I_\alpha^*} \frac{\partial I_\alpha^*}{\partial E_{kl}} + \frac{\partial W}{\partial J}. \tag{5}$$

We invoke the condition that I_α^* have no direct contribution to p and thus demand

$$\frac{\partial I_\alpha^*}{\partial E_{kl}} C_{kl} = 0, \quad \alpha = 1, 2. \tag{6}$$

In view of the above relationship, the first term in the expression of p vanishes. Therefore, the hydrostatic pressure, p, assumes a simpler expression

$$p = \frac{\partial W(J, I_1^*, I_2^*)}{\partial J}. \tag{7}$$

In this way, the isochoric and dilatational responses of the material can be entirely separated. It is noted that Eq. (7) is similar to Eq. (20b) in the reference by Atluri and Reissner (1989). Also note in the above that the modified strain invariants $I_\alpha^*, 1, 2$ govern the isochoric deformation and the volume change ratio J governs the dilational deformation.

In order to find a relationship for J in terms of the variable p, i.e. the inverse relation of Eq. (7), and also to write a variational equation for finite element approximations, we introduce a quantity W_c defined by

$$W_c = p(J-1) - W(J, I_1^*, I_2^*). \tag{8}$$

Taking the total differentiation of W_c in Eq. (8), one can readily find a relationship for J, which is designated by J^p and has the form

$$J^p = \frac{\partial W_c(p, I_1^*, I_2^*)}{\partial p} + 1. \tag{9}$$

The significance of Eq. (9) is as follows. The parameter J, a volume change ratio, is defined by either Eq. (4), which is related to the displacement field, or Eq. (9), related to the pressure field. In this case, some sort of constraint must be imposed on the definition of J and this constraint will be dealt within a variational sense.

A variational principle. The finite element equations will be derived from a variational principle involving both the displacement and pressure fields. To account for the effect of large deformation, a total Lagrangian description is employed. With this reference frame in mind, one may construct a functional having displacements u_i and pressure p as the primal variables by making use of the dilatational complementary energy function W_c defined in Eq. (8), i.e.

$$\pi = \int_{V^0} [-W_c(p, I_1^*, I_2^*) + p(J-1)] dV^0 + W_{\text{ext}}. \tag{10}$$

In the above equation, the variables, J, I_α^*, $\alpha = 1, 2$, are derived from the displacements u_i and the hydrostatic pressure p is treated as an independent field variable; also W_{ext} is the work done by external forces. Equation (10) represents a special case, i.e. two-field theorem of the generalized variational principles given by Atluri and Reissner (1989).

It is noted at this point that Eq. (10) is applicable to problems involving either compressible or incompressible materials undergoing large strain elastic deformations. One may specialize the above equation for nearly incompressible materials by making further approximation on the form of strain energy density function W. That is, from Eq. (7) a bulk modulus is defined by

$$\kappa = \frac{\partial^2 W}{\partial J^2}. \tag{11}$$

Generally, κ should be a function of both J and I_α^*, $\alpha = 1, 2$. But for rubber-like materials, it is reasonable to assume that the material response is linearly elastic in bulk. Hence, κ can be treated merely as a constant. It is equivalent to say that the hydrostatic pressure p is linearly proportional to the volume change $(J-1)$, i.e.

$$p = \kappa(J-1). \tag{12}$$

Integrating Eq. (11) and making use of Eq. (12), one can easily find the expression for W of the form

$$W(J, I_\alpha^*) = \bar{W}(I_\alpha^*) + \tfrac{1}{2}\kappa(J-1)^2. \tag{13}$$

In the above equation, it is clear to see that the strain energy density function W is decoupled into two distinct parts: the deviatoric part $\bar{W}$ and dilatational part, second term in Eq. (13); each one is independent of the other.

Following Eqs. (8) and (13), the complementary energy function W_c becomes

$$W_c(p, I_\alpha^*) = -\bar{W}(I_\alpha^*) + \frac{1}{2\kappa} p^2. \tag{14}$$

With Eq. (14), the functional π in Eq. (10) is rewritten as

$$\pi = \int_{V^0} \left[\bar{W}(I_1^*, I_2^*) + p(J-1) - \frac{1}{2\kappa} p^2 \right] dV^0 + W_{\text{ext}}. \tag{15}$$

This equation is sometimes referred as a "perturbed Lagrangian formulation". In the above, if we take the limiting value $\kappa \to \infty$, Eq. (15) will represent the potential of a (perfectly) incompressible material.

3 Constitutive models

Basically, rubber is considered to be a nonlinear, incompressible hyperelastic material which often experiences large deformations upon loading; Treloar (1975). For hyperelastic materials, it is known that the stress–strain equations are derivable from a strain energy density function W which is generally expressed as a function of strain invariants or principal stretches. Within this framework, a number of constitutive models have been proposed to represent the stress–strain responses of rubbers, e.g. those in references: Ogden (1972), Alexander (1986), and Finny and Kumar (1988). Each model was trying to correlate closely with the experimental data within the range of loading for a particular application. Some discussions on the merits and drawbacks of various constitutive models can be found, e.g. in the text by Treloar (1975) and the paper by Alexander (1968).

In the present paper, we consider two constitutive models in our proposed finite element analysis procedures: the generalized Rivlin model and Ogden's model. The former model is a classical theory which has been in existence for quite sometime (Treloar 1975). The latter model is chosen due to its the ability in representing the stress–strain behavior of rubber in an extended deformation range (large strain in the order of exceeding 100%). It is noted here that other constitutive models could also be implemented into the finite element analysis by following the similar procedure.

3.1 Generalized Rivlin model

In order to specify the mechanical response of rubber, the form of strain energy density function W in Eq. (3) must be defined explicitly. By assuming that the material is incompressible and isotropic in the undeformed state, Rivlin proposed a form of W being the sum of terms involving only I_1 and I_2, i.e. Rivlin (1951) and Treloar (1975)

$$W_r(I_1, I_2) = \sum_{m=0,n=0}^{M,N} C_{mn}(I_1 - 3)^m (I_2 - 3)^n. \tag{16}$$

A special case of the above equation is the two constant Mooney model having the expression (Green and Zerna 1966)

$$W_m(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3). \tag{17}$$

As defined in the above equations, the strain energy function contains both isochoric and hydrostatic deformations. In order to decouple these two parts for the finite element formulation of nearly incompressible materials, the two strain invariants are modified in the following manner:

$$I_1^* = I_1 I_3^{-1/3}, \quad I_2^* = I_2 I_3^{-2/3}. \tag{18}$$

Within the above definitions, the strain invariants satisfy the condition specified in Eq. (6). Thus, one can construct the distortional part of strain energy, $\bar{W}$ by replacing I_1 and I_2 by I_1^* and I_2^*, respectively in Eqs. (16) and (17). If, however, for a perfectly incompressible material, setting $I_3 \to 1$, the expression of Eq. (16) or (17) is restored.

3.2 Ogden's model

While the use of first-order terms in I_1 and I_2 in Rivlin's model may have inhibited its ability in predicting the response of a rubber at large strain level, Ogden (1972) proposed a model for which the strain energy is assumed to be a function of the principal stretches. One obvious advantage of using the principal stretches is that they are directly measurable quantities. According to Ogden, one writes

$$W_o = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) \tag{19}$$

or equivalently,

$$W_o(v_1, v_2, v_3) = \sum_{n=1}^{N} \frac{\mu_n}{2\theta_n}(v_1^{\theta_n} + v_2^{\theta_n} + v_3^{\theta_n} - 3) \tag{20}$$

where λ_1, λ_2 and λ_3 are the 3 principal stretches of deformation; v_1, v_2 and v_3 are the principal values of Green deformation tensor; they are related as the following

$$v_r = \lambda_r^2, \quad r = 1, 2, 3 \tag{21}$$

and $\mu_n, \alpha_n, \theta_n$ are material constants which are defined by fitting the experimental data; also, $\theta_n = \alpha_n/2$.

In order to decouple the isocharic and hydrostatic deformations in the expression of strain energy function W_o given in Eq. (20), we follow a modification procedure employed by Ogden (1976) for compressible materials. That is, the squares of principal stretches are modified by

$$v_1^* = v_1 I_3^{-1/3}, \quad v_2^* = v_2 I_3^{-1/3}, \quad v_3^* = v_3 I_3^{-1/3}. \tag{22}$$

With the above definitions, we specify the modified principal stretches appeared in Eq. (3) by setting: $I_1^* = v_1^*, I_2^* = v_2^*$. Thus, these quantities satisfy the condition in Eq. (6), i.e. they involve only the isochoric deformation. Replacing v_r by v_r^* respectively in Eq. (20), $r = 1, 2, 3$, a strain energy function for distortion can then be obtained for a nearly incompressible material and this is expressed symbolically as the following

$$\bar{W} = W_o(v_1^*, v_2^*, v_3^*). \tag{23}$$

Based on the variational principle with the use of a perturbed Lagrangian multiplier given in Eq. (15), one can readily formulate a two-field finite element model, for which both the displacement and pressure fields are interpolated independently. In our approximations, the displacement field is assumed to be C^0 continuous whereas a discontinuous pressure field is assumed for improved computational efficiency. Thus, the final stiffness equation of an element involves only the nodal displacements. For the analysis of two-dimensional problems, it is possible to derive either the low-order or higher order elements by assuming appropriate displacement and pressure fields. However, from the mathematical standpoint, an optimal element is the one which satisfies the so called Babuska (1977) and Brezzi (1974) conditions. For finite element implementation, we have chosen a two-dimensional 9-node Lagrange element with three-term pressure parameters defined in the global coordinates and this element has been incorporated into a computer program by Chang (1988) for computational purpose.

4 Numerical examples

In this section, the numerical results of five benchmark problems are presented. These problems are selected with the following premises: (1) verification check for the constitutive models implemented and numerical algorithm presented in this paper, (2) comparison of solution accuracy with independent results published in the literature, and (3) a quantitative judgement on the convergence rate of the present algorithm. It is noted that all problems are calculated by employing the full Newton–Raphson iterations.

4.1 Uniaxial and biaxial extensions of a rubber sheet

A rubber sheet of the dimensions (4 cm) × (1 cm) as shown in Fig. 1 is subjected to a uniform vertical load. Two cases of deformation are considered: uniaxial extension (axisymmetric deformation) and biaxial extension (plane strain state), respectively. A patch of 5-elements is used in the

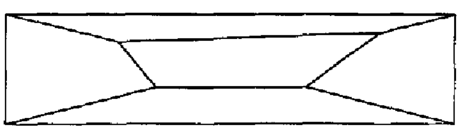
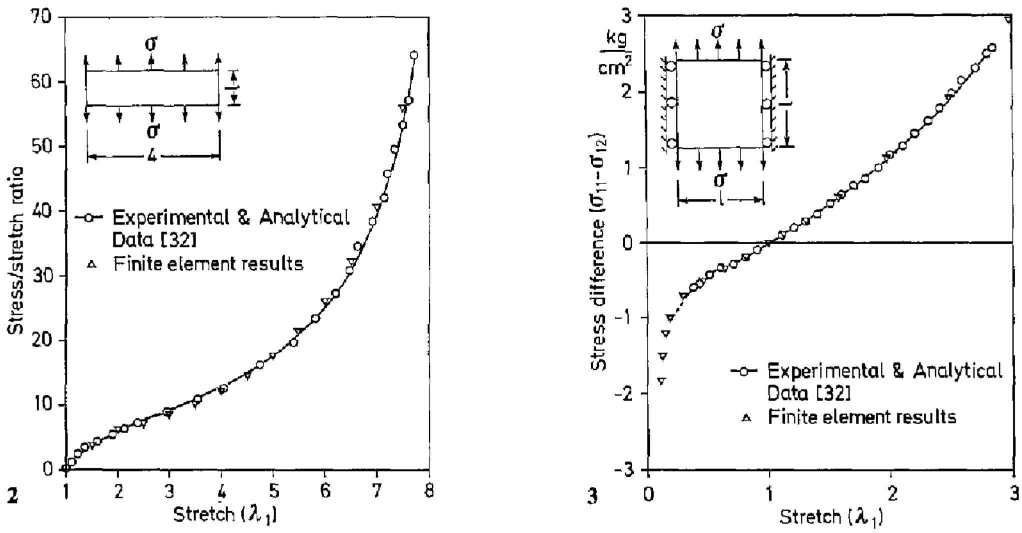


Fig. 1. A patch of elements under uniaxial and biaxial tensions



Figs. 2 and 3. Uniaxial (2) and biaxial (3) tensile response of a rubber sheet

analysis and Ogden's model is adopted with the following material constants:

$$\mu_1 = 6.27, \quad \mu_2 = 0.036, \quad \mu_3 = -0.054, \quad \mu_4 = 0.8 \times 10^{-15} \quad (\mathrm{kg \cdot cm^{-2}})$$

$$\alpha_1 = 1.23, \quad \alpha_2 = 4.44, \quad \alpha_3 = 19.49 \quad \alpha_4 = 19.49$$

$$\kappa = 10^5.$$

The purposes of this example are: (1) to verify the coding implementation of Ogden's model, especially its predictability in the large strain range, and (2) to check if the mixed Lagrange element does satisfy the patch test (there is no reason to believe it will not).

For both cases, the analyses are completed with 20 load steps and each load step requires about 4 equilibrium iterations. The results in the form of Cauchy stress vs. stretch are plotted for uniaxial extension (Fig. 2) and biaxial extension (Fig. 3). In both cases, the predictions by Ogden's model follow closely with the experimental measurements for the entire range of deformations (Ogden 1972).

4.2 *A cylinder under internal pressure*

This is an example which has been analyzed by several investigators, e.g. Sussman and Bathe (1987) and Scharnhorst and Pian (1978), due to the availability of analytical solution; Green and Zerna (1966) and Rivlin (1951). The cylinder is assumed to be infinite in length with an inner radius $r_i = 7$ in. and an outer radius $r_o = 18\frac{5}{8}$ in. We consider the Mooney–Rivlin material model with $C_1 = 80$ psi and $C_2 = 20$ psi for the rubber. The analytical solution of the problem was obtained by Rivlin (1951) as the following:

$$p = (C_1 + C_2)\frac{(r^2 - r_o^2)[(r_o + t)^2 - r_o^2]}{r^2[r^2 + (r_o + t)^2 - r_o^2]} + \ln\left[\frac{r^2}{r^2 + (r_o + t)^2 - r_o^2}\right] + 2\ln\left(\frac{r_o + t}{r_o}\right)$$

where $t = r_o - r_i$.

In finite element analysis, we consider only one section of the cylinder under axisymmetric deformation. The cylinder is modeled by ten 9-node Lagrange elements. A maximum internal pressure of 150 psi is applied to the inner wall in 10 equal increments. Each load step took between 1–2 iterations to reach solution convergence. Both the finite element and analytical solutions for pressure vs. the displacement of inner wall are shown in Fig. 4. It is clear to see that excellent agreement between the two solutions was obtained. In fact, this problem was re-analyzed with

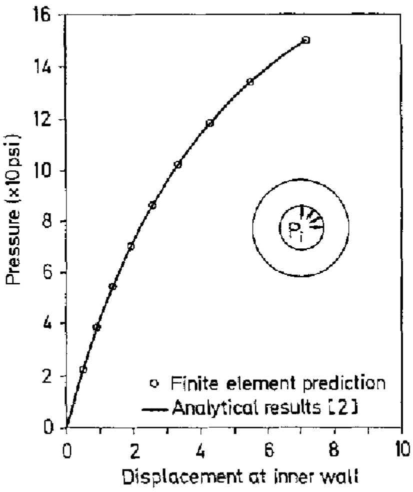


Fig. 4. Pressure vs. displacement response of a thick wall cylinder

two load steps (4-iterations per step) and one load step (6-iterations); identical results were obtained in all cases.

4.3 *Simple shear of a rubber block*

A rubber block, with its two opposite faces attached to metal plates, are subjected to a simple shear in the form of specified displacements as indicated in Fig. 5. The rubber block is deformed under the condition of plane strain. The purposes of considering this problem are: (1) it is possible to develop an approximate analytical solution, and (2) numerically, this is a rather challenging problem due to the plane strain constraint, especially in the large strain range. In fact, this problem was analyzed using a similar formulation without splitting the isochoric and hydrostatic energies. However, difficulty in arriving at a convergent solution was encountered when the overall shear strain is in the order of 50%.

The rubber block is modeled by a 4 × 48 mesh (shown in Fig. 5) and a maximum horizontal displacement of 0.5 (or equivalent to 250% average shear strain) is imposed on the top surface. In our analysis, we employed the full Newton–Raphson iteration procedure in which the tangent structural stiffness is reformed for every load step and every iteration. The maximum displacement

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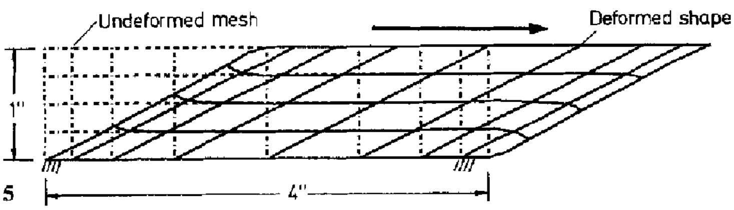

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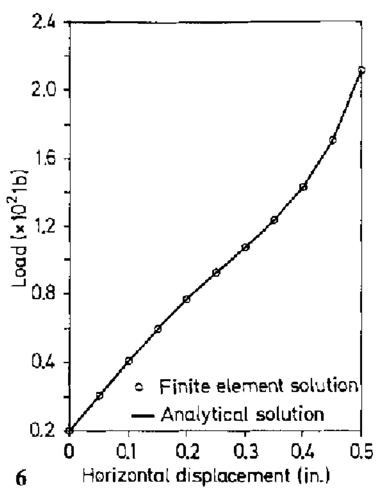


Figs. 5 and 6. 5 Simple shear of a rubber block under plane strain. **6** Load-displacement response of a rubber block under simple shear

Table 1. Change of residual energy per iteration for the 5-load step solution of a simple shear problem

Iteration number	Load step				
	1	2	3	4	5
1	0.9229E + 03	0.6328E + 03	0.2537E + 03	0.8894E + 02	0.2124E + 03
2	0.2998E + 00	0.3005E + 00	0.7534E + 00	0.6212E + 00	0.1932E + 00
3	0.3731E − 05	0.3787E − 06	0.3173E − 03	0.1259E − 02	0.6111E − 03
4	0.3765E − 14	0.4560E − 16	0.4947E − 09	0.5862E − 08	0.7161E − 09

Table 2. Change of residual energy per iteration for the 2-load step solution of a simple shear problem.

Iteration number	Load step	
	1	2
1	0.3605E + 5	0.4827E + 04
2	0.1031E + 4	0.1570E + 04
3	0.1257E + 2	0.6493E + 03
4	0.6452E + 1	0.5510E + 01
5	0.3103E + 0	0.4035E − 01
6	0.5334E − 04	0.1727E − 05
7	0.7785E − 11	0.3737E − 13

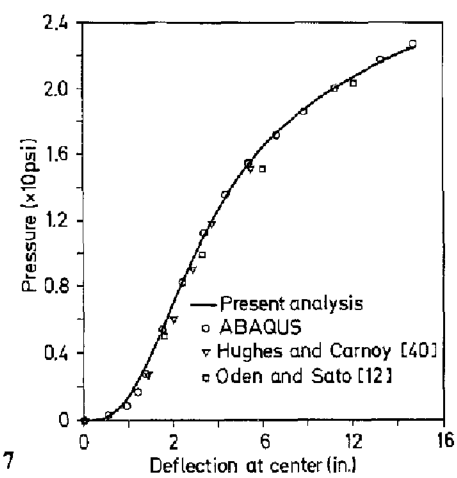
was applied to the block for three different cases of equal load steps: (a) 10 steps, (b) 5 steps, and (c) 2 steps. In all cases, the same final convergent solution was reached with different number of iterations required per load step. For example, in the case of 10 equal load steps, solution converges with 2 iterations almost for every load step. For the case of 5-load steps, the average number of iterations per load step is around 4 and the convergence rate is nearly quadratic as seen in Table 1, the change of residual energy per iteration. For the case of two load steps, obviously more iterations (7) is required to reach solution convergence and the change of residual energy is shown in Table 2. As seen in this table, once the residual energy is reduced to a level in the order of less than 10, quadratic rate of asymptotic convergence prevails.

The load vs. horizontal displacement of the top surface obtained from the finite element analysis is shown in Fig. 6 against the analytical solution. From the plots, the two sets of solution are almost indistinguishable. It is also noted that stress singularities exist at the four corners of the block. If one is interested in detailed stress distribution of the problem, the adopted mesh may not be fine enough for this purpose. However, for obtaining the displacement response it was found that the present mesh is sufficient by comparing the present result with that of a coarse mesh.

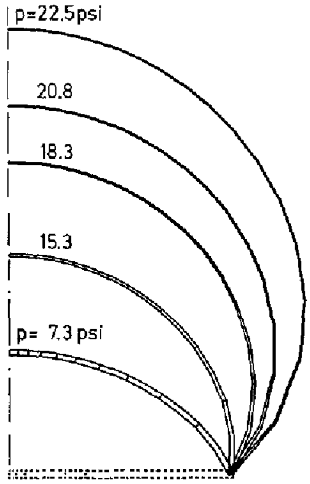
4.4 *Inflation of a circular disk*

Considered here is a circular rubber disk of flat geometry initially. The disk is subjected to a uniform normal pressure (always normal to the deformed configuration) and it deforms into a bubble shape. A Mooney–Rivlin material with two constants in considered: $C_1 = 80$ psi and $C_2 = 20$ psi, $\kappa = 100000$. A maximum pressure of 22.5 psi, divised into 290 increments, is applied to the disk. Note that the applied pressure is always normal to the deformed surface and the surface area of the disk changes significantly as a result of deformation. Since in our present analysis, we do not consider the stiffness changes due to deformation. Consequently, small load steps have to be used in order to achieve solution convergence. Our analysis result in the form of pressure vs. vertical displacement at the center of the disk is shown in Fig. 7 in conjunction with the published results by Oden and Sato (1967) and Hughes and Carnoy (1981).

Figure 8 shows the deformed shapes of the membrane under pressures $P = 7.3$, 15.3, 19.3, 20.8, 22.5 psi. It is noted that the boundary of the disk is allowed to rotate freely during the analysis



7



8

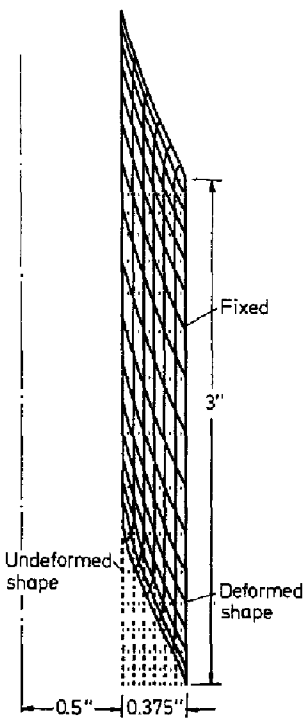
Figs. 7 and 8. 7 Central deflection vs. applied pressure of a rubber disk. **8** Inflated geometries of a rubber disk at various pressure levels

and this boundary condition is achieved by imposing multipoint constraint conditions during the analysis.

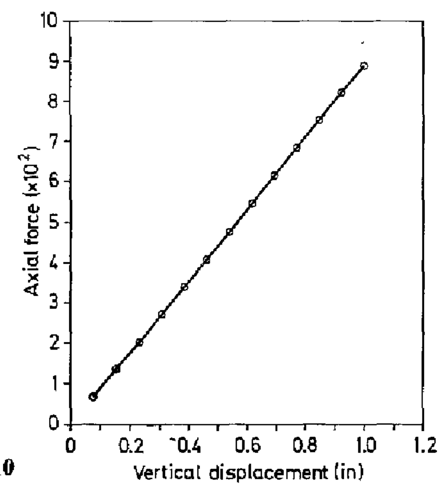
4.5 Shear of a bushing

A rubber bushing, similar to the geometry of the preceding problem, with different dimensions, is subjected to a vertical force around the inner steel tube and the outer tube is fixed. Since the rubber is contained between the inner and outer tubes, a highly confined situation, the material

9



10



Figs. 9 and 10. 9 Undeformed and deformed shapes of a rubber bushing subjected to an axial force. **10** load-deflection response of a rubber bushing is thus deformed under constraint condition. Both the inner and outer tubes are assumed to be rigid and only a sector of the bushing is modeled by a 6×16 mesh. Due to the anticipated high stress gradients at the four corners of the rubber ring, a fine mesh is focused at the upper and lower parts of the model (see Fig. 9).

A vertical displacement of 0.4 in is imposed to the inner steel tube in 20 equal increments. The mechanical response of the rubber is represented by the Mooney–Rivlin model: $C_1 = 66.67$ psi, $C_2 = 16.67$ psi, $\kappa = 10^5$. This problem was perceived to be difficult to analyze numerically due to the confined condition. In our analysis, solution converges rather rapidly (2–3 iterations per load step) and this is probably resulted from the mixed formulation in conjunction with the splitting of hydrostatic and isochoric deformations. The vertical load vs. displacement plot is presented in Fig. 10. From the figure, it appears that the load-displacement response of the bushing is rather linear, although the effects of both material and geometric nonlinearities are present in the analysis. The linearity is again probably cause by the extreme confining of the rubber material between the two steel tubes.

5 Conclusion

A mixed finite element method is presented for the nonlinear analysis of rubber or rubber-like materials involving large strains, nonlinear material behavior, and incompressibility constraint. The main thrust of this paper is to present a consistent mixed formulation based on a perturbed Lagrange approach from which both the displacement and pressure fields are independently assumed. Further, the strain energy is decomposed into two distinct parts: the hydrostatic and isochoric parts. It is believed that with this decomposition, a quadratic convergence rate is achieved in the solution algorithm employed. Moreover, our two-dimensional formulation is evolved around a 9-node Lagrange element with 3-term pressure parameters which is known to satisfy the Brezzi and Babuska conditions. Our immediate future effort is to extend to present formulation to three-dimensional finite element analysis and such extension should be rather straight forward.

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