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CONSTANT ISOTROPIC SURFACES IN 5-DIMENSIONAL SPACE FORMS

0. Introduction

The definition of isotropic immersion was first given in [8]. An isometric immersion $f: M^n \to \bar{M}^m$ is said to be λ-isotropic if, for each $x \in M^n$, the ellipse of curvature $\{H(X, X): X$ is a unit vector tangent to M^n at $x\}$ in a normal space is contained in a sphere with radius $\lambda(x)$, where H denotes the second fundamental form of f. Moreover, if $\lambda(x)$ does not depend on x, then f is called a constant isotropic immersion.

Isotropic immersions have been studied by several authors (for instance, see [5]–[9]). They seem to form a wide class. For example, the composition of isotropic immersions and their Euclidean products are also isotropic, and furthermore there are many constant isotropic immersions in equivariant isometric immersions of homogeneous spaces (cf. [7]). However, all isotropic submanifolds which appeared in [5], [7], [8], [9], [10] are homogeneous. It is certain that an isotropic non-homogeneous surface exists in a sphere. This can be seen in the study of minimal surfaces homeomorphic to a sphere. If $f: M^2 \to S^m$ is a full minimal immersion (with m necessarily even) of a differentiable sphere M^2, then the complex vector $\zeta = H(X, X) + iH(X, Y)$ is isotropic in virtue of the Riemann–Roch theorem, $\{X, Y\}$ being any orthonormal basis for the tangent space of M^2, and hence f is isotropic in our sense (see [3] and [6, p. 140]) and it is certain that many non-homogeneous minimal 2-spheres are immersed in S^m (cf. [3, Th. 3]). But we do not know whether a constant isotropic non-homogeneous surface exists in S^m or not.

Constant isotropic surfaces immersed in a 4-dimensional real space form $\bar{M}^4(c)$ were determined in Theorem (5c) of [6]. More generally, constant isotropic submanifolds, all of whose geodesics are contained in 4-dimensional totally geodesic submanifolds of $\bar{M}^m(c)$, were classified in [9]. However, constant isotropic surfaces, with no other assumptions, immersed in an odd-dimensional space form have not yet been studied.

This paper is concerned mainly with the two circumstances mentioned above, and we shall study constant isotropic surfaces immersed in a 5-dimensional space form. We shall prove that if $f: M^2 \to \bar{M}^5(c)$ is a constant isotropic immersion, then f is totally geodesic, totally umbilical, the second standard immersion of a 2-sphere into a totally geodesic or umbilical hypersurface of

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$\bar{M}^5(c)$, or a certain immersion of the Euclidean space R^2. Unfortunately, a constant isotropic non-homogeneous surface does not exist in $\bar{M}^5(c)$, so it seems to be of interest that we study constant isotropic surfaces in $\bar{M}^6(c)$ (especially S^6). Is there a constant isotropic non-homogeneous surface in S^6?

In Section 1 we prepare notations and basic equations that we use later. Moreover, we show that if the dimension of the first normal space N_x^1 is not greater than 2 for every x, then it is independent of $x \in M^2$. As a result, we have two non-trivial cases: (A) $\dim N_x^1 = 2$ for every x (minimal case), (B) $\dim N_x^1 = 3$ for some x. In Section 2 we study case (A) in which the Gauss curvature K is constant ($\geqslant 0$) and $c > 0$. If $K > 0$, $f: M^2 \to S^5(c)$ is determined by Calabi's theorem [2]. If $K = 0$, then $f: R^2 \to S^5(c)$ is the immersion given in [1]. In Section 3 we study case (B). Under the assumption that K is not constant, we shall obtain second-order differential equations satisfied by K in two ways and derive a contradiction from these equations.

1. Isotropic immersions of surfaces

Let M be a Riemannian manifold and let $\bar{M}(c)$ be a simply connected real space form of curvature c. According to $c > 0$, $c = 0$ and $c < 0$, $\bar{M}(c)$ is isometric to a Euclidean sphere, Euclidean space and hyperbolic space. Let $f: M \to \bar{M}(c)$ be an isometric immersion with differential f_*. The tangent bundle TM of M may be considered as a subbundle of the induced bundle $f^* T\bar{M}$ and the orthogonal complement NM of TM in $f^* T\bar{M}$ is called the normal bundle of the immersion f. The Levi–Civita connection on $T\bar{M}$ induces a connection $\bar{D}$ on $f^* T\bar{M}$, which is decomposed to connections D on TM and $D^\perp$ on NM. Then D coincides with the Levi–Civita connection of M and $D^\perp$ is called the normal connection. Connections D and $D^\perp$ induce a connection on the bundle $(\Sigma \otimes^r T^* M) \otimes NM$. We shall also denote it by D. Thus the covariant derivative of the second fundamental form H of the immersion f is defined as

$$(D_X H)(Y, Z) = D_X(H(Y, Z)) - H(D_X Y, Z) - H(Y, D_X Z)$$

for vector fields X, Y, Z tangent to M. The Weingarten map A_ξ corresponding to a normal vector ξ at x is a symmetric transformation of $T_x M$ defined by $\langle A_\xi X, Y\rangle = \langle H(X, Y), \xi\rangle$. Let R and $R^\perp$ be the curvature tensors of D and $D^\perp$, respectively. We have the following basic equations:

(1.1) $\bar{D}_X Y = D_X Y + H(X, Y)$,

(1.2) $\bar{D}_X \xi = -A_\xi X + D_X \xi$,

(1.3) $R(X, Y)Z = c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} + A_{H(Y,Z)} X - A_{H(X,Z)} Y$,

(1.4) $(D_X H)(Y, Z) = (D_Y H)(X, Z)$,

(1.5) $R^{\perp}(X,Y)\xi = H(X, A_{\xi}Y) - H(A_{\xi}X, Y)$

for vector fields X, Y, Z tangent to M and ξ normal to M.

An isometric immersion $f: M \to \bar{M}$ is said to be (λ-)*isotropic* if

(1.6) $\|H(X,X)\| = \lambda(\pi(X))$

for every $X \in UM$, where $\pi: UM \to M$ is the projection of the unit tangent bundle UM of M and λ is a function defined on M (cf. [8]). Moreover, if λ is constant, then immersion is said to be *constant isotropic*. It is easily proved that f is isotropic if and only if

(1.7) $\langle H(X,X), H(X,Y)\rangle = 0$

for all orthogonal vectors X and Y tangent to M at every point. For orthonormal vectors X and Y, we have, from (1.6),

(1.8) $\langle H(X,X), H(Y,Y)\rangle + 2\|H(X,Y)\|^2 = \lambda^2.$

In the sequel, *we assume that M is a connected surface and f a constant λ-isotropic immersion into a 5-dimensional space form $\bar{M}(c)$.* Let K be the Gauss curvature of M. By (1.3) and (1.8), we have

(1.9) $K = c + \lambda^2 - 3\mu^2 = c - \dfrac{\lambda^2}{2} + \tfrac{3}{2}\nu,$

where $\mu = \|H(X,Y)\|$ and $\nu = \langle H(X,X), H(Y,Y)\rangle$ for orthonormal vectors X and Y. Thus K always satisfies $K(x) \leqslant c + \lambda^2$ at $x \in M$ and when $\lambda \neq 0$, equality holds if and only if x is an umbilical point. Let N_x^1 be the first normal space at x which is the subspace spanned by $\{H(V,W): V, W \in T_x M\}$.

LEMMA 1.1 ([8, 1]). *We see that* $c\text{-}2\lambda^2 \leqslant K \leqslant c + \lambda^2$ *and*

$\dim N_x^1 = 0 \Leftrightarrow \lambda = 0 \Leftrightarrow$ *totally geodesic,*
$\dim N_x^1 = 1 \Leftrightarrow K(x) = c + \lambda^2 \Leftrightarrow x$ *is umbilical,*
$\dim N_x^1 = 2 \Leftrightarrow K(x) = c - 2\lambda^2 \Leftrightarrow \eta(x) = 0 \quad (\lambda \neq 0),$
$\dim N_x^1 = 3 \Leftrightarrow (K(x) - c - \lambda^2)(K(x) - c + 2\lambda^2) \neq 0,$

where η is the mean curvature vector.

Proof. Let X and Y be orthonormal vectors tangent to M at x. N_x^1 is spanned by $H(X,X)$, $H(X,Y)$ and $H(Y,Y)$. Computing the Gramian of these vectors, we see that Gramian $= (\tfrac{4}{27})(K - c - \lambda^2)^2(K - c + 2\lambda^2)$. Thus $c - 2\lambda^2 \leqslant K \leqslant c + \lambda^2$, and $(K(x) - c - \lambda^2)(K(x) - c + 2\lambda^2) \neq 0$ if and only if $\dim N_x^1 = 3$. Assume that $\lambda \neq 0$. We have already seen the case $K(x) = c + \lambda^2$. $K(x) = c - 2\lambda^2$ if and only if $\lambda = \mu$ because of (1.9). Using (1.8), it follows that $K(x) = c - 2\lambda^2$ if and only if $H(X,X) = -H(Y,Y)$ for orthonormal vectors

X and Y. Since the mean curvature vector is defined by $\eta = \{H(X,X) + H(Y,Y)\}/2$, we have the assertion. Q.E.D.

Denote the open subset $\{x \in M: \dim N_x^1 = 3\}$ by U. Since λ is constant, if U is empty, then we see from Lemma 1.1 that K is constant. Totally geodesic and umbilical surfaces in $\bar{M}(c)$ are well known (cf. [10]). So, in the subsequent sections, we shall consider the following two cases:

(A) M is a non-totally geodesic minimal surface of constant curvature $c - 2\lambda^2$,

(B) $U \neq \emptyset$.

2. Case (A)

In this section we deal with case (A). Thus f is a constant isotropic minimal immersion of the connected surface M of constant curvature $c - 2\lambda^2$ ($\lambda \neq 0$) into $\bar{M}(c)$. To begin with, we prove

LEMMA 2.1. *Let $\|DH\|$ be the length of the covariant derivative of the second fundamental form. It is given by $\|DH\|^2 = 8\lambda^2(3\lambda^2 - c)$. In particular, $\lambda^2 \geqslant c/3$.*

Proof. Let $\{X, Y\}$ be an orthonormal basis for the tangent space of M. Taking account of the fact that $\lambda = \mu$, we set $\xi_1 = H(X,X)/\lambda$ and $\xi_2 = H(X,Y)/\lambda$. They are orthonormal because of (1.7). Take the unit normal vector ξ_3 so that $\{X, Y, \xi_1, \xi_2, \xi_3\}$ give the oriented basis for the tangent space of $\bar{M}(c)$. Then we have

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \qquad A_3 = 0,$$

where we have put $A_1 = A_{\xi_i}$ for $i = 1, 2, 3$. These are easily verified by the equations $\langle A_i X, X\rangle = \langle H(X,X), \xi_i\rangle$, $\langle A_i X, Y\rangle = \langle H(X,Y), \xi_i\rangle$ and $H(X,X) = -H(Y,Y)$. Thus the length of the second fundamental form $\|H\|$ is given by $\|H\|^2 = \operatorname{trace}(A_1^2 + A_2^2 + A_3^2) = 4\lambda^2$. By a routine calculation, we have

$$\begin{aligned}
\tfrac{1}{2}\Delta\|H\|^2 &= \langle \Delta H, H\rangle + \|DH\|^2 \\
&= 2\langle DD\eta, H\rangle - 4c\|\eta\|^2 \\
&\quad + 2c\|H\|^2 + 2\langle \eta, \sum_j \operatorname{tr}\Big(\sum_i A_i^2 A_j\Big)\xi_j\rangle \\
&\quad - \sum (\operatorname{tr}(A_i A_j))^2 + 2\sum \operatorname{tr}(A_i A_j A_i A_j) - 2\operatorname{tr}\Big(\sum A_j^2\Big)^2 \\
&\quad + \|DH\|^2
\end{aligned}$$

(cf. [10, p. 42]. Substituting $\eta = 0$, $\|H\|^2 = 4\lambda^2$, the fifth term $= 8\lambda^4$, the sixth term $= 0$ and the seventh term $= 8\lambda^4$ in the above equation, we obtain $\|DH\|^2 = 8\lambda^2(3\lambda^2 - c)$. Q.E.D.

Let $\{X, Y\}$ be an orthonormal local frame field and let $\{\xi_1, \xi_2, \xi_3\}$ be the orthonormal normal field defined as in the proof of Lemma 2.1.

LEMMA 2.2. *Denote* $(D_Z H)(V, W)$ *by* $(DH)(Z, V, W)$. *The subspace spanned by* $\{(DH)(Z, V, W)\colon Z, V, W \in T_x M\}$ *is orthogonal to the first normal space. If we set*

$$(DH)(X,X,X) = -(DH)(X,Y,Y) = \alpha\xi_3,$$
$$(DH)(Y,Y,Y) = -(DH)(X,X,Y) = \beta\xi_3,$$

then $\alpha^2 + \beta^2 = 2\lambda^2(3\lambda^2 - c)$.

Proof. Let x be an arbitrarily fixed point of M. We may assume that $DX = 0$ at x. Since $\langle H(X,X), H(X,X)\rangle$ is constant we have

$$\langle (DH)(X,X,X), H(X,X)\rangle = \langle (DH)(Y,X,X), H(X,X)\rangle = 0$$

at x. Since X may be arbitrarily chosen at x, the equation $\langle (DH)(X,X,X), H(X,X)\rangle = 0$ holds for every X tangent to M at x. By symmetrization, we have

$$3\langle (DH)(X,X,Y), H(X,X)\rangle + 2\langle (DH)(X,X,X), H(X,Y)\rangle = 0$$

and hence $\langle (DH)(X,X,X), H(X,Y)\rangle = 0$. Noting that f is minimal and using Lemma 2.1, we have the assertion. Q.E.D.

Let $\pi\colon \hat{M} \to M$ be the universal Riemannian covering. The preceding lemmas are valid for the immersion $f \circ \pi$. We assume that M is complete.

LEMMA 2.3. *Assume that* K *is non-positive. Then* $K = 0$ *and* c *must be positive.*

Proof. If $\lambda^2 = c/3$, then $K = \lambda^2 > 0$ which contradicts the assumption. Let $x \in M$ and let $\{X, Y\}$ be an orthonormal basis for $T_x M$. Define X_θ and θ_0 $(0 \leqslant \theta_0 \leqslant \pi)$ by $X_\theta = \cos\theta X + \sin\theta Y$ and $\cos\theta_0 = \alpha/(2\lambda^2(3\lambda^2 - c))^{1/2}$, respectively, where $\alpha = \langle (DH)(X,X,X), \xi_3\rangle$ (see Lemma 2.2). Then we have, from Lemma 2.2,

$$(DH)(X_\theta, X_\theta, X_\theta) = \{2\lambda^2(3\lambda^2 - c)\}^{1/2} \cos(3\theta + \theta_0)\xi_3.$$

Therefore, we see that $\|(DH)(X_\theta, X_\theta, X_\theta)\|^2$ attains a maximum $2\lambda^2(3\lambda^2 - c)$ at $\theta = (m\pi - \theta_0)/3$ and a minimum 0 at $\theta = \{(2m+1)\pi/2 - \theta_0\}/3 (m \in Z)$. In other words, the maximum points of the function $\Phi_x(\cdot) = \|(DH)(\cdot,\cdot,\cdot)\|^2$ on the unit circle in $T_x M$ are vertices of the regular hexagon, one of whose vertices is $\cos(\theta_0/3)X - \sin(\theta_0/3)Y$ and the minimum is attained at their middle points.

Now, we construct globally defined parallel orthonormal vector fields V and Z on $\hat{M}$ such that the function Φ_x attains a maximum at $V(x)$ a minimum at $Z(x)$ for every $x \in \hat{M}$. Since $\hat{M}$ is isometric to R^2 or a hyperbolic surface H^2 of constant curvature there exists an orthonormal frame field $\{X, Y\}$ defined on $\hat{M}$. Let ξ_3 be the unit normal vector field defined as in Lemma 2.1 and put $\alpha = \langle (DH)(X,X,X), \xi_3 \rangle$. Then α is a smooth function on $\hat{M}$ but $\theta_0 (0 \leqslant \theta_0 \leqslant \pi)$ is defined only continuously. If we define V by $V = \cos(\theta_0/3)X - \sin(\theta_0/3)$ Y, then V is a continuous unit vector field such that Φ_x attains a maximum at $V(x)$ for every $x \in \hat{M}$. In fact, V is smooth. This is shown as follows. Let $x_0 \in \hat{M}$ be arbitrarily fixed and let X^* be a unit vector field on a neighborhood of x_0 such that $\|(DH)(X^*,X^*,X^*)\|^2 \neq 2\lambda^2(3\lambda^2 - c)$. Let Y^* be a unit vector field orthogonal to X^*. Since $\cos\theta_0^* = \alpha^*/\{2\lambda^2(3\lambda^2 - c)\}^{1/2} \neq \pm 1$, θ_0^* is smooth around x_0. V is written as $V = \cos\theta^* X^* + \sin\theta^* Y^*$, where $0 \leqslant \theta^* \leqslant \pi$, and hence $3\theta^* + \theta_0^* = m\pi$ $(m \in Z)$. Since θ^* and θ_0^* are continuous, m is constant. Thus θ^* is smooth. If we choose a unit vector field Z orthogonal to V, then $\{V, Z\}$ is an orthonormal frame field on $\hat{M}$ such that $(DH)(V,V,V) = \lambda\kappa\xi_3$ and $(DH)(Z,Z,Z) = 0$, where $\kappa = \{2(3\lambda^2 - c)\}^{1/2}$.

We shall prove that V and Z are parallel. Let ϕ and ψ be functions defined by $D_V V = \phi Z$ and $D_Z Z = \psi V$. Covariantly differentiating $(DH)(V,V,Z) = 0$ in the direction V, we have

$$(D^2H)(V,V,V,Z) + 2(DH)(D_V V, V, Z) + (DH)(V, V, D_V Z) = 0.$$

It follows that $(D^2H)(V,V,V,Z) = 3\lambda\phi\kappa\xi_3$. In order to prove that $\phi = 0$ we have only to show that

$$\langle (D^2H)(V,V,V,Z), (DH)(V,V,V) \rangle = 0.$$

By Ricci identity, (1.5) and Lemma 2.2,

$$\begin{aligned}
&\langle (D^2H)(V,Z,V,V) - (D^2H)(Z,V,V,V), (DH)(V,V,V) \rangle \\
&= \langle R^\perp(V,Z)H(V,V) - 2H(R(V,Z)V,V), (DH)(V,V,V) \rangle \\
&= \langle H(V, A_{H(V,V)}Z) \\
&\quad - H(A_{H(V,V)}V, Z) - 2H(R(V,Z)V,V), (DH)(V,V,V) \rangle \\
&= 0.
\end{aligned}$$

Thus it suffices to show that $\langle (D^2H)(Z,V,V,V), (DH)(V,V,V) \rangle = 0$. Since $\|(DH)(V,V,V)\|^2$ is constant, we have

$$\begin{aligned}
0 &= Z \cdot \langle (DH)(V,V,V), (DH)(V,V,V) \rangle \\
&= 2\langle (D^2H)(Z,V,V,V) + 3(DH)(D_Z V, V, V), (DH)(V,V,V) \rangle \\
&= 2\langle (D^2H)(Z,V,V,V), (DH)(V,V,V) \rangle.
\end{aligned}$$

Similarly, covariantly differentiating $(DH)(V,V,Z)=0$ in the Z direction, we have $(D^2H)(Z,V,V,Z) = -3\psi\lambda\kappa\xi_3$. Since

$$\begin{aligned}
&\langle (D^2H)(Z,V,V,Z),(DH)(V,V,V)\rangle \\
&= \langle (D^2H)(V,Z,V,Z) + R^{\perp}(Z,V)H(V,Z) - H(R(Z,V)V,Z) \\
&\quad - H(V,R(Z,V)Z),(DH)(V,V,V)\rangle \\
&= -\langle (D^2H)(V,V,V,V),(DH)(V,V,V)\rangle \\
&= -\tfrac{1}{2}V\cdot \|(DH)(V,V,V)\|^2 \\
&= 0,
\end{aligned}$$

we obtain $\psi = 0$. Therefore V and Z are parallel and hence $K = c - 2\lambda^2 = 0$.
Q.E.D.

If $K > 0$, c must be positive. We see that case (A) occurs only when $c > 0$.

There is a minimal immersion of $S^2(c/3)$ into $S^4(c)$ which is called the second standard immersion (or Veronese surface, cf. [5], [10]). This is constructed by making use of an orthonormal basis for the second eigenspace of the Laplace operator. We can also construct a minimal immersion of Euclidean surface R^2 into $S^5(c)$. This is defined by

$$\begin{aligned}
T(u,v) = \frac{1}{\sqrt{6c}}&(2\cos x\cos y,\ 2\cos x\sin y,\ \sqrt{2}\cos 2x, \\
&2\sin x\cos y,\ 2\sin x\sin y,\ \sqrt{2}\sin 2x) \\
\in S^5(c) &\subset R^6,
\end{aligned}$$

where $(u,v)\in R^2$, $x = \sqrt{c/2}\,u$ and $y = (\sqrt{6c}/2)v$ (cf. [1], [10]). In fact, T gives rise to a minimal imbedding of a flat torus into $S^5(c)$. Moreover, we can easily examine that T is $\sqrt{c/2}$-isotropic.

PROPOSITION 2.4. *Let f be a constant $\lambda(\neq 0)$-isotropic minimal immersion of a connected complete surface M into a 5-dimensional simply connected space form $\bar{M}(c)$. If $c \leqslant 0$, then such an immersion does not exist. If $c > 0$, $f\circ\pi\colon \hat{M}\to S^5(c)$ is one of the following*:

(1) $\hat{M} = S^2(c/3)$, $f\circ\pi \approx \iota\circ$ *(second standard immersion)*,
(2) $\hat{M} = R^2$, $f\circ\pi \approx T$,

where the notation $F\approx G$ means that there exists an isometry Ψ of $S^5(c)$ such that $F = \Psi\circ G$ and $\iota\colon S^4(c)\to S^5(c)$ is a totally geodesic imbedding.

Proof. If $K>0$, then we conclude (1) by Calabi's theorem ([2, p. 123]). Next suppose that $K = 0$. For two $\sqrt{c/2}$-isotropic minimal immersions $F = f\circ\pi$

and T of R^2 into $S^5(c)$, we have parallel orthonormal frame fields $\{V, Z\}$ and $(\tilde{V}, \tilde{Z})$, respectively, as in the proof of Lemma 2.3. Without loss of generality, we may assume $V = \tilde{V}$ and $Z = \tilde{Z}$. Denote Ψ by the isometry of $S^5(c)$ which maps the adapted frame $\{\sqrt{c}\,F, F_*V, F_*Z, \xi_1, \xi_2, \xi_3)_O$ at O to $\{\sqrt{c}\,T, T_*V, T_*Z, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)_O$, where F and T are considered as position vectors, the ξ_i's are defined in the proof of Lemma 2.1 and O is the origin of R^2. We can show $F = \Psi \circ T$ by the following standard argument. Let $x \in R^2$ be any point, $\xi_i^* = \Psi_* \tilde{\xi}_i$ and $G = \Psi \circ T$. Then the adapted frame fields $\{\sqrt{c}\,F, F_*V, F_*Z, \xi_1, \xi_2, \xi_3\}$ and $\{\sqrt{c}\,G, G_*V, G_*Z, \xi_1^*, \xi_2^*, \xi_3^*\}$ coincide at O and their restriction to the segment $\overline{Ox}$ are the solution of a system of ordinary linear differential equations with constant coefficients:

$$v_0' = \sqrt{c}(pv_1 + qv_2),\ v_1' = -\sqrt{c}\,pv_0 + \lambda(pv_3 + qv_4),$$
$$v_2' = -\sqrt{c}\,qv_0 + \lambda(-qv_3 + pv_4),\ v_3' = \lambda(-pv_1 + qv_2) + \kappa pv_5,$$
$$v_4' = -\lambda(qv_1 + pv_2) - \kappa qv_5,\ v_5' = \kappa(-pv_3 + qv_4),$$

where the v_i's are R^6-valued functions and p, q are constants satisfying $\mathbf{Ox}/\|\mathbf{Ox}\| = pV + qZ$. Therefore, we have $F = G$ at x. Q.E.D.

3. Case (B)

We assume that M is orientable and complete. In case (B), $U = \{x \in M: \dim N_x^1 = 3\} \neq \emptyset$. Denote by U' the open set $\{x \in M: (\operatorname{grad} K)(x) \neq 0\}$. Then $U' \subset U$ by Lemma 1.1. Let $X = \operatorname{grad} K/\|\operatorname{grad} K\|$ and take the unit vector field Y orthogonal to X so that the frame $\{X, Y\}$ gives the orientation of M. The frame field $\{X, Y\}$ is defined on U' and satisfies $Y\cdot K = Y\cdot \mu = Y\cdot \nu = 0$ because of (1.9).

LEMMA 3.1. *Define unit normal fields ξ_1, ξ_2 and ξ_3 by*

$$\xi_1 = \frac{\eta}{\|\eta\|}, \quad \xi_2 = \frac{H(X,X) - H(Y,Y)}{2\mu}, \quad \xi_3 = \frac{H(X,Y)}{\mu}.$$

Then they are orthonormal. On U' we have

$$(3.1)\qquad (DH)(X,X,X) = \frac{3}{8h\mu}(X\cdot\nu)(\mu\xi_1 - h\xi_2),$$

$$(3.2)\qquad (DH)(X,X,Y) = -\frac{1}{4\mu}(X\cdot\nu)\xi_3,$$

$$(3.3)\qquad (DH)(X,Y,Y) = \frac{1}{8h\mu}(X\cdot\nu)(\mu\xi_1 + h\xi_2),$$

(3.4) $$(DH)(Y, Y, Y) = 0,$$

where we have put $h = \|\eta\|$.

Proof. The first assertion is easily derived from (1.8). In the proof of Lemma 2.2, we proved that

(3.5) $$\begin{aligned}\langle (DH)(X,X,X), H(X,X)\rangle &= \langle (DH)(Y,X,X), H(X,X)\rangle \\ &= \langle (DH)(X,X,X), H(X,Y)\rangle = 0.\end{aligned}$$

Let X^* (resp. Y^*) be a vector field around x such that $X^*(x) = X(x)$ (resp. $Y^*(x) = Y(x)$), $DX^* = 0$ (resp. $DY^* = 0$) at x and $X^* \perp Y^*$. Differentiating $\langle H(X^*, X^*), H(X^*, Y^*)\rangle = 0$ in the Y^* direction, we have

$$\langle (DH)(Y,X,X), H(X,Y)\rangle + \langle H(X,X), (DH)(X,Y,Y)\rangle = 0$$

at x. Since $\langle (DH)(V,V,V), H(V,W)\rangle = 0$ for every orthonormal vectors V and W, we have

$$3\langle (DH)(Y,X,X), H(X,Y)\rangle + \langle (DH)(X,X,X), H(Y,Y)\rangle = 0.$$

It follows that

$$\langle (DH)(X,X,X), H(Y,Y)\rangle = 3\langle H(X,X), (DH)(X,Y,Y)\rangle$$

at x. Thus

$$\begin{aligned}(X\cdot\nu)(x) &= X^*\cdot\langle H(X^*,X^*), H(Y^*,Y^*)\rangle(x) \\ &= 4\langle H(X,X), (DH)(X,Y,Y)\rangle(x).\end{aligned}$$

We have proved

(3.6) $$\begin{aligned}&\langle (DH)(X,Y,Y), H(X,X)\rangle = -\langle (DH)(X,X,Y), H(X,Y)\rangle = \tfrac{1}{4}X\cdot\nu, \\ &\langle (DH)(X,X,X), H(Y,Y)\rangle = \tfrac{3}{4}X\cdot\nu.\end{aligned}$$

Note that (3.5) and (3.6) also hold when we exchange X for Y. We use (3.5) and (3.6) to calculate the components of $(DH)(X,X,X), (DH)(X,X,Y), \ldots$ with respect to the frame $\{\xi_1, \xi_2, \xi_3\}$. Then we have (3.1) and (3.4).

LEMMA 3.2. *Let* ϕ *and* ψ *be functions on* U *defined by* $D_X X = \phi Y$ *and* $D_Y Y = \psi X$. *Then* $\phi = 0$ *and* ψ *is given by*

$$\psi = \frac{1}{X\cdot\nu}\left\{\left(\frac{\lambda}{4h\mu} X\cdot\nu\right)^2 + 8\mu^2(K - \mu^2)\right\}.$$

Proof. By Ricci identity, (1.5) and (1.8),

(3.7) $$(D^2H)(X,Y,Y,Y) - (D^2H)(Y,X,Y,Y) = 2(\mu^2 - K)\mu\xi_3.$$

The first term of the left-hand side is given by

$$(3.8)\qquad (D^2H)(X, Y, Y, Y) = -3(DH)(D_X Y, Y, Y) = 3\phi(DH)(X, Y, Y) = \frac{3\phi}{8h\mu}(X\cdot\nu)(\mu\xi_1 + h\xi_2),$$

where we have used (3.3) and (3.4). We next compute the second term. Since $2h^2 = \lambda^2 + \nu$, $Y\cdot h = 0$. It is easily verified from (3.2) and (3.4) that

$$(3.9)\qquad D_Y\xi_1 = -\frac{1}{8h\mu}(X\cdot\nu)\xi_3,$$

$$(3.10)\qquad D_Y\xi_2 = -\left\{\frac{1}{8\mu^2}(X\cdot\nu) + 2\psi\right\}\xi_3.$$

Therefore we can easily compute the right-hand side of

$$(D^2H)(Y, X, Y, Y) = D_Y((DH)(X, Y, Y)) - (DH)(D_Y X, Y, Y) - 2(DH)(X, D_Y Y, Y).$$

Using (3.2) and (3.4), $Y\cdot X\cdot\nu = (D^2\nu)(X, Y)$ and $\mu^2 + h^2 = \lambda^2$, we have

$$(3.11)\qquad (D^2H)(Y, X, Y, Y) = \frac{1}{8h\mu}(D^2\nu)(X, Y)(\mu\xi_1 + h\xi_2) + \left\{\frac{1}{4\mu}(X\cdot\nu)\psi - \frac{\lambda^2}{64h^2\mu^3}(X\cdot\nu)\right\}\xi_3.$$

Substituting (3.8) and (3.11) into (3.7), we see that

$$3\phi X\cdot\nu = (D^2\nu)(X, Y),$$

$$\psi = \frac{1}{X\cdot\nu}\left\{\left(\frac{\lambda}{4\mu h}X\cdot\nu\right)^2 + 8\mu^2(K - \mu^2)\right\}.$$

By the first equation above,

$$3\phi X\cdot\nu = X\cdot(D\nu)(Y) - (D\nu)(D_X Y) = \psi X\cdot\nu$$

and since $3X\cdot\nu = 2X\cdot K = 2\|\text{grad } K\| \neq 0$ on U', we have $\phi = 0$. Q.E.D.

Since $D_X X = 0$, the integral curves of X are geodesics. Let s be the arc-length parameter of the integral curves.

LEMMA 3.3. *The function μ satisfies*

$$\mu'' = \frac{3\nu}{4h^2\mu}(\mu')^2 + 2\mu(K - \mu^2)$$

along each integral curve of X, *where* $\mu' = d\mu/ds$.

Proof. By Ricci identity (1.5) and (1.8), we have

$$(3.12)\qquad (D^2H)(X,Y,X,X) - (D^2H)(Y,X,X,X) = 2\mu(K-\mu^2)\xi_3.$$

Firstly, we compute the first term of the left-hand side of (3.12). Noting that $D_X X = D_X Y = 0$, we have

$$(D^2H)(X,Y,X,X) = D_X((DH)(X,X,Y)).$$

Moreover, it is easily verified from (3.2) and (1.8) that $D_X\xi_3 = 0$. We differentiate covariantly both sides of (3.2) in the X direction. Then

$$(3.13)\qquad (D^2H)(X,Y,X,X) = \left(\frac{1}{4\mu^2}\mu'v' - \frac{1}{4\mu}v''\right)\xi_3.$$

Secondly, we compute the second term of the left-hand side of (3.12). We use (3.1), (3.9) and (3.10) to compute the right-hand side of

$$(D^2H)(Y,X,X,X) = D_Y((DH)(X,X,X)) - 3(DH)(D_Y X,X,X).$$

Taking account of the equations $Y\cdot X\cdot v = 0$ and $h^2 = v + \mu^2$, we have

$$(3.14)\qquad (D^2H)(Y,X,X,X) = \frac{3v}{64h^2\mu^3}(v')^2\xi_3.$$

Substitute (3.13) and (3.14) into (3.12). Then we obtain the desired equation.

Q.E.D.

By the definition $R(X,Y) = [D_X, D_Y] - D_{[X,Y]}$ of the curvature tensor, $\psi' - \psi^2 = K$. Thus the equation obtained in Lemma 3.2 gives a second-order differential equation satisfied by μ. Lemma 3.3 implies that

LEMMA 3.4. *If we put* $y = \mu^2$, *then* y *satisfies*

$$\lambda^2(y')^2 = 32yh^2\{-5y^2 + 9\lambda^2 y - \lambda^2(c+\lambda^2)\}$$

along each integral curve of X.

Proof. Note that $K' = -6\mu\mu'$, $hh' = -\mu\mu'$ and $v' = -4\mu\mu'$. By Lemma 3.2 and a routine calculation,

$$(3.15)\qquad \psi' = \frac{\lambda^2}{4}\left\{\frac{h^2-2\mu^2}{\mu^2h^4}(\mu')^2 - \frac{1}{\mu h^2}\mu''\right\} - 2(K-\mu^2) + 16\mu^2$$
$$+ 2\mu(K-\mu^2)\frac{\mu''}{(\mu')^2}.$$

Substituting the equation given in Lemma 3.3 into (3.15), we have

$$(3.16)\qquad \psi' = \lambda^2 \frac{h^2 - 5\mu^2}{16h^4\mu^2}(\mu')^2 + (K - \mu^2)\frac{3\nu - \lambda^2}{2h^2} + \frac{4\mu^2}{(\mu')^2}(K - \mu^2)^2 - 2(K - \mu^2) + 16\mu^2.$$

On the other hand,

$$(3.17)\qquad \psi^2 = \frac{\lambda^4}{16\mu^2 h^4}(\mu')^2 + \frac{\lambda^2}{h^2}(K - \mu^2) + \frac{4\mu^2}{(\mu')^2}(K - \mu^2)^2.$$

It follows from (3.16) and (3.17) that

$$(3.18)\qquad K = -\frac{3\lambda^2}{8h^4}(\mu')^2 + 16\mu^2 - \frac{2h^2 + 3\mu^2}{h^2}(K - \mu^2).$$

Equation (3.18) can be rewritten as

$$(3.19)\qquad \lambda^2(\mu')^2 = 8h^2\{(\lambda^2 + 5h^2)\mu^2 - \lambda^2 K\}.$$

We multiply both sides of (3.19) by $4\mu^2$ and obtain the desired equation.

Q.E.D.

PROPOSITION 3.5. *Let f be a constant $\lambda(\neq 0)$-isotropic immersion of a connected complete surface M into a 5-dimensional simply connected space form $\bar{M}(c)$. If $U = \{x \in M : \dim N_x^1 = 3\} \neq \emptyset$, then $\lambda^2 > \max\{c/3, -c\}$, the universal Riemannian covering manifold $\hat{M}$ is isometric to $S^2((c + \lambda^2)/4)$ and $f \circ \pi \approx \tilde{\imath} \circ$ (second standard immersion: $\hat{M} \to S^4(3(c + \lambda^2)/4)$), where $\tilde{\imath}: S^4(3(c + \lambda^2)/4) \to \bar{M}(c)$ is an umbilical immersion* (cf. [10, p.28]).

Proof. Suppose that $U' = \{x \in M : (\operatorname{grad} K)(x) \neq 0\}$ is not empty. Then U' is contained in U. Consider the integral curve of $X = \operatorname{grad} K/\|\operatorname{grad} K\|$ passing through a point in U'. The image is contained in U' on a small interval I and we have equations obtained in Lemmas 3.3 and 3.4 on I. The equation in Lemma 3.3 is rewritten as

$$(3.20)\qquad y'' = \frac{7\lambda^2 - 10y}{8y(\lambda^2 - y)}(y')^2 + 4y(c + \lambda^2 - 4y),$$

where we have used $(\mu')^2 = (y')^2/(4y)$, $2\mu\mu'' = y'' - (y')^2/(2y)$. If we differentiate both sides of the equation in Lemma 3.4, then

$$(3.21)\qquad \lambda^2 y'' = 16\{20y^3 - 42\lambda^2 y^2 + 2\lambda^2(c + 10\lambda^2)y - \lambda^4(c + \lambda^2)\}.$$

Using the equation of Lemma 3.4 and (3.21), we eliminate y'' and $(y')^2$ from (3.20). Then

$$10y^3 - 13\lambda^2 y^2 + \lambda^2(2\lambda^2 - c)y + \lambda^4(c + \lambda^2) = 0,$$

which implies that y is constant on I. Thus K is constant on I. This is a contradiction. Therefore K is constant on M and hence, from Lemma 1.1, we conclude that $U = M$. Since (3.5) and (3.6) in the proof of Lemma 3.1 do not depend on the particular choice of orthonormal vector fields X and Y, we see that the second fundamental form is parallel. Isotropic submanifolds with their parallel second fundamental forms immersed in real space forms were classified in [10] (see also [4]). In particular, if such a submanifold with constant sectional curvature is not totally geodesic, umbilical and minimal, then it is Veronese submanifold contained in a totally umbilical submanifold of positive curvature. Let $\tilde{c}$ (>0) denote the sectional curvature of the totally umbilical submanifold N in which $f(M)$ is contained as a Veronese surface. By Lemma 1.1 and Proposition 2.4, we see that $K = \tau^2 = \tilde{c}/3$, where τ is the isotropy constant of the immersion $M \to N$. If ζ denotes the mean curvature normal vector of N in $\bar{M}(c)$, then $\tilde{c} = c + \|\zeta\|^2$, $\lambda^2 = \tau^2 + \|\zeta\|^2$ and so $\tilde{c} = 3(\lambda^2 + c)/4$. Q.E.D.

Finally, we state the main theorem. By virtue of Propositions 2.4 and 3.5, we have

THEOREM. *Let $f\colon M \to \bar{M}(c)$ be a constant λ-isotropic immersion of a connected, complete, simply connected surface into a 5-dimensional simply connected real space form of curvature c. Then it is one of the following*:

(A1)	$M = S^2(c/3)$,	$f \approx \iota \circ$ *(second standard immersion)*,
(A2)	$M = R^2$,	$f \approx T$,
(B)	$M = S^2((c + \lambda^2)/4)$,	$f \approx \tilde{\iota} \circ$ *(second standard immersion)*,
(C)	*totally umbilical immersion,*	
(D)	*totally geodesic immersion,*	

where ι (resp. $\tilde{\iota}$) denotes a totally geodesic immersion (resp. totally umbilical immersion $S^4(3(c + \lambda^2)/4) \to \bar{M}(c)$). We note that cases (A1) *and* (A2) *occur only when $\bar{M}(c) = S^5(c)$.*

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