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ISOMETRIC, HOLOMORPHIC AND SYMPLECTIC REFLECTIONS

ABSTRACT. As an extension of local geodesic symmetries we study here local reflections with respect to a topologically embedded submanifold P in a Riemannian manifold (M, g). First we derive a criterion for isometric reflections. Then we study holomorphic and symplectic reflections on an almost Hermitian manifold. In particular we focus on the influence of these reflections on the intrinsic and extrinsic geometry of the submanifold. Finally we treat these three kinds of reflections and their relationship when the ambient manifold is a locally Hermitian symmetric space. The results are derived by the use of Jacobi vector fields.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold and B a connected embedded submanifold. B, (M, g) and the embedding of B in (M, g) are assumed to be analytic and B is assumed to be relatively compact. Our aim is to study *local reflections* with respect to B. These local diffeomorphisms generalize reflections with respect to a linear subspace in Euclidean space E^n. When B is a point, we obtain the local geodesic symmetries. These symmetries have been studied extensively and Jacobi vector fields play an important role in this study. (See [14] for a short survey.) Jacobi vector fields, combined with Fermi coordinates, also provide a useful tool for the study of reflections with respect to a submanifold B when $\dim B > 0$. (In what follows we shall always restrict ourselves to this case.)

It is clear that geometric restrictions on a reflection will give restrictions on the geometry of the submanifold B and on the curvature of the ambient manifold (M, g). We shall study these restrictions in several special cases. In Section 3 we derive the necessary and sufficient conditions for an *isometric* reflection. The criterion becomes particularly simple and useful for locally symmetric spaces and leads to several examples. In Section 4 we study *holomorphic* and *symplectic* reflections on almost Hermitian manifolds. It turns out that in both cases B must be a holomorphic submanifold. Moreover, if the reflection is holomorphic and the ambient space a Kähler manifold, then B is also totally geodesic. This is not the case for symplectic reflections. Any reflection with respect to an arbitrary holomorphic submanifold in a Kähler manifold of constant holomorphic sectional curvature is symplectic. Such ambient spaces may be characterized using symplectic reflections with respect to holomorphic surfaces (Section 6).

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In Section 5 we derive a criterion for holomorphic and symplectic reflections in locally Hermitian symmetric spaces and we study the relationship between them and the isometric reflections.

We refer to [6] for some results about *global symmetries* and *global reflections* and to [7], [12] for some additional results concerning *harmonic* and *volume-preserving* reflections.

2. Preliminaries

We start with a brief description of the basic material we shall need. (See [9], [11] for further details.)

Let $m \in B$ and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field of (M, g) defined along B in a neighborhood of m. We specialize our moving frame such that $E_1, \ldots, E_p$ are tangent vector fields and $E_{p+1}, \ldots, E_n$ are normal vector fields of the submanifold B of M, where $p = \dim B$ and $n = \dim M$. Let $(y_1, \ldots, y_p)$ be a system of coordinates in a neighborhood of m in B such that $(\partial/\partial y_i)(m) = E_i(m)$, $i = 1, \ldots, p$, and let $(x_1, \ldots, x_n)$ be a system of Fermi coordinates with respect to m, $(y_1, \ldots, y_p)$ and $\{E_{p+1}, \ldots, E_n\}$. These coordinates are defined in an open neighborhood U_m of m in M. This means that for the normal bundle $\nu = T^\perp B$ of B we have

$$x_i\left(\exp_\nu\left(\sum_{p+1}^{n} t_\beta E_\beta\right)\right) = y_i, \quad i = 1, \ldots, p,$$

$$x_\alpha\left(\exp_\nu\left(\sum_{\beta=p+1}^{n} t_\beta E_\beta\right)\right) = t_\alpha, \quad \alpha = p+1, \ldots, n.$$

Choose a fixed normal unit vector u at m, $u \in T_m^\perp B \subset T_m M$, and consider the geodesic $\gamma(t) = \exp_m(tu)$. We have

$$\gamma(0) = m, \quad \gamma'(0) = u.$$

We specialize the frame field $\{E_1, \ldots, E_n\}$ in such a way that

$$E_n(m) = u = \gamma'(0).$$

Next, consider the frame field $\{e_1(t), \ldots, e_n(t)\}$ along $\gamma(t)$ obtained by parallel transport of $\{E_1(m), \ldots, E_n(m)\}$. Further, let Y_i, Y_a, $i = 1, \ldots, p$, $a = p+1, \ldots, n-1$, denote the Jacobi vector fields along γ with initial conditions

$$Y_i(0) = E_i(m), \qquad Y_i'(0) = \nabla_u \frac{\partial}{\partial x_i},$$

$$Y_a(0) = 0, \qquad Y_a'(0) = E_a(m),$$

where ∇ denotes the Riemannian connection of (M, g). Note that

$$(2.1)\qquad Y_i(t) = \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}, \qquad Y_a(t) = t\left.\frac{\partial}{\partial x_a}\right|_{\gamma(t)}.$$

Define the endomorphism-valued function $t \mapsto D_u(t)$ by

$$(2.2)\qquad Y_\alpha(t) = D_u(t)e_\alpha, \quad \alpha = 1, \dots, n-1.$$

Then the Jacobi equation implies

$$(2.3)\qquad D_u'' + R \circ D_u = 0,$$

where $t \mapsto R(t)$ is the endomorphism-valued function on $(\gamma'(t))^\perp \subset T_{\gamma(t)}M$ defined by

$$R(t)x = R_{\gamma'(t)x}\gamma'(t), \quad x \in (\gamma'(t))^\perp.$$

R denotes the Riemann curvature tensor on (M, g) defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all tangent vectors X, Y of M.

To write down the initial conditions for $D_u(t)$, where u is fixed, we need some facts about submanifolds. Denote by $\tilde{\nabla}$ the Riemannian connection of B. Further, let X, Y be tangent vector fields and N a unit normal vector field along an open domain in B. Then we have the orthogonal decompositions

$$\nabla_X Y = \tilde{\nabla}_X Y + T_X Y, \qquad \nabla_X N = T(N)X + \nabla_X^\perp N,$$

where $T_X Y = T(X, Y)$ is the second fundamental form operator of B, $T(N)$ the shape operator of B corresponding to the normal vector N, and $\nabla^\perp$ is the normal connection along B. Note that

$$g(T(N)X, Y) = -g(T(X, Y), N).$$

Also, we shall use the operator $\perp$ defined by [9], [11]

$$\perp_X N = \nabla_X^\perp N.$$

Now, using the initial conditions for Y_α, we easily obtain the following initial conditions (in matrix form with respect to the basis $\{E_1, \dots, E_{n-1}\}_m$ of $(u)^\perp \subset T_m M$):

$$(2.4)\qquad D_u(0) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_u'(0) = \begin{bmatrix} T(u) & 0 \\ -{}^t\!\perp(u) & I \end{bmatrix},$$

where

$$T(u)_{ij} = g(T(u)E_i, E_j)(m),$$

$$\perp(u)_{ia} = g(\perp_{E_i} E_a, E_n)(m).$$

In what follows we consider the local diffeomorphism

$$\varphi_B\colon p \to \varphi_B(p), \qquad \exp_m(tu) \mapsto \exp_m(-tu)$$

for $u \in T_m^\perp B$, $\|u\| = 1$. φ_B is called *the reflection with respect to the submanifold* B. Using Fermi coordinates, φ_B is locally given by

$$\varphi_B\colon (x_1, \ldots, x_p, x_{p+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_p, -x_{p+1}, \ldots, -x_n).$$

3. Isometric reflections

We start by determining the necessary and sufficient conditions for an *isometric reflection* φ_B.

THEOREM 1. *Let (M, g) be a Riemannian manifold and B a submanifold. Then the reflection φ_B is a local isometry if and only if*

(i) *B is totally geodesic;*

(ii) *$(\nabla^{2k}_{u\ldots u}R)_{uv}u$ is normal to B,*
$(\nabla^{2k+1}_{u\ldots u}R)_{uv}u$ is tangent to B and
$(\nabla^{2k+1}_{u\ldots u}R)_{ux}u$ is normal to B

for all normal vectors u, v of B, any tangent vector x of B and all $k \in \mathbb{N}$.

Proof. First, we suppose that φ_B is an isometry. Then, since B belongs to the fixed point set of φ_B, B is totally geodesic [10]. Further, φ_B preserves R and its covariant derivatives. Hence we have

$$(\nabla^p_{\varphi_* X,\ldots,\varphi_* X}R)_{\varphi_* Y \varphi_* Z \varphi_* U \varphi_* W} = (\nabla^p_{X\ldots X}R)_{YZUW}.$$

Now, let $u, v, w \in T_m^\perp B$, $x, y \in T_m B$. Then

$$\varphi_* u = -u, \quad \varphi_* v = -v, \quad \varphi_* w = -w, \quad \varphi_* x = x, \quad \varphi_* y = y$$

and hence

$$(\nabla^p_{\varphi_* u\ldots\varphi_* u}R)_{\varphi_* u \varphi_* v \varphi_* u \varphi_* x} = (-1)^{p+3}(\nabla^p_{u\ldots u}R)_{uvux},$$
$$(\nabla^p_{\varphi_* u\ldots\varphi_* u}R)_{\varphi_* u \varphi_* v \varphi_* u \varphi_* w} = (-1)^{p+4}(\nabla^p_{u\ldots u}R)_{uvuw},$$
$$(\nabla^p_{\varphi_* u\ldots\varphi_* u}R)_{\varphi_* u \varphi_* x \varphi_* u \varphi_* y} = (-1)^{p+2}(\nabla^p_{u\ldots u}R)_{uxuy}.$$

This implies the result.

Next, we prove the converse. Let $p = \exp_m(tu)$, $\|u\| = 1$, $u \in T_m^\perp B$. Then we have, using (2.1), (2.2),

$$(3.1)\qquad \begin{aligned} g_{ij}(p) &= g(D_u(t)e_i, D_u(t)e_j),\\ g_{ia}(p) &= g\left(D_u(t)e_i, \frac{1}{t}D_u(t)e_a\right),\\ g_{ab}(p) &= g\left(\frac{1}{t}D_u(t)e_a, \frac{1}{t}D_u(t)e_b\right),\end{aligned}$$

for $i, j = 1, \ldots, p$ and $a, b = p+1, \ldots, n-1$. Since B is totally geodesic we have from (2.4)

$$(3.2)\qquad D_u(0) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_u'(0) = \begin{bmatrix} 0 & 0 \\ -{}^t\perp & I \end{bmatrix}$$

and so, (2.3) and (3.2) yield

$$(3.3)\qquad D_u^{(l+2)}(0) = -\sum_{k=0}^{l} C_l^k R^{(l-k)}(0) D_u^{(k)}(0).$$

Further, conditions (ii), together with (3.2), (3.3), imply

$$D_u^{(2l)}(0)v \text{ is tangent}, \qquad D_u^{(2l)}(0)x \text{ is tangent},$$

$$D_u^{(2l+1)}(0)v \text{ is normal}, \quad D^{(2l+1)}(0)x \text{ is normal},$$

for $v \in T_m^\perp B$, $x \in T_m B$. Hence we have

$$D_u(t)e_i = \alpha_i(t) + \beta_i(t),$$

$$\frac{1}{t}D_u(t)e_a = \alpha_a(t) + \beta_a(t),$$

where α_i, α_a are tangent and β_i, β_a normal along B. Moreover, α_i, β_a are even functions of t and β_i, α_a are odd functions of t. Hence

$$\begin{aligned} g_{ij}(p) &= g(\alpha_i(t), \alpha_j(t)) + g(\beta_i(t), \beta_j(t)),\\ g_{ia}(p) &= g(\alpha_i(t), \alpha_a(t)) + g(\beta_i(t), \beta_a(t)),\\ g_{ab}(p) &= g(\alpha_a(t), \alpha_b(t)) + g(\beta_a(t), \beta_b(t))\end{aligned}$$

and so we get

$$g_{ij}(\varphi_B(p)) = g_{ij}(p), \quad g_{ia}(\varphi_B(p)) = -g_{ia}(p), \quad g_{ab}(\varphi_B(p)) = g_{ab}(p)$$

which proves the desired result.

This criterion becomes much simpler for locally symmetric spaces. We have

COROLLARY 2. *Let (M, g) be a locally symmetric Riemannian manifold and B a submanifold. Then the reflection φ_B is an isometry if and only if*

(i) *B is totally geodesic;*
(ii) *$R_{uv}u$ is normal to B for all $u, v \in T^{\perp}B$.*

Next, we give a useful geometric interpretation for conditions (ii) in Corollary 2.

THEOREM 3. *Let B be a totally geodesic submanifold in a locally symmetric space (M, g). Then $R_{uv}u$ is normal to B for each $u, v \in T^{\perp}B$ if and only if through each $m \in B$ there exists a totally geodesic submanifold $\bar{B}$ such that $T_m\bar{B} = T_m^{\perp}B$.*

Proof. First, suppose that such a $\bar{B}$ exists for each $m \in B$. Then the Codazzi equation [1] implies at once that

$$R_{uv}u \in T_m\bar{B} = T_m^{\perp}B$$

for all normal vectors u, v.

To prove the converse we first note that Corollary 2 yields that the reflection φ_B is an isometry. Since (M, g) is locally symmetric, the geodesic symmetry s_m centered at $m \in B$ is an isometry and hence $s_m \circ \varphi_B$ is also an isometry. Moreover, m is a fixed point of $s_m \circ \varphi_B$. Further, at m we have

$$(s_m \circ \varphi_B)_* = s_{m*} \circ \varphi_{B*} = -\varphi_{B*}. \tag{3.4}$$

Now, let $\bar{B}$ be the connected component of the fixed point set of $s_m \circ \varphi_B$ through m. Then $\bar{B}$ is a totally geodesic submanifold of M through m. Moreover (3.4) yields that $T_m\bar{B} = T_m^{\perp}B$.

Finally we treat some applications of Corollary 2. First, let (M, g) be a space of constant curvature c. Then we have

$$R_{XYZW} = c\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}$$

and hence

$$R_{uv}u = c\{g(u, u)v - g(u, v)u\}. \tag{3.5}$$

For a Kähler manifold of constant holomorphic sectional curvature c we have

$$\begin{aligned} R_{XYZW} = \frac{c}{4}\{&g(X, Z)g(Y, W) - g(Y, Z)g(X, W) \\ &+ 2g(JX, Y)g(JZ, W) + g(JX, Z)g(JY, W) \\ &- g(JX, Z)g(JX, W)\} \end{aligned}$$

and so

$$R_{uv}u = \frac{c}{4}\{g(u, u)v - g(u, v)u + 3g(Ju, v)Ju\}. \tag{3.6}$$

Further, we note that a totally geodesic submanifold in a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$ is either a holomorphic or a totally real submanifold [2]. This remark, (3.4) and (3.5) prove at once

COROLLARY 4. (a) *Let (M, g) be a space of constant curvature. Then φ_B is an isometry if and only if B is totally geodesic.*

(b) *Let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$. Then φ_B is an isometry if and only if either B is a holomorphic totally geodesic submanifold or a totally real totally geodesic submanifold of dimension $\frac{1}{2}\dim M$.*

Further, we have the following result already proved in [16]:

THEOREM 5. *Let (M, g) be a Riemannian manifold. Then (M, g) is a space of constant curvature if and only if the reflections with respect to all geodesics are isometries.*

Proof. Let (M, g) be a space of constant curvature and σ an arbitrary geodesic. Corollary 4 implies that φ_σ is an isometry.

Conversely, let φ_σ be the reflection with respect to a geodesic σ and suppose that φ_σ is an isometry for all σ. Then, from Theorem 1, we have that $R_{uv}u$ is normal to σ and

$$(3.7) \qquad \nabla_u R_{uvuv} = 0$$

for all $u, v \in T^\perp\sigma$. Since this must hold for all geodesics, (3.7) yields

$$\nabla_x R_{xyxy} = 0$$

for all $x, y \in T_m M$ and all $m \in M$. Hence (M, g) is locally symmetric [8], [16]. This proves the result for $\dim M = 2$. For $\dim M > 2$ we use

$$R_{uvux} = 0$$

for all $u, v \in T^\perp\sigma$ and $x \in T\sigma$. Hence

$$R_{xyxz} = 0$$

for any orthogonal triple of tangent vectors x, y, z of M. Now the result follows from Cartan's characterization of spaces of constant curvature [1].

REMARK. The situation of Theorem 3 occurs in many examples. We mention the M_+ and M_- submanifolds in compact symmetric spaces as introduced by *T.* Nagano and the first author in their study about geodesic submanifolds and related problems [3], [4], [5]. Our theory implies that any reflection with respect to an M_+ or an M_- submanifold is a (global) isometry. But this follows also easily from the theory developed in these papers. For further results concerning global symmetries and reflections we refer to [6].

4. Holomorphic and symplectic reflections on almost Hermitian manifolds

Let (M, g, J) be an almost Hermitian manifold and B an arbitrary submanifold. The reflection φ_B with respect to B is said to be *holomorphic* if

$$\varphi_{B*} \circ J = J \circ \varphi_{B*} \tag{4.1}$$

and φ_B is said to be *symplectic* if

$$\varphi_B^* \Omega = \Omega \tag{4.2}$$

where Ω denotes the Kähler form on (M, g, J) determined by $\Omega(X, Y) = g(X, JY)$, for all tangent vector fields X, Y of M. We have

THEOREM 6. *If the reflection φ_B is holomorphic, then B is a holomorphic submanifold.*

Proof. Let X be a tangent vector field of B. Then (4.1) implies

$$\varphi_{B*} JX = J\varphi_{B*} X = JX.$$

Hence, JX is also tangent.

THEOREM 7. *If the reflection φ_B is symplectic, then B is a holomorphic submanifold which is minimal.*

Proof. Let X, Y be vector fields on M along B. Then (4.2) implies

$$g(\varphi_{B*} X, J\varphi_{B*} Y) = g(X, JY). \tag{4.3}$$

First, let X be normal to B. Then $\varphi_{B*} X = -X$ and so (4.3) yields

$$J\varphi_{B*} Y + JY \text{ is tangent to } B.$$

So, for Y tangent, we get

$$J\varphi_{B*} Y + JY = 2JY$$

and hence, JY is also tangent.

Moreover, since φ_B is symplectic, φ_{B*} preserves the volume and so B is a minimal submanifold [12], [15] (see also Theorem 9 for a proof).

COROLLARY 8. *If φ_B is holomorphic and symplectic, then B is a totally geodesic holomorphic submanifold.*

Proof. The result follows from Theorem 6 or Theorem 7 and the fact that φ_B is an isometry.

The theorems already indicate that the conditions 'φ_B is holomorphic' and 'φ_B is symplectic' must have a strong influence on the extrinsic geometry of the submanifold B. We now determine some of the consequences on the second fundamental form.

THEOREM 9. *Let (M, g, J) be an almost Hermitian manifold and B a submanifold such that the reflection φ_B is symplectic. Then B is a holomorphic submanifold and the second fundamental form operator satisfies*

$$T(X, Y) + T(JX, JY) = 0$$

for all tangent X, Y to B. Moreover, $(\nabla_u J)X$ is normal to B for all $u \in T^\perp B$, $X \in TB$.

Proof. From (4.2) we get

$$\Omega_{ij}(\varphi_B(p)) = \Omega_{ij}(p), \tag{4.4}$$

where $\Omega_{ij} = \Omega(\partial/\partial x_i, \partial/\partial x_j)$, $i, j = 1, \ldots, p$. Now we use the technique developed in Section 2 and state an expression for $\Omega_{ij}(p)$ using the endomorphism $D_u(t)$. Let $p = \exp_m(tu)$, $u \in T_m^\perp B$, $\|u\| = 1$. Then

$$\Omega_{ij}(p) = g\left(\frac{\partial}{\partial x_i}, J\frac{\partial}{\partial x_j}\right) = g(D_u(t)e_i, JD_u(t)e_j).$$

Using the initial conditions for $D_u(t)$ we have

$$D_u(t)e_i = E_i(m) + t(TE_i - {}^t\perp E_i)(m) + O(t^2).$$

Note that ${}^t\perp E_i(m) \in T_m^\perp B$. Hence we have

$$\Omega_{ij}(p) = g(E_i, JE_j)(m) + t\{g(TE_i, JE_j) + g(E_i, JTE_j) + g(E_i, J'E_j)\}(m) + O(t^2). \tag{4.5}$$

So (4.4) and (4.5) yield

$$g(TE_i, JE_j) + g(E_i, JTE_j) = -g(E_i, J'E_j)$$

or, equivalently,

$$g(u, T(E_i, JE_j) - T(JE_i, E_j)) = -g(E_i, J'E_j).$$

Since B is holomorphic (Theorem 6), this is also equivalent to

$$g(u, T(JE_i, JE_j) + T(E_i, E_j)) = -g(JE_i, J'E_j).$$

So we have

$$g(u, T(JX, JY) + T(X, Y)) = -g(JX, J'Y)$$

for all tangent vectors X, Y. Note that the left-hand side is symmetric in X and Y. But, as is easily seen from $g(JX, JY) + g(X, Y) = 0$, the right-hand side is skew-symmetric. This yields the desired result.

REMARK. The minimality in Theorem 7 follows at once from $T(X, Y) + T(JX, JY) = 0$.

THEOREM 10. *Let (M, g, J) be an almost Hermitian manifold and B a submanifold such that the reflection φ_B is holomorphic. Then B is a holomorphic submanifold and the second fundamental form operator satisfies*

$$T(JX, JY) - T(X, Y) = 0 \tag{4.6}$$

for all X, Y tangent to B. Moreover, $(\nabla_u J)X$ is normal to B for all $u \in T^{\perp}B$, $X \in TB$.

Proof. First we note that

$$-J_\alpha^\beta = \Omega_{\alpha\gamma} g^{\gamma\beta}, \quad \alpha, \beta = 1, \ldots, n.$$

Hence

$$-J_i^j = \Omega_{ik} g^{kj} + \Omega_{ia} g^{aj} \tag{4.7}$$

where $i, j, k = 1, \ldots, p$; $a = p + 1, \ldots, n - 1$, since $g^{nj} = 0$.

Now, once again we use power series expansions. Note that

$$\Omega_{ia}(p) = g(D_u(t)e_i, \frac{1}{t} D_u(t)e_a) = g(E_i, JE_a)(m) + O(t)$$

and since B is holomorphic,

$$\Omega_{ia}(p) = O(t). \tag{4.8}$$

Further, we easily get

$$\begin{aligned} g_{ij}(p) &= g(E_i, E_j)(m) + 2tg(TE_i, E_j)(m) + O(t^2), \\ g_{ia}(p) &= O(t), \\ g_{ab}(p) &= g(E_a, E_b)(m) + O(t) \end{aligned} \tag{4.9}$$

and hence,

$$\begin{aligned} g^{ij}(p) &= g(E_i, E_j)(m) - 2tg(TE_i, E_j)(m) + O(t^2), \\ g^{ia}(p) &= O(t), \\ g^{ab}(p) &= g(E_a, E_b)(m) + O(t). \end{aligned} \tag{4.10}$$

Using (4.5), (4.7), (4.8), (4.9) and the fact that B is holomorphic, we get

$$\begin{aligned} -J_i^j(p) = g(E_i, JE_j)(m) + t\{&g(E_i, J'E_j) + g(TE_i, JE_j) \\ &+ g(E_i, JTE_j) + 2g(TJE_i, E_j)\}(m) + O(t^2). \end{aligned}$$

Hence, when φ_B is holomorphic, one must have

$$g(E_i, J'E_i) + g(TE_i, JE_j) + g(TJE_j, E_j) = 0.$$

Proceeding in the same way as in Theorem 9, we get the desired result.

Note that the second author proved in [13] that if (M, g, J) is a quasi-Kähler manifold, i.e.

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0,$$

then

$$T(X, Y) + T(JX, JY) = 0$$

for a holomorphic submanifold. Hence, Theorem 10 yields

THEOREM 11. *Let (M, g, J) be a quasi-Kähler manifold and let B be a submanifold such that φ_B is holomorphic. Then B is a totally geodesic holomorphic submanifold.*

COROLLARY 12. *Let (M, g, J) be a Kähler manifold and B a submanifold such that φ_B is holomorphic. Then B is a totally geodesic holomorphic submanifold.*

5. Holomorphic and symplectic reflections on locally Hermitian symmetric spaces

In this section we concentrate on Kähler manifolds which are in addition locally symmetric. For this case we have more complete results. The fundamental reason for this is that we can write a complete solution of the Jacobi equation (2.3) with initial conditions (2.4).

Indeed, let (M, g) be a locally symmetric space. Then it is easy to see that

$$D_u(t) = (\cos t\sqrt{R_m})\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}\begin{bmatrix} T & 0 \\ -{}^t\bot & I \end{bmatrix}, \tag{5.1}$$

where $R_m = R(0)$. From this we get

$$D_u(t)e_i = (\cos t\sqrt{R_m})E_i(m) + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_i - {}^t\bot E_i)(m), \tag{5.2}$$

$$D_u(t)e_a = \frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m),$$

for $i = 1, \ldots, p$, $a = p + 1, \ldots, n - 1$. Hence we obtain the following useful expressions:

$$g_{ij}(p) = g\Bigg((\cos t\sqrt{R_m})E_i(m) + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_i - {}^t\bot E_i), \tag{5.3}$$

$$(\cos t\sqrt{R_m})E_j(m) + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_j - {}^t\bot E_j)(m)\Big),$$

$$g_{ia}(p) = g\Big((\cos t\sqrt{R_m})E_i(m) + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_i - {}^t\bot E_i)(m),$$

$$\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m)\Big),$$

$$g_{ab}(p) = g\left(\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m), \frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_b(m)\right).$$

Next, let (M, g, J) be a locally symmetric Kähler manifold. Then we have from (5.2), since J is parallel,

$$\Omega_{ij}(p) = g\Big((\cos t\sqrt{R_m})E_i(m) + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_i - {}^t\bot E_i)(m), \tag{5.4}$$

$$J(\cos t\sqrt{R_m})E_j(m) + J\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_j - {}^t\bot E_j)(m)\Big),$$

$$\Omega_{ia}(p) = g\Big((\cos t\sqrt{R_m})E_i(m) + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_i - {}^t\bot E_i)(m),$$

$$J\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m)\Big),$$

$$\Omega_{ab}(p) = g\left(\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m), J\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_b(m)\right),$$

$$\Omega_{in}(p) = g\left((\cos t\sqrt{R_m})E_i(m) + \frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_i - {}^t\bot E_i)(m), Ju\right),$$

$$\Omega_{an}(p) = g\left(\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m), Ju\right).$$

Now we prove

THEOREM 13. *Let (M, g, J) be a locally Hermitian symmetric space and B a submanifold. Then φ_B is symplectic if and only if*

(i) *B is holomorphic;*
(ii) *$R_{uv}u$ is normal to B for all $u, v \in T^\bot B$;*
(iii) *$RTJ + JRT = 0$ and $RTRJ + JRTR = 0$ on tangent vectors.*

Proof. We start with the condition

$$\Omega_{in}(\varphi_B(p)) = -\Omega_{in}(p).$$

From (5.4) we see that this is equivalent to

$$g((\cos t\sqrt{R_m})E_i(m), Ju) = 0$$

for all sufficiently small t. So, the first condition we get is

(5.5) $$R_{uJuux} = 0$$

for all normal u and tangent x. We linearize (5.5) and use the first Bianchi identity and the Kähler identity $R_{xyJzJw} = R_{xyzw}$ to get

(5.6) $$3R_{uJvux} - R_{uvuJx} = 0,$$

or replacing x by Jx:

(5.7) $$3R_{uJvuJx} + R_{uvux} = 0.$$

Replace v by Jv and x by Jx in (5.7) to get

(5.8) $$3R_{uvux} + R_{uJvuJx} = 0.$$

So, (5.7) and (5.8) yield

$$R_{uvux} = 0$$

or equivalently, $R_{uv}u$ is normal to B along B.

Using this, the expressions (5.4) become

(5.9) $$\Omega_{ij}(p) = g\left((\cos t\sqrt{R_m})E_i(m), J\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_j)(m)\right)$$
$$+g\left(\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}(TE_i)(m), J(\cos t\sqrt{R_m})E_j(m)\right)$$
$$+g((\cos t\sqrt{R_m})E_i(m), J(\cos t\sqrt{R_m})E_j(m))$$
$$+g\left(\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}({}^t\bot E_i)(m), J\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}({}^t\bot E_j)(m)\right),$$

$$\Omega_{ia}(p) = -g\left(\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}({}^t\bot E_i)(m), J\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m)\right),$$

$$\Omega_{ab}(p) = g\left(\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}E_a(m), J\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}E_b(m)\right),$$

$$\Omega_{in}(p) = -g\left(\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}\,({}^t\bot E_i)(m),\ Ju\right),$$

$$\Omega_{an}(p) = g\left(\frac{\sin t\sqrt{R_m}}{t\sqrt{R_m}}\,E_a(m), Ju\right).$$

From this we have

$$\Omega_{ia}(\varphi_B(p)) = -\Omega_{ia}(p), \quad \Omega_{ab}(\varphi_B(p)) = \Omega_{ab}(p),$$
$$\Omega_{in}(\varphi_B(p)) = -\Omega_{in}(p), \quad \Omega_{an}(\varphi_B(p)) = \Omega_{an}(p).$$

Hence, the remaining condition is

$$\Omega_{ij}(\varphi_B(p)) = \Omega_{ij}(p)$$

and this is equivalent to

$$\text{(5.10)} \qquad g\left((\cos t\sqrt{R_m})E_i(m), J\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}\,(TE_j)(m)\right)$$
$$+ g\left(\frac{\sin t\sqrt{R_m}}{\sqrt{R_m}}\,(TE_i)(m), J(\cos t\sqrt{R_m})E_j(m)\right) = 0,$$

for all sufficiently small t.

The first condition derived from (5.10) is $TJ + JT = 0$. This is always satisfied since (M, g, J) is Kählerian and B holomorphic. The next condition turns out to be

$$3R_{uJTxuy} - 3R_{uxuJTy} + R_{uJxuTy} - R_{uTxuJy} = 0$$

for all tangent vectors x, y. Now replace x by Jx:

$$\text{(5.11)} \qquad 3R_{uTxuy} + 3R_{uJxuTJy} - R_{uxuTy} - R_{uTJxuJy} = 0.$$

Next, we replace x by Jx and y by Jy in (5.11):

$$\text{(5.12)} \qquad 3R_{uTJxuJy} + 3R_{uxuTy} - R_{uJxuTJy} - R_{uTxuy} = 0.$$

So, from (5.11) and (5.12) we get

$$R_{uTJxuJy} + R_{uxuTy} = 0$$

or equivalently,

$$\text{(5.13)} \qquad RTJ + JRT = 0$$

on tangent vectors.

Using this, an easy calculation shows that the next condition becomes

(5.14) $RTRJ + JRTR = 0$

on tangent vectors.

Finally, (5.13) and (5.14) imply

(5.15) $R^kTR^lJ + JR^lTR^k = 0$

for all $k, l, \in \mathbb{N}$. An easy calculation then shows from (5.15) that (5.10) is satisfied completely. This completes the proof.

Now, we derive some corollaries from Theorem 13. First we have

COROLLARY 14. *Let (M, g, J) be a locally Hermitian symmetric space and B a totally geodesic submanifold. Then the reflection φ_B is symplectic if and only if*

(i) *B is holomorphic;*
(ii) *$R_{uv}u$ is normal to B for all u, $v \in T^\perp B$.*

Using Corollary 2 we get

COROLLARY 15. *Let (M, g, J) be a locally Hermitian symmetric space and B a totally geodesic holomorphic submanifold. Then the reflection φ_B is symplectic if and only if it is an isometry.*

Moreover, we have

COROLLARY 16. *Let (M, g, J) be a locally Hermitian symmetric space and B a holomorphic submanifold. If the reflection φ_B is an isometry, then it is symplectic.*

It is easy to see that the converse does not hold since we have

THEOREM 17. *Let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature. Then a reflection with respect to an arbitrary holomorphic submanifold is always symplectic.*

Proof. From the expression for R (see Section 3) we get

$$R_{ux}u = \frac{c}{4}x$$

for all tangent vectors x. The result follows now at once from Theorem 13.

To finish this section we derive a criterion for holomorphic reflections φ_B on a locally Hermitian symmetric space. Recall that B must necessarily be a totally geodesic holomorphic submanifold. We have

THEOREM 18. *Let (M, g, J) be a locally Hermitian symmetric space. Then the reflection φ_B with respect to the submanifold B is holomorphic if and only if B is a totally geodesic holomorphic submanifold such that $R_{uv}u$ is normal for all $u, v \in T^{\perp}B$.*

Proof. We express that

$$J_i^a(\varphi_B(p)) = -J_i^a(p), \quad i = 1, \ldots, p, \quad a = p+1, \ldots, n-1.$$

Here

$$-J_i^a(p) = (\Omega_{ik} g^{ka} + \Omega_{ib} g^{ba})(p).$$

We use again power series expansions to obtain

$$\begin{aligned}
(5.16) \qquad & g_{ij}(p) = g(E_i, E_j)(m) + O(t^2), \\
& g_{ia}(p) = -tg({}^t\bot E_i, E_a)(m) - \tfrac{2}{3}t^2 g(RE_i, E_a)(m) + O(t^3), \\
& g_{ab}(p) = g(E_a, E_b)(m) - \tfrac{1}{3}t^2 g(RE_a, E_b)(m) + O(t^3);
\end{aligned}$$

$$\begin{aligned}
(5.17) \qquad & g^{ij}(p) = g(E_i, E_j)(m) + O(t^2), \\
& g^{ia}(p) = tg({}^t\bot E_i, E_a)(m) + \tfrac{2}{3}t^2 g(RE_i, E_a)(m) + O(t^3), \\
& g^{ab}(p) = g(E_a, E_b)(m) + O(t^2).
\end{aligned}$$

Further, we have

$$\begin{aligned}
(5.18) \qquad & \Omega_{ij}(p) = g(E_i, JE_j)(m) + O(t^2), \\
& \Omega_{ia}(p) = -tg({}^t\bot E_i, JE_a)(m) + \tfrac{1}{6}t^2 g((RJ + 3JR)E_i, E_a)(m) + O(t^3).
\end{aligned}$$

Hence, (5.17) and (5.18) yield

$$\begin{aligned}
-J_i^a(p) = t\left\{ \sum_{k=1}^{p} g(E_i, JE_k)(m) g({}^t\bot E_k, E_a)(m) - g({}^t\bot E_i, E_a)(m) \right\} \\
+ \tfrac{1}{2}t^2 g((JR - RJ)E_i, E_a)(m) + O(t^3).
\end{aligned}$$

So, when φ_B is holomorphic, we must have

$$g((JR - RJ)E_i, E_a) = 0$$

and hence

$$R_{uJuux} = 0$$

along B. As before this implies $R_{uvux} = 0$ for all $u, v \in T^{\perp}B$, $x \in TB$.

The converse follows easily from the fact that the conditions imply that φ_B is isometric and symplectic (Theorem 13 and Corollary 2).

From this and Corollary 2 we get

THEOREM 19. *Let (M, g, J) be a locally Hermitian symmetric space and B a holomorphic submanifold. Then the reflection is holomorphic if and only if it is an isometry.*

From this and Corollary 15 we derive

COROLLARY 20. *Let (M, g, J) be a locally Hermitian symmetric space and B a totally geodesic holomorphic submanifold. Then the following statements are equivalent:*

(i) φ_B *is an isometry;*
(ii) φ_B *is holomorphic;*
(iii) φ_B *is symplectic.*

REMARK. Corollary 20 implies that the reflections with respect to an M_+ and M_- submanifold in a compact Hermitian symmetric space are holomorphic and symplectic. This may also be proved easily using the theory developed in [3], [4], [5].

6. SYMPLECTIC REFLECTIONS AND KÄHLER MANIFOLDS OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

The main purpose of this final section is to give a characterization of Kähler manifolds of constant holomorphic sectional curvature by using symplectic reflections.

First we derive from Theorem 13:

THEOREM 21. *Let (M, g, J) be a locally Hermitian symmetric space and B a submanifold such that φ_B is symplectic. Then there exists for each $m \in B$ a totally geodesic submanifold $\bar{B}_m$ through m such that $T_m\bar{B}_m = T_m^\perp B$.*

Proof. From Theorem 13 we get

$$R_{uvux} = 0$$

for all u, v $\in T^\perp B$ and $x \in TB$. Hence

$$R_{uvwx} + R_{wvux} = 0.$$

Using the first Bianchi identity, this yields

$$(6.1) \qquad 2R_{uvwx} - R_{uwvx} = 0.$$

Hence we also have

$$(6.2) \qquad 2R_{uwvx} - R_{uvwx} = 0.$$

So, (6.1) and (6.2) imply

(6.3) $R_{uvwx} = 0.$

Now the theorem follows easily from (6.3) by using Lie triple systems [10].

We use this result to prove

THEOREM 22. *Let (M, g, J) be a locally Hermitian symmetric space. Then (M, g, J) is a space of constant holomorphic sectional curvature if and only if the reflection with respect to any holomorphic surface is symplectic.*

Proof. First, let (M, g, J) be a space of constant holomorphic sectional curvature. Then the result follows from Theorem 17.

Conversely, let $m \in M$ and let $u \in T_m M$. There always exists a holomorphic surface B tangent to the subspace spanned by $\{u, Ju\}$. Since φ_B is symplectic, Theorem 21 implies that there exists through m a totally geodesic holomorphic hypersurface tangent to $T_m^\perp B$. Since u is arbitrary, this implies that for all m and any holomorphic $(n-2)$-plane through m, there exists a totally geodesic holomorphic submanifold tangent to the $(n-2)$-plane. Hence the axiom of holomorphic $(n-2)$-planes is satisfied and so (M, g, J) is a space of constant holomorphic sectional curvature [17].

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