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CREATING CONSTRUCTIVIST ENVIRONMENTS AND CONSTRUCTING CREATIVE MATHEMATICS

ABSTRACT. Constructivism is not reducible to a set of rules to follow or actions to perform. However, we suggest that it is possible to define beliefs that must be held by teachers if they are to create constructivist environments for learning. In the first part of this paper we put forward as critical four tenets of belief and follow this with descriptions and analyses of classrooms in which the teachers' intentions are to create environments based on these beliefs. We examine the mathematical understanding actions of pupils in these classrooms to determine the relevance and validity of our claims.

INTRODUCTION

Over the last decade "constructivism" has become the watchword for good teaching with many teachers, researchers, and mathematics educators. As can so often happen when an idea is popularized, the original notion is in danger of being distorted and deprived of its power by users who wish to be seen to be doing the "right thing" in their teaching or research. Whereas the term "constructivism" was intended to convey the Piagetian notion of the nature of building cognitive structures, for many people it has come to mean a way of teaching embodied by one of two classroom methods. For some, having students use manipulatives is the necessary and sufficient condition for "doing constructivist teaching". They see children acting on objects and by this means constructing for themselves given mathematical ideas. Such activity is seen as leading all children to construct the mathematics which the manipulatives are intended to model. As Steffe and von Glasersfeld (1988) and others have shown, however, even young children use their own created mental objects, some of which do have a figural quality but others of which are abstract, to build their mathematical knowledge. Thus, while people's mathematical understanding depends on their experience, the objects of this experience and the instruction do not need to be physical.

The other quality of the instructional environment often thought to be *per se* "constructivist" is the use of group discussion. While it is true that students' verbal interacting with one another about mathematics may be useful to their learning, there is much about mathematics which is most powerfully represented, even for and by young learners, symbolically or diagramatically. Since the sphere of possible mathematical actions can and, indeed, for the growth of mathematical understanding, eventually must include working on mental objects, mathematical knowledge and understanding can certainly be built with-

Educational Studies in Mathematics **23**: 505–528, 1992.

out any such verbal interaction.

The beliefs outlined above are, furthermore, indicative of a more general, underlying problem associated with the term "constructivist teaching". Many educationalists would like to have a list of specific behaviors that they could perform or that they could see in their classroom which would demonstrably label them as "constructivist teachers". Such a program of constructivist teaching behaviors does not exist. Just as there are no mathematical understandings "out there" waiting to be acquired, there is no "constructivist teaching model" out there waiting to be implemented.

Were that the end of the story, writing this paper and doing the research on which its discussion is based would be folly. It is clear that teachers can and do create environments, based on the belief that all knowledge has to be constructed by the individual, in which students' mathematical knowledge-building and understanding is fostered. Such environments are the creations of teachers and follow from elements of their beliefs and knowledge put into action. They are not dependent on the specifics of classroom activities, materials, problems, or exercises. Given the necessary tenets of a teacher's beliefs elaborated later in this paper, it is quite possible for what might be observed as totally didactic teaching to create a constructivist environment. Sfard (1991) hints at this notion, and evidence of the mathematical understanding of students in Oriental countries such as Taiwan may well suggest that, in that culture, an instructional system in which deliberate teaching for understanding is preceded by rote learning can be constructivist.

Notice that from now on we shall intentionally not use the term "constructivist teaching" to avoid any association with an acquirable list of activities or behaviors. We have chosen instead to use the phrase "creating a constructivist environment". In doing so we are consciously using "environment" not in terms of the teacher creating a given external reality which will enable children to learn specific mathematics, but with the knowledge that the understanding that students draw from their experiences within the environment is determined by their own structures and histories, by their individual ways of perceiving and acting and organizing. We intend the phrase "creating a constructivist environment" to point to the fact that it is the teacher's intentions, not any specific activities which are done or not done, which determine the constructivist nature of the teaching. The environment in that sense is the result of deliberate, active behaviors by the teacher in the full knowledge that constructivism pertains to the actions of the learners.

> The task of education ... becomes a task of first inferring models of the students' conceptual constructs and then generating hypotheses as to how the students could be given the opportunity to modify their structures so that they lead to mathematical actions that might be considered compati-

ble with the instructor's expectations and goals. (von Glasersfeld, 1990, p. 34)

In providing such opportunities for students, the teacher must be conscious that these opportunities exist only as an environment in which each student makes personal interpretations.

The purpose of this paper is twofold. We intend first to describe four beliefs about teaching, the classroom, and the children which provide a basis for creating a constructivist environment in which to learn mathematics. We will then draw illustrative examples from classrooms on which we have extensive data to show some of the results of providing such environments.

PART 1. CREATING A CONSTRUCTIVIST ENVIRONMENT: FOUR NECESSARY, UNDERLYING TENETS OF BELIEF

What are the qualities of the intentions of a teacher who attempts to create a constructivist environment for mathematical learning and understanding? We see four tenets of belief as critical:

1) *Although a teacher may have the intention to move students towards particular mathematics learning goals, she will be well aware that such progress may not be achieved by some of the students and may not be achieved as expected by others.* Regardless of the environment, children build their own knowledge and mathematical understanding. In exhibiting their own understanding, children may be seen to form images or notice mathematical properties which are false or even incompatible in the eyes of the teacher. Yet that is the understanding which the student is showing at that moment and therefore the understanding with which the teacher must work. This tenet suggests that under constructivist principles a teacher must be continually re-creating the environment, not only in the light of individual student constructions but also for the class as a whole. There can be no intention to plan a teaching sequence and then simply apply that plan. The teacher must be constantly reappraising the learning taking place within the classroom environment as it evolves.

2) *In creating an environment or providing opportunities for children to modify their mathematical understanding, the teacher will act upon the belief that there are different pathways to similar mathematical understanding.* This belief in different routes to mathematical understanding entails a realization that each student comes to his or her current state of understanding through a unique pattern of engagement in the various kinds of activities offered. (Again we must stress that we are not necessarily talking about physical activities.) There is no unique or even best path for growth in understanding. As a direct consequence of this, there is also no particular form or sequence of instruction which can be

positively associated with growth in understanding in a constructivist environment.

3) *The teacher will be aware that different people will hold different mathematical understandings*. From this a number of implications follow. The teacher cannot think that his or her own understanding, the understanding of a given mathematician, the understanding underlying the writing of particular texts and materials, and the students' understanding will all be the same for any particular mathematical topic. Indeed the students themselves will all possess their own understandings which will be inherently different from one another. Thus, "creating a constructivist environment" means that the teacher will be oriented to account for this variation. This inter-student difference is not simply a matter of rate or style in reaching a given understanding of a given mathematical topic. There is no such thing as, for example, an "understanding of fractions" to eventually be passed on to, or even gained by, students. An understanding of a topic is not an acquisition. Understanding is an ongoing process which is by nature unique to that student. Holding this tenet implies that the teacher believes in and, just as importantly, acts on this difference in understanding.

4) *The teacher will know that for any topic there are different levels of understanding, but that these are never achieved 'once and for all'*. This tenet is, as were the preceding two, concerned with the growth of a student's mathematical understanding. Here we are interested in the teacher's intentions in terms of allowing for this growth. We see mathematical understanding as entailing the continual organization of self-built knowledge structures. Our joint research and theorizing over the past four years have focussed on mathematical understanding as an ongoing, dynamic process. Although it is beyond the limitations of this paper to describe our theory of understanding in detail, this has been done elsewhere (Pirie and Kieren, 1989, 1990, 1991; Kieren, 1990; Kieren and Pirie, 1991). In brief, our theory posits the notion that there are eight potential levels in the growth of mathematical understanding — namely, primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventing. We see each of these layers of understanding as embedding, but allowing access to, all previous layers. We see growth in a person's mathematical understanding with respect to a topic as a back-and-forth movement between activities at different levels. This we term "folding back". At certain stages, such as the transition from the image-oriented first three layers to the level of formalizing, understanding is said to have crossed a "don't need" boundary. The implication of these boundaries is that, although one can easily fold back to previous levels, such activity is no longer necessary to function mathematically in a particular topic. Growth in understanding is thus a dynamic,

organizing, and re-organizing process.

As with the previous three tenets, this fourth statement has implications for how a teacher reacts to what he or she observes in the classroom. He or she will not only be aware that students will come to understanding in different ways, but will also expect that different students will exhibit different kinds of understanding in the face of the same mathematical task. Possibly the most important consequence of this and the previous tenet, however, is the need to be aware that although two students may appear to exhibit the same understanding this may not be the case. The implication of this is that simply examining what a student does in the face of a mathematical task is not enough. If a teacher is to really observe the kind of understanding exhibited by a student, she must prompt students to justify what they say or do and thus reveal their thinking and logic. In order to expose different levels of understanding, tasks need to be used which allow for varying levels of response.

Not only should a teacher allow for and validate differences in levels of understanding between students, but she must also function in the awareness of the different levels of understanding within any one individual student. Human beings, unlike computers, understand things at many levels at once (Minsky, 1986). A teacher cannot think, "Oh, John is now at a formalizing level of understanding fractions, and hence will use formal algorithms from now on to handle tasks". Our theory of the growth of understanding referred to above suggests that a teacher must be aware that a student will fold back to less formal, less sophisticated actions as part of the normal growth process. In fact, a teacher who is trying to create a constructivist environment might deliberately try to invoke folding back to such previous level action as a means of promoting growth.

Summary of Part 1

So far, we have been arguing that a constructivist environment for mathematics learning is not a product of a particular program of classroom or individual activity. Such an environment is created by a teacher through a set of constructivist beliefs in action. These include the belief that there is no mathematical understanding "out there" to be acquired or achieved by students. Students act to develop their own unique understanding. In observing, and forming models of students' understanding, and in designing opportunities for growth of understanding, the teacher will take cognizance of the different levels of understanding to be exhibited by different students and the different pathways to such understandings which may be taken. Because a teacher is consciously responding to the diversity of student constructions, any of a variety of instructional acts might be appropriate. However careful the preparation, the teacher who is creating a constructivist environment will know that, since it is each student's system of knowing and acting which determines what that student will achieve, his or her goals for a student or class may not be achieved as intended.

PART 2. MATHEMATICAL ACTIVITY IN A CONSTRUCTIVIST ENVIRONMENT

Suppose one observed students in a classroom where the teacher created an environment based on the constructivist beliefs outlined above. What would one see? Would the mathematical understanding actions of the students support these beliefs?

We intend now to consider the above questions with reference to data collected during detailed observations taken in classes of 8 year olds and 12 year olds working on the topic of fractions. Audio recordings of the verbal contributions of the students and teachers were made, records of the ongoing student activities were kept, and the written work of all students was available. For the sake of simplicity within this paper, we have elected to consider classes, both of whom were engaged in building understandings of the rational numbers. We wish to make it clear that we do not intend that either our remarks and analyses with respect to constructivism or our theory of the growth of mathematical understanding upon which some of the analyses are based should be seen as stemming solely from consideration of the mathematical content of this particular topic.

Because we are arguing that a constructivist environment is the ongoing creation of the teacher, in each of the episodes offered below, a description of the constructivist intentions of the teacher is given. This is followed by a record of the incident and an analysis in the light of the tenets developed in Part 1.

Episode 1

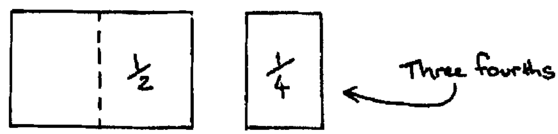
Background. The teacher had given the children sheets of paper (units) to be folded into halves, fourths, eighths, and sixteenths. The students had done this and discussed many aspects of their actions with one another. The teacher then, wanting to capitalize on his observation that most of the children seemed to have an image of fractions as amounts, provided each child with a "kit" containing numerous unit, half, fourth, eighth, and sixteenth pieces. He observed that most children could actively combine and compare "fractions" in this context but he believed that there were a variety of understandings and sought to validate this.

Record. The teacher gave the following written directions:

Here are some things children wrote or drew about three fourths ($^3/_4$).

Three fourths is less than one whole.

$^1/_2 + {}^1/_4$ equals $^3/_4$.



<u>YOUR TURN</u> : On this sheet or on the back write down or draw pictures of <u>5 things about three fourths</u>. You can do more if you want to.

Fig. 1. The writing task.

Of 24 students responding to this item, 21 produced at least five correct responses. Here are the responses from five of the class members.

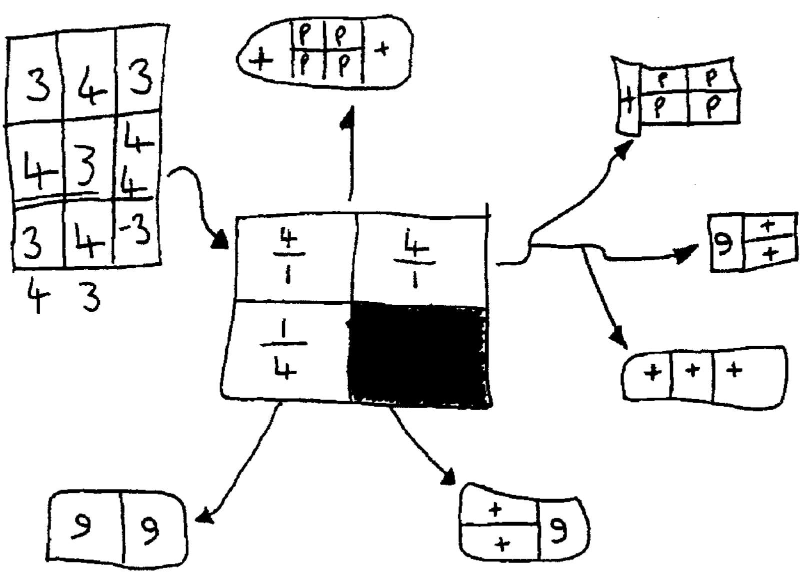


Fig. 2a. Brent's idea of 3/4.

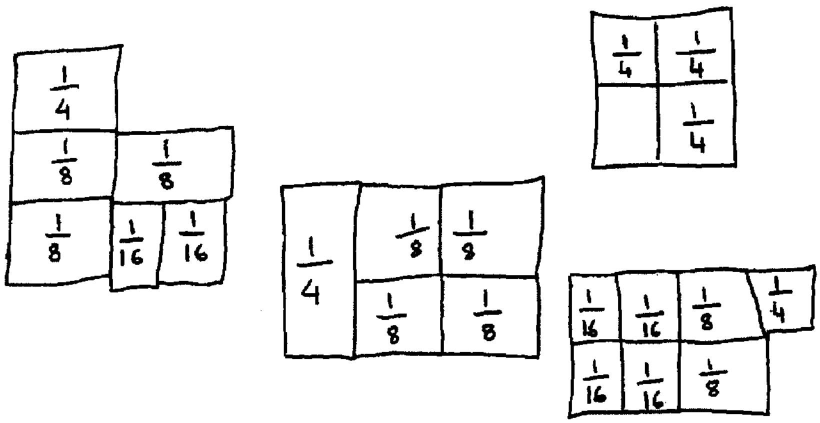


Fig. 2b. Judy's idea of 3/4.

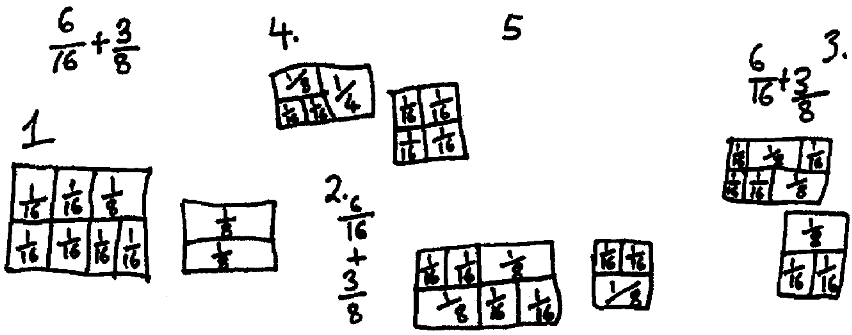


Fig. 2c. Pat's idea of 3/4.

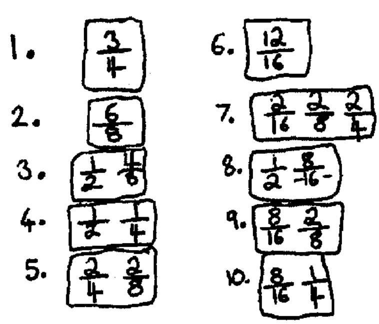


Fig. 2d. Don's idea of 3/4.

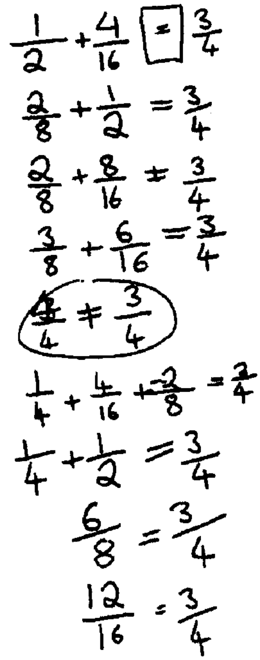


Fig. 2e. April's idea of 3/4.

Fig. 2. Five eight year olds' ideas about 3/4.

Analysis. We start with this episode in order to give a "snapshot" of a teacher creating a constructivist environment. The teacher has offered the students a task which will allow them to record and represent their current understandings of the "half" fractions. Up to this point the students have been working mainly with the "kit", although they have been writing as they felt it necessary. The task in no way suggests that the materials should no longer be used; it merely seeks to provoke some form of written record of their thinking. Because the intent is not to have the students necessarily conform to standard mathematical practices at this point, the wording of the task itself encourages diversity of response. Indeed, this episode typifies, through one small incident, the creation of a constructivist environment. Knowing full well that his students' primitive knowing and the images they have formed from the experiences that he has offered them will be different for each student, the teacher seeks to have some of this understanding articulated so that he may modify and direct his future interactions with these students.

What can we say from these student responses? Brent (Figure 2a) is clearly trying to respond to the task and has possibly picked up on some of the symbolism given in the question, but his personal codification is incomprehensible to the observer. From his actions, however, it is clear that there is "sense" here for Brent and *he* is satisfied that he has completed the task. It is also clear that the teacher needs to investigate more closely the images of combining fractions that Brent is working with. Further insight into Brent's thinking is seen in Episode 6.

Judy (Figure 2b) uses a spatial representation for her fraction combinations, with some demonstration of relative size — the 1/16ths are roughly half the size of the 1/8ths and in one case the 1/8ths are half the size of the 1/4th. (It should be noted that this representation of relative size is *not* demonstrated in the set task.)

Pat (Figure 2c) seems to be using "coverings" as her image of ways to make 3/4ths and shows three different physical arrangements of 1/8ths and 1/16ths on a "half piece and a quarter piece".

Don (Figure 2d) is working in a more symbolic way and using (not always correctly) both the idea of equivalence and combination, but his representation still speaks of the physical "kit".

April (Figure 2e), on the other hand, is confident in her use of traditional symbolism and has the image that combination of quantities is equivalent to addition. Discussion with her partner and the teacher on the legitimacy of the statement "3/4 = 3/4" showed a questioning of her image that "=" means "makes".

The students' responses here represent their own interpretation of the task and reveal something of their understanding of 3/4. It is clear that the teacher's assumption of the existence of different constructions of meaning between children within the topic of "half" fractions is justified.

Episode 2

Background. The students described in Episode 1 had been working in what might be called a "halving world". They had folded paper into fractions and combined fractions of the form m/2n, making images, noticing properties of these images, and, in the case of one child, working formally with these fractional quantities. Later, in order to have the children broaden their understanding, the teacher invoked new image-making activities by having the students fold a rectangular unit into three equal parts. The children already knew the fractional name for one third.

Record. The teacher demonstrated an "envelope fold" as a means of folding thirds.

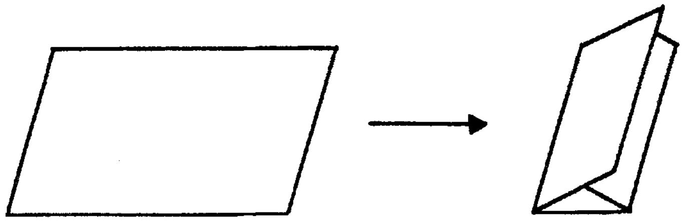


Fig. 3a. Folding into thirds — demonstration.

Even at this point the children followed their own thinking with about half the children doing the folding as follows:

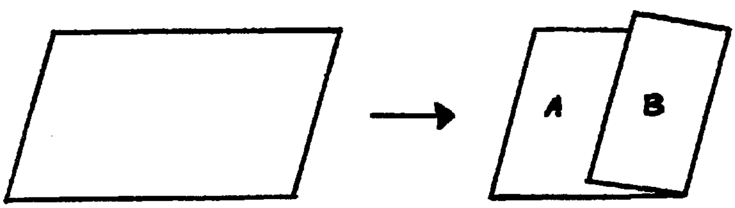


Fig. 3b. Folding into thirds — students' method.

They adjusted the fold-over part B until it equalled the width of the visible unfolded part A.

Once the children had mastered (to their own satisfaction) folding into three equal parts and had discussed related fractional words: such as "one third"; "three thirds is one whole"; "five thirds" and so on, all the children generated sixths — as a half of a third — through folding and most went on to create and discuss, at least informally, twelfths and their relationships to sixths and thirds. Todd, Walter, and Beaver carried this fraction family out to 96ths. These three in particular worked at getting an image of this new set of fractions, with Walter concluding that 1 1/8 was 108/96 while Todd described previously known fractions, for example 3/4 and 3/2, in terms of 48ths.

When challenged, half the class came up with a way to fold 9ths, and six of the students carried on to fold 27ths as well and developed this into a study of the "thirds" family.

Sandy, a gifted eight year old, who had shown formalized understanding with respect to "half fractions", appeared to apply a formal method to generate new fractions: He offered the statement "I have noticed an interesting fraction [writing on the blackboard] 1/270". When asked how he got that fraction, he was flustered and (still writing on the board) tried to use some symbolic algorithms for multiplying fractions but could not do so correctly. He finally erased 1/270 from the board and sat down. A short while later, when asked what 3/2 would be if it were folded into 48ths, Sandy hesitatingly said 54/48 — only to be corrected by his partner Russell: "That's only six 48ths more than one — that's not enough for a half more".

Analysis. Here we offer a different illustration of the creation of a constructivist environment. At the very beginning of the episode, we saw that even in a directed, image-making activity it was the children's primitive knowing and not simply the outside directions which brought forth the activities observed. As the students moved on to further exploration, many of the features of a constructivist environment manifested themselves. The images, properties, and formalizations which the children carried with them about the "half-fractions" as they folded back to the "one third" image-making triggered a visible, wide variety of new knowledge and understanding.

Sandy attempted to use his existing, formalized understanding of "half" fractions and his primitive knowing of the factors of 270 to communicate his new ideas in a formalized way but realized, for himself, that he needed to fold back to further image making before he was able to articulate what he had "seen". This need was further underlined when he responded to the problem of representing 3/2 as 48ths. He was attempting to compute an answer with no reference to his understanding of the nature of 3/2 as one whole and one half. That he was able to fold back, unaided, and resolve these problems was evident in his subsequent work.

Thus, in this constructivist environment, the teacher presented an activity with the intention of having the children extend their understanding to new fractions, and the children used actions, and understandings that they already had, to respond in a wide variety of ways: some worked almost totally in a physical-pictoral way, others worked with words and symbols, and one child used formalizing acts in a new setting. The children showed varied, unexpected understandings, reflecting their own effective level of understanding.

Episode 3

Background. The background to Episode 3 is the same as that of Episode 1.

Record. Sandy responded to the question concerning 3/4 in the following way:

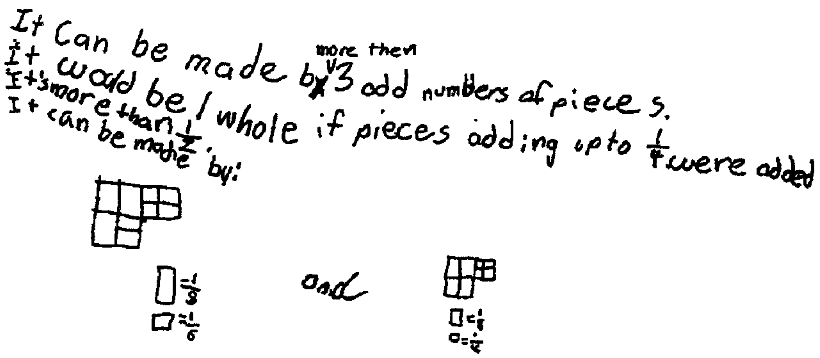


Fig. 4a. Sandy's idea of 3/4.

On the previous day, Sandy had been seen apparently trying to write down in a "chart" (his term) the combinations of fractions that would produce 5/4. As a result of this, the teacher tried to prompt Sandy to do a similar activity for 3/4, but he refused and continued to participate in the general class activities. Later in the same lesson he casually handed the "chart", reproduced here, to his teacher.

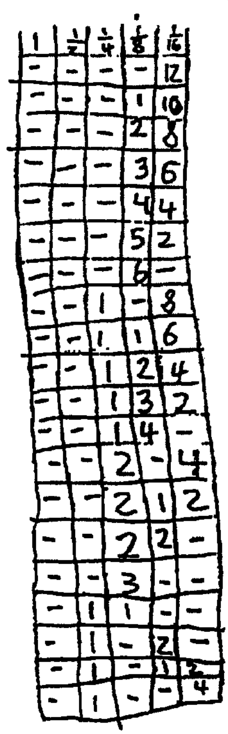


Fig. 4b. Sandy's chart.

Analysis. The purpose of this episode is to show that differences in levels of understanding can be seen, almost simultaneously, within the work of a single student. The response to the diagnostic task posed by the teacher shows use of image-oriented ways of combining fractions to yield 3/4 and simple comparisons of 3/4 with other known quantities. The "chart", however, produced during the same lesson, is illustrative of a child who is able to systematically present his formalized mathematical understanding. We see evidence of Sandy's working flexibly between several different levels of understanding. The creation of a constructivist environment for this student allowed the teacher to realize the fact that Sandy could and did function at many different levels of understanding. He demonstrated his understanding of fractions in a formalized way, but also worked at the level of noticing properties of his image and simply observing and playing with a physical image. For teachers to realize fully the potential of the constructivist environment they are trying to create, they themselves need to be able to easily fold back across the "don't need" boundary between image-orientated thinking and formalized mathematical thinking and see ways to invoke and support this movement between levels that leads to growth in understanding.

Episode 4

Background. Three eight year old children were using one-meter paper strips, known as "dragons", folded into halves, fourths, eighths, and sixteenths to measure things. The teacher intervened in their dialogue with the intention of offering appropriate language to aid their growth of understanding through communication using conventional terminology.

Record. In this episode, the students had marked off on the wall the height of one of the girls and were trying to find its exact measurement. Molly tried using a whole strip and one fourth but this was less than the marked height, and Rebecca said, "Let's get out an eighth dragon". When adding an eighth proved too big, they tried adding on a sixteenth instead. This was still too long. Walter then said, "It's a half a sixteenth more". Rebecca responded, "a thirty-two".

At this point, the teacher, who was observing all of this, interjected, "It's called a thirty-second".

Walter persisted, "She's one and a fourth and half of a sixteenth [units tall]".

Rebecca, quietly, "a thirty-twoth". (She followed this with a chuckle, quite clearly indicating her preference for her own language logic!).

Analysis. In this measurement situation, while Molly does not know what to do when a simple combination of measures does not match the height, both Walter and Rebecca exhibit the understanding that one can find a measure by combining further smaller units. In fact, Walter expresses the measurement not in terms of quantities but in terms of a problem-solving process: "half of a sixteenth". As a teacher frequently does, this teacher offers "correct" language for the amount. In this case, however, Walter and Rebecca act in a way that the teacher did not anticipate — both reject her proffered suggestion but for quite different reasons. Walter wished to describe a process and did not want to use a quantity name. Rebecca, while wishing to use a quantity name, persisted in using a name which fitted in with the logical system which she could see in use: four*th*, eigh*th*, sixteen*th*, ...

This is a brief but clear example of the children's intentions determining their, in this case verbal, actions. They had different purposes for the understanding they were at that moment trying to construct, different both from each other and from the teacher. While the teacher offered correct and useful information, she could not assume that the children would use it.

Episode 5

Background. In this episode, June, Shiela, and Cathy have engaged in using the fraction "kit" referred to earlier and have done exercises such as those illustrated in Episode 1. June and Shiela have both demonstrated that they see fractions as quantities and understand that fractions can be made up from many different combinations of other fractions.

Because to this point the students had only worked with fractions of continuous units, such as sheets of paper, the teacher wished to invoke image-making activities in which a unit was composed of discrete objects. To this end he selected a dozen eggs in an ordinary rectangular carton to act as the unit.

Record. The students were given a worksheet with questions such as the following:

Put eggs into a dozen carton so that each of four persons gets a fair share. Color in the diagram with four colors showing each person's share.

Cathy: So how many do you have to color for each person?
June: How many each? [*Looking and counting after sharing*] Three.
Shiela: I know! Each gets one fourth!
J: Uh?
C: O.K. What's one fourth?
J: You know, it's a quarter.

C: So how many eggs is one fourth?
S: See [*pointing to her diagram*].

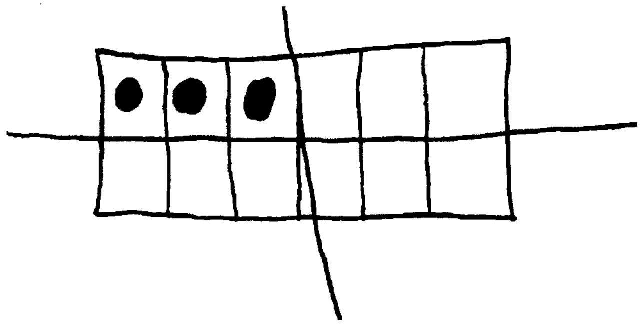


Fig. 5a. Shiela's egg carton fourth.

S: Three. One, two, three. One, two, three...
J: Three is what fraction of a dozen?
S: I think I explained this a little bit harder. I took out one half and split halves into one quarter of a dozen.
J: You could just put in half then, 'cause it's half [*relating result to halving action*].
S: But you already had half before you split it again.
J: When you split half into half, what do you get?
S: Only one fourth. I split the halves into one fourths.
J: But would that still be right here [*with egg cartons and not paper or pizzas*]?

Analysis. June was very puzzled by the meaning of fractions in this discrete setting and felt constrained to follow the directions step by step, frequently asking for help, clarification, and reassurance. The change from continuous to this unitized, discrete model of fractions invoked image-making behavior with its need for direction. June could understand fractions in the new setting only to the extent that she could follow directions and get results. Her response here to questions about simple half fractions was to use counting and corresponding competencies from her primitive knowing to work on individual tasks under the direction of the worksheet. We notice that while she could successfully use her competencies to generate responses such as "each of the four shares is three eggs", these responses were in no way connected or co-ordinated with her other half-fraction knowledge.

Shiela, on the other hand, actually saw the egg carton dozens in terms of her previous continuous image of fractions. She simply "divided up" the rectangular egg carton to get fractional parts — working with the eggs as an afterthought. Shiela did not fold back to image making but applied her image of fractions as continuous amounts, and related actions in a new setting. Her "answer" was in terms of fractions of a dozen and then only incidentally did she give a response in terms of numbers of eggs.

The point we are trying to make is that while the teacher made a decision to introduce at least a new model of fractions, the growth of understanding for the two girls was essentially different. Though it might appear so from this brief excerpt, it is not true that Shiela was on the pathway to understanding while June was not. Each was exhibiting her own growth pattern. That this was the case can be seen in an excerpt from an interview with the girls ten days later.

Teacher: Here is a box for candy. Pretend these chips are candies. Can you put in enough candies to show one fourth of a box?

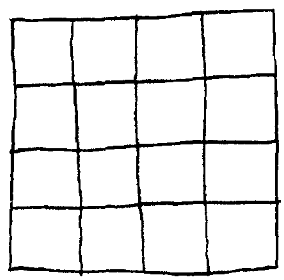


Fig. 5b. The candy box.

Shiela: Here is one fourth.

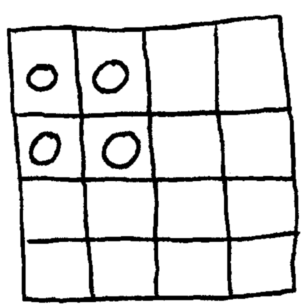


Fig. 5c. Shiela's candy box fourth

June: So is this one fourth.

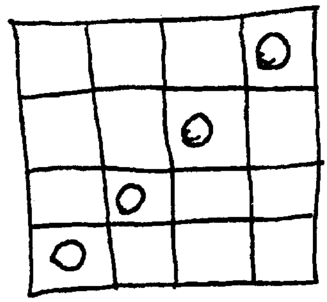


Fig. 5d. June's candy box fourth.

Shiela: No it can't be! [*30 second pause*] Well, there are four in my fourth, so it [*June's result*] is the same.

Using our model we can analyze the growth of understanding as follows. Both girls had developed an image of fractions as detailed at the start of this episode. Their reactions to the teacher's curricular intervention with unitized discrete models were, however, entirely different. June folded back to image making using sorting and counting. From this activity, which started in a very local way, simply accomplishing individual tasks, she came to see new patterns and changed her image of fractions to include discrete sets. Shiela, on the other hand, viewed the egg carton as a continuous thing and simply elaborated on her old image to include this new situation. She noticed a new property of an already held image. This episode well illustrates two different pathways to understanding. In the final interview, both girls show that they have an image of fractions which is based on numerical as well as geometric comparisons with a unit. Clearly for June, this numerical understanding has grown from her image making in parallel to her previous geometric image. But for Shiela her numerical image is a derivative of her geometric understanding.

In other work (Pirie and Kieren, 1990) we have suggested that a teacher can use provocative teaching acts (those which push students to outer levels of understanding) or invocative acts (those which encourage students to fold back to less sophisticated ways of understanding) in order to extend their knowledge. Whether the intervention is provocative or invocative, however, is determined by the reaction of the student and cannot be pre-determined by the teacher. This is clearly illustrated in this episode where, while the teacher's intention was invocative, it had that effect only for June. It proved to be provocative for Shiela.

Episode 6

Background. As a means of having children demonstrate and consolidate their knowledge of continuous models of half fractions, the teacher had the class play a game, the object of which was to cover a unit using fractional tiles. The choice of tiles was determined by the roll of two dice on which the fractions 1/2, 1/4, 1/8, and 1/16 appeared. The children played this game in groups of three or four for approximately 30 minutes.

Record. At the beginning of the next class, although it had not been suggested by the teacher, two students, Sandy and Todd, each produced fraction games that they had invented and wished to discuss and share. Sandy's game involved sophisticated matching of sets of equivalent fractions. Todd's game was a covering game, but involved setting a fractional target (e.g., 5/8, 5/4, 9/16) to cover. Both games were well thought out with clear rules and strategies which involved images and properties of the half fractions in their play. Todd unfortunately was frustrated in the execution of his game because he could not accurately cut out the fractional pieces he needed and thus his physical "coverings" did not match his theoretical ideas and game rules.

A third boy, Brent, having observed the other two invented games, worked for 15 to 20 minutes to invent a game of his own. He then called the teacher and research assistant over to demonstrate his game to them. Although Brent's game involved pieces and dice, it was just an incoherent series of moves. It involved fractions only in the sense that he used fractional words to describe elements of his game such as, "You start here. If you roll an eighth then you land on one fourth. Then you try get to sixteenth".

Analysis. In this case, the classroom mathematical activity of game playing served to provoke three students to attempt to express their knowledge and understanding of fractions in the form of their own game. The teacher, not unnaturally, expected that such personal constructions would lead to the expression of useful mathematics by these children and therefore deviated from the planned class activities and encouraged them in their enterprise. In this open, self-chosen activity, we have a situation where the three boys clearly exhibited different mathematical understandings which neither matched the teacher's expectations nor were necessarily useful in the ultimate building of mathematical knowledge. Sandy and Todd validated their understood images and the properties of fractions that they had noticed by using them coherently in a game. Todd's reaction to his own game was particularly interesting because when his expected "coverings" did not work he displayed total confidence in the mathematical theory on which they were based and knew that the fault lay in his lack of manual dexterity when cutting out the pieces. Brent's activity revealed that although he had a very clear image of the nature of "a game", his personal creation did not lead to the construction or expression of mathematical understanding at the level expected by the teacher. Unlike the games of Todd or Sandy, Brent's invention showed only nominal fraction knowledge and revealed the possibility that he did not even have a usable image of the half fractions. This had not been so evident to the teacher before.

Episode 7

Background. In the several days prior to this episode these 12 year old students had been working with image making for the equivalence of fractions, addition, and subtraction using continuous rectangular units in the context of pizzas which were cut into halves, thirds, fourths, sixths, eighths, twelfths, and so on. Students had been asked to write down five things they knew about fractional numbers. Some responses which show images held and properties noticed were:

Fractions make amounts of things. Two fractions could look different but be the same amount. (Miguel)

Each fraction has unlimited equal fractions. (Leanne)

Any fraction can go on for ever. For example: 1/2, 2/4, 3/6, 4/8, 5/10, 6/12, 7/14, 8/16, 9/18, 10/20, 11/22, ... (Jose)

Record. The teacher's intention in the following episode was to review orally the responses to seatwork in which the students had been asked to generate fractions of an hour from given numbers of minutes.

Teacher: What about 15 minutes? Miguel?
Miguel: Two eighths.
T: [*A little surprised by the answer*] Two eighths? Two eighths of what?
M: Two eighths of [*hesitation*] a pizza.
T: Well, we're dealing with time here.
M: Oh yeah! Two eighths of an hour.
T: Any other fraction of an hour which tells us about 15 minutes? Bonnie?
Bonnie: One-fourth hour.
T: Good. Let's start a list. [*writing*] Two eighths, one fourth ... Others?
Sarah: Three 12ths.

T: Why is that?

K: Well 5, 5, and 5 are 15. They are one 12th so three 12ths of an hour.

T: Yes, we could see that on the clock — in five-minute pieces. One, two, three 12ths. [*Writes 3/12.*] Any other?

Pete: Four 16ths.

T: [*Adding 4/16 to the list*] Other?

Dan: How about seven 28ths.

[*The list now reads: 2/8, 1/4, 3/12, 4/16, 7/28.*]

Leanne: [*Interrupting*] Wait a minute. Some of those fractions are less equivalent than others. [*Pause*] No, no they're all equivalent to one fourth.

Jose: No, Leanne. Go! Don't stop! That's a good idea. See: two 8ths and one 4th, and one 4th and three 12ths — they're *really* equivalent. But two 8ths and three 12ths aren't as equivalent and three 12ths and seven 28ths, they're hardly equivalent at all!

Analysis. This episode illustrates many of the features of a constructivist environment. First, it shows that in an environment where different individual understandings are expected, even the act of checking up on a simple exercise can provide a rich interactive mathematical environment. We see the validity of the teacher's assuming different understandings, yet, what is evident is that these different individual understandings are compatible and contribute to the growth of understanding within the class. Sarah's statement reveals that her understanding of this situation entails observing aspects of a physical image. The offering of 7/28 by Dan, on the other hand, shows him using a property of constructing equivalent fractions that he has noticed — namely, multiplying the numerator and denominator of 1/4 by 7. The creation of the list of equivalent fractions allowed the teacher to observe the different related understandings held by the various children. Because students expected to have to explain or validate their answers, this activity also allowed students to participate based on their own mental structures.

The striking aspect of this episode, though, is the way that this simple list of equivalent fractions in response to a routine exercise also triggers the expression of an original, and rather different, understanding. Leanne, who has previously exhibited formalized understanding with respect to equivalence — she expects any fraction to be part of an infinite set of equivalent fractions — is now prompted to fold back and notice a new property — "less equivalent than". Although Leanne appears to retreat to her more formal concept of equivalence, her remark invokes Jose to also fold back and he starts a discussion on this new noticed property.

In a constructivist environment, the assumption of different understandings and different levels of understanding leads a teacher not merely to look for simple answers to routine questions, but to allow for and seek and even be surprised by the different mathematical understandings shown by students.

CONCLUDING REMARKS

In this paper we have attempted to portray a constructivist environment in the mathematics classroom, not in terms of the use of specific materials, microworlds, or teaching styles but as a creation arising out of the teacher's ongoing process of acting from a constructivist belief in the nature of mathematical learning. Such a teacher knows that there is no external mathematical understanding to be acquired or even attained by students. Each person's mathematical understanding is unique. Indeed, since we believe that all knowledge is personally constructed and organized, students in any environment will construct understanding in some form. What we are interested in is the creation by a teacher of an environment which is consciously based on optimizing the opportunities for the construction of mathematical understanding. For this reason, such a teacher is free to choose from the many different kinds of instructional acts available to her, knowing what they can contribute to students' construction of mathematics.

We focused on four tenets of a constructivist standpoint which we felt could be investigated in the classroom and analyzed our data from this perspective. Any act of the teacher or feature of the environment will not necessarily lead to a student's constructing or exhibiting the mathematical understanding expected by the teacher. Furthermore, students exhibiting similar mathematical behaviors will in fact have different understandings since there are different levels of understanding within any one topic and these can be reached by students through different pathways.

So, what happens when a teacher acts to create an environment in this manner? The episodes analyzed above appear to validate our choice of foci. In these constructivist environments, children did indeed show individual understandings of the mathematics being taught. In Episodes 1 and 2, we illustrated teachers deliberately creating constructivist environments. In Episodes 4 and 7, we saw different students construct and show different, but compatible and coherent, understandings of fractions. Episode 3 illustrated how a teacher in such an environment must also anticipate that a student will understand a piece of mathematics at many levels and in many modes at once, and must encourage this diversity. In addition, as seen in Episodes 4, 5, 6, and 7, despite the intended goals of the teacher, students are free to construct mathematics based on their own structures and ideas, and even in this environment, students can still arrive at incomplete or profound understandings unanticipated by the teacher.

While in a constructivist environment, the knowledge and understanding built by students is based on each one's own primitive knowing, and the teacher needs to observe carefully the understanding displayed by his or her students and provide opportunities for validation, together with provocative and invocative challenges for them. As shown in Episode 5, however, it is the student's response to the situation rather than the nature of the situation which determines the student's pathway to understanding.

We have tried to argue and illustrate that the teacher's continuing constructive act of "creating a constructivist environment" can have observable consequences in the growth of knowing and mathematical understanding by students. We have tried also to show some of the ongoing demands of such an environment for the teacher and the richness and texture of such an environment for the students.

NOTE

This paper and the associated research have been done in part with the support of Social Sciences and Humanities Research Council Grant 410 900738 and a Canadian Commonwealth Research Fellowship.

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