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Probability Theory and Related Fields

Central Limit Theorems for Quadratic Forms in Random Variables Having Long-Range Dependence

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1. Introduction

Let $f(x)$ and $g(x)$ be integrable real symmetric functions on $[-\pi, \pi]$ that are bounded on subintervals that exclude the origin. Let $X_1, X_2, \ldots$ be a mean zero stationary Gaussian sequence with spectral density $f(x)$, and let $\ldots, -a_1, a_0, a_1, \ldots$ be the Fourier coefficients of $g(x)$. We prove that the distribution of the normalized quadratic form

$$Z_N = \frac{1}{\sqrt{N}} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i-j} X_i X_j - E \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i-j} X_i X_j \right\}$$

converges to a normal distribution if there exist constants $\alpha < 1$ and $\beta < 1$ with $\alpha + \beta < 1/2$ such that for each $\delta > 0$, $f(x) = O(|x|^{-\alpha-\delta})$ and $g(x) = O(|x|^{-\beta-\delta})$ as $x \to 0$.

Of particular interest are the cases where $f(x) \sim x^{-\alpha} L_1(x)$ and $g(x) \sim x^{-\beta} L_2(x)$ as $x \to 0$ with L_1 and L_2 slowly varying. The exponents α and β are allowed to be positive, zero or negative. The sequence $\{X_j\}$ is said to exhibit a long-range dependence when $\alpha > 0$. When $\alpha < 0$, the covariances $r_k = EX_j X_{j+k}$ satisfy $\sum_{k=-\infty}^{+\infty} r_k = 0$.

Suppose $f(x) \sim x^{-\alpha} L_1(x)$ and $g(x) \sim x^{-\beta} L_2(x)$ as $x \to 0$. Rosenblatt (1961) showed that in the special case $1/2 < \alpha < 1$ and $a_{i-j} = \delta_{ij}$, the quadratic form $\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i-j} X_i X_j$, adequately normalized, converges to a *non-normal* distribution. The assumption $a_{i-j} = \delta_{ij}$ implies $g(x)$ constant and thus $\beta = 0$. Our result shows that the normalized quadratic form Z_N converges to a *normal* distribu-

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tion when $1/2<\alpha<1$ and $\beta<1/2-\alpha<0$. If $\alpha\leqq 1/2$, it is even possible to choose $\beta>0$ as long as $\beta<\min(1/2-\alpha, 1)$.

These results are used in the study of the asymptotic behavior of maximum likelihood type estimators related to the sequence $\{X_j\}$ (Fox and Taqqu 1986). Examples of sequences $\{X_j\}$ satisfying $f(x)\sim x^{-\alpha}L_1(x)$ that are of special interest include fractional Gaussian noise and fractional ARMA.

A sequence $\{X_j\}$ is *fractional Gaussian noise* (Mandelbrot and Van Ness 1968) if its covariance satisfies

$$r(k)=EX_jX_{j+k}=\frac{\sigma^2}{2}\{||k|-1|^{2H}-2|k|^{2H}+(|k|+1)^{2H}\}$$

for $0<H<1$. In that case (Sinai 1976)

$$f(x)=\frac{\sigma^2}{\int_{-\infty}^{+\infty}(1-\cos y)|y|^{-1-2H}dy}(1-\cos x)\sum_{k=-\infty}^{+\infty}|x+2k\pi|^{-1-2H},$$

so that $\alpha=2H-1\in(-1,1)$.

A sequence $\{X_j\}$ is *fractional ARMA* (Hoskings 1981) if its spectral density is

$$f(x)=|e^{ix}-1|^{-d}\frac{|\varphi(e^{ix})|^2}{|\psi(e^{ix})|^2}$$

where φ and Ψ are polynomials having no zeroes on the unit circle and $d<1$. In that case $\alpha=d$. Heuristically, fractional ARMA is the sequence, which, when differenced $d/2$ times, yields an autoregressive-moving average (ARMA) sequence with spectral density $|\varphi(e^{ix})|^2/|\Psi(e^{ix})|^2$.

Our main results are in Sect. 2. Sections 3 through 7 are devoted to the proof of Theorem 1. That proof uses "power counting" arguments in the sense of mathematical physics. In Sect. 3 we introduce the power counting set-up and state an extension of a power counting theorem of Lowenstein and Zimmerman (1975). Preliminary lemmas are proven in Sect. 4 and, together with the results of Sect. 5, they are used to establish Propositions 6.1 and 6.2 of Sect. 6. These propositions describe the asymptotic behavior of certain multiple integrals. Section 7 contains the proof of Theorem 1. Theorem 4 is proven in Sect. 8.

2. Main Results

Let $f(x)$ and $g(x)$ be integrable real symmetric functions on $[-\pi,\pi]$, not necessarily non-negative. Define the Fourier coefficients

$$r_n=\int_{-\pi}^{\pi}e^{inx}f(x)dx$$

and

$$a_n=\int_{-\pi}^{\pi}e^{inx}g(x)dx.$$

Let R_N and A_N be the $N \times N$ matrices with entries $(R_N)_{j,k} = r_{j-k}$ and $(A_N)_{j,k} = a_{j-k}$, $0 \leqq j, k \leqq N-1$. Let $\operatorname{Tr} M$ denote the trace of a matrix M.

We say that f satisfies the *regularity condition* if the discontinuities of f have Lebesgue measure 0 and f is bounded on the interval $[\delta, \pi]$ for all $\delta > 0$.

Theorem 1. *Suppose that f and g each satisfy the regularity condition. Suppose in addition that there exist $\alpha < 1$ and $\beta < 1$ such that for each $\delta > 0$*

$$|f(x)| = O(|x|^{-\alpha-\delta}) \quad \text{as } x \to 0$$

and

$$|g(x)| = O(|x|^{-\beta-\delta}) \quad \text{as } x \to 0.$$

Then

a) *If $p(\alpha+\beta) < 1$,*

$$\lim_{N\to\infty} \frac{\operatorname{Tr}(R_N A_N)^p}{N} = (2\pi)^{2p-1} \int_{-\pi}^{\pi} [f(x)g(x)]^p dx.$$

b) *If $p(\alpha+\beta) \geqq 1$,*

$$\operatorname{Tr}(R_N A_N)^p = o(N^{p(\alpha+\beta)+\varepsilon}) \text{ for every } \varepsilon > 0.$$

The theorem is proven in Sect. 7. The proof of Part a) amounts to showing that

$$\lim_{N\to\infty} \int_{[-\pi,\pi]^{2p}} Q(y)\, d\mu_N(y) = \int_{[-\pi,\pi]^{2p}} Q(y)\, d\mu(y)$$

where

$$Q(y) = f(y_1)\, g(y_2)\, f(y_3)\, g(y_4) \dots b(y_{2p-1})\, g(y_{2p}),$$

$$d\mu_N(y) = \frac{1}{(2\pi)^{2p-1} N} \sum_{j_1=0}^{N-1} \dots \sum_{j_{2p}=0}^{N-1} e^{i(j_1-j_2)y_1} e^{i(j_2-j_3)y_2}$$

$$\dots e^{i(j_{2p-1}-j_{2p})y_{2p-1}} e^{i(j_{2p}-j_1)y_{2p}} dy_1 \dots dy_{2p},$$

and where μ is Lebesgue measure concentrated on the diagonal of $[-\pi, \pi]^{2p}$.

Introduce now a stationary Gaussian sequence X_j, $j \geqq 1$ with mean 0 and spectral density $f(x) \geqq 0$, so that

$$E X_j X_{j+k} = r_k = \int_{-\pi}^{\pi} e^{ikx} f(x)\, dx.$$

Let x_N denote the random vector $(X_1, X_2, \dots, X_N)$. Put $\mu_N = E x_N' A_N x_N$.

Theorem 2. *Suppose that f and g each satisfy the regularity condition. Suppose in addition that there exist $\alpha < 1$ and $\beta < 1$ such that $\alpha + \beta < 1/2$ and such that for each $\delta > 0$*

$$f(x) = O(|x|^{-\alpha-\delta}) \quad \text{as } x \to 0$$

$$g(x) = O(|x|^{-\beta-\delta}) \quad \text{as } x \to 0.$$

Then

$$\frac{x_N' A_N x_N - \mu_N}{\sqrt{N}}$$

tends in distribution to a normal random variable with mean 0 *and variance* $16\pi^3 \int_{-\pi}^{\pi} [f(x)g(x)]^2 dx.$

Proof. Since the sequence X_j is Gaussian, the p^{th} cumulant of $x_N' A_N x_N$ is equal to $2^{p-1}(p-1)!\operatorname{Tr}(R_N A_N)^p$. (See, for example, Grenander and Szego 1958, p. 218). Thus the p^{th} cumulant of

$$\frac{x_N' A_N x_N - \mu_N}{\sqrt{N}}$$

is given by

$$c_p(N) = \begin{cases} 0 & \text{if } p=1 \\ 2^{p-1}(p-1)!\,\dfrac{\operatorname{Tr}(R_N A_N)^p}{N^{p/2}} & \text{if } p \geqq 2. \end{cases}$$

An application of Theorem 1 yields

$$\lim c_p(N) = \begin{cases} 0 & \text{if } p \neq 2 \\ 16\pi^3 \int_{-\pi}^{\pi} [f(x)g(x)]^2 dx & \text{if } p=2. \end{cases}$$

This implies the conclusion of Theorem 2. □

The following is an immediate consequence of Theorem 2.

Theorem 3. *Suppose that* f *and* g *each satisfy the regularity condition. Suppose in addition that there exist* $\alpha<1$ *and* $\beta<1$ *such that* $\alpha+\beta<1/2$,

$$f(x) \sim |x|^{-\alpha} L_1(x) \quad \textit{as } x \to 0$$

and

$$g(x) \sim |x|^{-\beta} L_2(x) \quad \textit{as } x \to 0,$$

where L_1 *and* L_2 *are slowly varying at* 0. *Then the conclusion of Theorem* 2 *holds.*

The next theorem, which is used in Fox and Taqqu (1986), is proven in Sect. 8. Define $\bar{X}_N = (1/N) \sum_{j=1}^{N} X_j$ and the random vector $\tilde{x}_N = (X_1 - \bar{X}_N, \ldots, X_N - \bar{X}_N)$.

Theorem 4. *If the conditions of Theorem* 2 *are satisfied, then*

$$\frac{\tilde{x}_N' A_N \tilde{x}_N - E\{\tilde{x}_N' A_N \tilde{x}_N\}}{\sqrt{N}}$$

tends in distribution to a normal random vector with mean 0 *and variance* $16\pi^3 \int_{-\pi}^{\pi} [f(x)g(x)]^2 dx.$

3. Power Counting Theorems

Power counting methods can be used to verify the convergence of multiple integrals whose integrands are products of powers of affine functionals. Let $b_1, \ldots, b_m$ and $\theta_1, \ldots, \theta_m$ be real constants and let $M_1(x), \ldots, M_m(x)$ be m linear functionals on $\mathbb{R}^n$. Put $L_j(x) = M_j(x) + \theta_j$, $j = 1, \ldots, m$. Define the function P: $\mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$P(x) = |L_1(x)|^{b_1} |L_2(x)|^{b_2} \ldots |L_m(x)|^{b_m}.$$

Define $T = \{L_1, \ldots, L_m\}$ and let $W \subset T$. Let $\operatorname{span}\{W\}$ denote the set of linear combinations of elements of W and $s(W)$ denote those linear combinations which coincide with elements of T. Thus

$$s(W) = T \cap \operatorname{span}\{W\}.$$

For each $W \subset T$ we define the quantity

$$d(P, W) = |W| + \sum_{\{j: L_j \in s(W)\}} b_j,$$

where $|W|$ denotes the cardinality of W. We refer to $d(P, W)$ as the dimension of P with respect to W. We say that $W = \{L_{i_1}, \ldots, L_{i_k}\}$ is *strongly independent* if $M_{i_1}, \ldots, M_{i_k}$ are linearly independent. Let S be the set of those L_j in T that have exponents $b_j < 0$. Finally, for each $t > 0$, let

$$U_t = [-t, t]^n = \{x \in \mathbb{R}^n : |x_i| \leqq t,\ i = 1, \ldots, n\}.$$

The next theorem extends a basic result of Lowenstein and Zimmermann (1975). It is proved at the end of Sect. 4.

Theorem 3.1. *Suppose that $d(P, W) > 0$ for every strongly independent set $W \subset S$. Then $\int_{U_t} P(x)\,dx < \infty$ for all $t > 0$.*

To illustrate the application of the theorem, let $n = 3$ and define $P(x)$: $\mathbb{R}^3 \to \mathbb{R} \cup \{\infty\}$ by

$$P(x) = |x_1 + x_2 + 2|^{b_1} |x_1 + x_2 + x_3 - 1|^{b_2} |x_3 - 3|^{b_3},$$

where $b_1, b_2, b_3 < 0$. Define $L_1(x) = x_1 + x_2 + 2$, $L_2(x) = x_1 + x_2 + x_3 - 1$ and $L_3(x) = x_3 - 3$. Then $S = T = \{L_1, L_2, L_3\}$. The strongly independent subsets of S are $\{L_1\}$, $\{L_2\}$, $\{L_3\}$, $\{L_1, L_2\}$, $\{L_1, L_3\}$ and $\{L_2, L_3\}$. We have $d(P, \{L_j\}) = 1 + b_j$, $j = 1, 2, 3$. The other three dimensions are all equal to $2 + b_1 + b_2 + b_3$ because for example $s(\{L_1, L_2\}) = \{L_1, L_2, L_3\}$. Therefore $\int_{U_t} P(x)\,dx$ will be finite provided that $b_1 + b_2 + b_3 > -2$ and $b_1, b_2, b_3 > -1$.

Remark. Suppose the condition of Theorem 3.1 is satisfied. Then in fact $d(P, W) > 0$ for every $W \subset T$. To see this, suppose first that $W \subset T$ contains only one element not in S, say L. Then $W = W_0 \cup \{L\}$, where $W_0 \subset S$ and $d(P, W_0) > 0$. If $d(P, W) \leqq 0$ then $s(W)$ must contain some element of S which is not in W_0, say L'. Then $W_1 = W_0 \cup \{L'\}$ satisfies $W_1 \subset S$ and $d(P, W_1) = d(P, W) < 0$, since $S(W_1) = S(W_2)$ and $|W_1| = |W_2|$. This contradicts the assumption. Hence any subset of T which differs from a subset of S by one element has positive dimension. The same method can be used inductively to show that all subsets of T have positive dimension.

4. Preliminary Lemmas

Retain the notation introduced in Sect. 3. Fix a permutation $\sigma=(\sigma_1,\ldots,\sigma_m)$ of $\{1,\ldots,m\}$ and let

$$E^t_\sigma=\{x\in U_t\colon |L_{\sigma_1}(x)|\leqq|L_{\sigma_2}(x)|\leqq\ldots\leqq|L_{\sigma_m}(x)|\}.$$

We use the greedy algorithm to construct a basis B_σ for T. The greedy algorithm proceeds as follows. We put $L_{\sigma_1}\in B_\sigma$. We put $L_{\sigma_2}\in B_\sigma$ if L_{σ_2} is not in the span of $\{L_{\sigma_1}\}$. On the j^{th} step we put $L_{\sigma_j}\in B_\sigma$ if L_{σ_j} is not in the span of $\{L_{\sigma_1},\ldots,L_{\sigma_{j-1}}\}$. It is well known that in this way we obtain a basis $B_\sigma=\{L_{\tau_1},\ldots,L_{\tau_r}\}$ for T, where r is the rank of T. We then have

$$|L_{\tau_1}|\leqq|L_{\tau_2}|\leqq\ldots\leqq|L_{\tau_r}|,\qquad x\in E^t_\sigma. \tag{4.1}$$

The functions $L_{\tau_1},\ldots,L_{\tau_r}$ are linearly independent but not necessarily strongly independent.

We use B_σ to construct the partition of T given by

$$T_1=s\{L_{\tau_1}\}$$

and

$$T_k=s\{L_{\tau_1},\ldots,L_{\tau_k}\}/s\{L_{\tau_1},\ldots,L_{\tau_{k-1}}\},\qquad k=2,\ldots,r.$$

Lemma 4.1. *For each permutation σ there is a constant C_σ (independent of x and t) such that if $L\in T_k$ then*

a) $|L|\leqq C_\sigma|L_{\tau_k}|$, $x\in E^t_\sigma$,

and

b) $|L_{\tau_k}|\leqq|L|$, $x\in E^t_\sigma$.

Proof. a) If $L\in T_k$ then $L=a_1L_{\tau_1}+\ldots+a_kL_{\tau_k}$ for some constants $a_1,\ldots,a_k$. Therefore

$$|L|\leqq|a_1||L_{\tau_1}|+\ldots+|a_k||L_{\tau_k}|,\qquad x\in\mathbb{R}^n.$$

Relation (4.1) implies that for $x\in E^t_\sigma$ the right hand side is less than $(|a_1|+\ldots+|a_k|)|L_{\tau_k}|$.

b) Suppose that $L\in T_k$. We must have either $L=L_{\tau_k}$ or else L was rejected by the greedy algorithm. In proving b) we can thus assume that L was rejected by the greedy algorithm. Since $L\in T_k$ it follows that $L\notin s\{L_{\tau_1},\ldots,L_{\tau_{k-1}}\}$. Therefore it must be that L was considered by the greedy algorithm after L_{τ_k}. But the greedy algorithm considers candidates in order of increasing absolute value on E^t_σ. Thus we must have $|L_{\tau_k}|\leqq|L|$, $x\in E_\sigma$. This completes the proof of Lemma 4.1. □

The next lemma provides a majorant for $P(x)$ involving only elements of B_σ.

Lemma 4.2. *For each permutation σ there is a constant C_1 (independent of x and t) such that*

$$P(x) \leqq C_1 |L_{\tau_1}|^{\Delta_1} \dots |L_{\tau_r}|^{\Delta_r}, \qquad x \in E^t_\sigma,$$

where

$$\Delta_1 = d(P, \{L_{\tau_1}\}) - 1,$$

and

$$\Delta_k = d(P, \{L_{\tau_1}, \dots, L_{\tau_k}\}) - d(P, \{L_{\tau_1}, \dots, L_{\tau_{k-1}}\}) - 1, \qquad k = 2, \dots, r.$$

Proof. We have

$$P(x) = \prod_{k=1}^{r} F_k(x),$$

where

$$F_k(x) = \prod_{\{j: L_j \in T_k\}} |L_j|^{b_j} = \Big(\prod_{\{j: L_j \in T_k \setminus S\}} |L_j|^{b_j}\Big)\Big(\prod_{\{j: L_j \in T_k \cap S\}} |L_j|^{b_j}\Big).$$

Fix $k \leqq r$ and consider the two products on the right hand side. In the first product all of the exponents are non-negative because the L_j's do not belong to S. Therefore Lemma 4.1a implies that the first product is majorized on E^t_σ by

$$\prod_{\{j: L_j \in T_k \setminus S\}} C_\sigma^{b_j} |L_{\tau_k}|^{b_j}.$$

In the second product all of the exponents are negative. Thus Lemma 4.1b implies that the second product is majorized on E^t_σ by

$$\prod_{\{j: L_j \in T_k \cap S\}} |L_{\tau_k}|^{b_j}.$$

Combining these facts we conclude that there is a constant C_2 such that

$$F_k(x) \leqq C_2 |L_{\tau_k}|^{p_k}, \qquad x \in E^t_\sigma, \ k \leqq r,$$

where

$$p_k = \sum_{\{j: L_j \in T_k\}} b_j.$$

Lemma 4.2 will follow from this inequality if we show that $\Delta_k = p_k$, $k = 1, \dots, r$. We have

$$d(P, \{L_{\tau_1}\}) = 1 + \sum_{\{j: L_j \in s(L_{\tau_1})\}} b_j = 1 + \sum_{\{j: L_j \in T_1\}} b_j = 1 + p_1.$$

Thus

$$\Delta_1 = d(P, \{L_{\tau_1}\}) - 1 = p_1.$$

If $k \geqq 2$ then

$$\begin{aligned} d(P, (L_{\tau_1}, \dots, L_{\tau_k}\}) &= k + \sum_{\{j: L_j \in s(L_{\tau_1}, \dots, L_{\tau_k})\}} b_j \\ &= 1 + ((k-1) + \sum_{\{j: L_j \in s(L_{\tau_1}, \dots, L_{\tau_{k-1}})\}} b_j) + \sum_{\{j: L_j \in T_k\}} b_j \\ &= 1 + d(P\{L_{\tau_1}, \dots, L_{\tau_{k-1}}\}) + p_k. \end{aligned}$$

Thus

$$\Delta_k = d(P, \{L_{\tau_1}, \ldots, L_{\tau_k}\}) - d(P, \{L_{\tau_1}, \ldots, L_{\tau_{k-1}}\}) - 1 = p_k.$$

This completes the proof of Lemma 4.2. □

Lemma 4.3. *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be given real numbers. Then for all $t > 0$*

$$\int_{|x_1| \leqq |x_2| \leqq \ldots \leqq |x_n| \leqq t} |x_1|^{\varphi_1} |x_2|^{\varphi_2} \ldots |x_n|^{\varphi_n} dx_1 dx_2 \ldots dx_n < \infty,$$

if $d_k = k + \sum_{j=1}^{k} \varphi_j > 0$ for $k = 1, \ldots, n$.

Proof. It clearly suffices to consider the case $t = 1$. We proceed by induction on n. The lemma is obviously true for $n = 1$. Now suppose that the lemma holds for $n - 1$ and that we are given $\varphi_1, \ldots, \varphi_n$ satisfying the hypotheses of the lemma. Choose $\delta \geqq 0$ such that $d_n - \delta > 0$ and $\varphi_n - \delta \neq -1$. (If $\varphi_n \neq -1$ we can take $\delta = 0$). Then the above integral (with $t = 1$) is less than

$$\int_{|x_1| \leqq |x_2| \leqq \ldots \leqq |x_n| \leqq 1} |x_1|^{\varphi_1} \ldots |x_{n-1}|^{\varphi_{n-1}} |x_n|^{\varphi_n - \delta} dx_1 \ldots dx_n$$
$$= \int_{|x_1| \leqq |x_2| \leqq \ldots \leqq |x_{n-1}|} |x_1|^{\varphi_1} \ldots |x_{n-1}|^{\varphi_{n-1}} \int_{|x_{n-1}| \leqq |x_n| \leqq 1} |x_n|^{\varphi_n - \delta} dx_n dx_1 \ldots dx_{n-1}$$

After evaluating the integral over x_n, we obtain

$$\frac{2}{\varphi_n - \delta + 1} \Big\{ \int_{|x_1| \leqq |x_2| \leqq \ldots \leqq |x_{n-1}| \leqq 1} |x_1|^{\varphi_1} \ldots |x_{n-1}|^{\varphi_{n-1}} dx_1 \ldots dx_{n-1}$$
$$- \int_{|x_1| \leqq |x_2| \leqq \ldots \leqq |x_{n-1}| \leqq 1} |x_1|^{\varphi_1} \ldots |x_{n-2}|^{\varphi_{n-2}} |x_{n-1}|^{\varphi_{n-1} + \varphi_n - \delta + 1} dx_1 \ldots dx_n \Big\}.$$

The induction hypothesis implies that the first integral in the braces is finite. To apply the induction hypothesis to the second integral, note that

$$(n-1) + \varphi_1 + \ldots + \varphi_{n-2} + (\varphi_{n-1} + \varphi_n - \delta + 1) = n + \varphi_1 + \ldots + \varphi_n - \delta = d_n - \delta > 0.$$

Thus the second integral is finite, which completes the proof of Lemma 4.3. □

Lemma 4.4. *Let σ be a permutation of $\{1, \ldots, m\}$ and let I be the largest index such that $\{L_{\tau_1}, \ldots, L_{\tau_I}\}$ is strongly independent. If*

$$d(P, \{L_{\tau_1}, \ldots, L_{\tau_k}\}) > 0, \qquad k = 1, \ldots, I,$$

then

$$\int_{E_\sigma^t} P(x) dx < \infty.$$

Proof. According to Lemma 4.2 it suffices to show

$$\int_{E_\sigma} |L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_r}|^{\Delta_r} dx < \infty,$$

where $\Delta_1, \ldots, \Delta_r$ are as defined in Lemma 4.2.

Case 1. $I=r$. Let $C_3=\max\{|L_j(x)|: x\in U_t,\ 1\leqq j\leqq m\}$. The last integral is majorized by

$$C_4 \int\limits_{|y_1|\leqq|y_2|\leqq\ldots\leqq|y_r|\leqq C_3} |y_1|^{\Delta_1}\ldots|y_r|^{\Delta_r}dy_1\ldots dy_r,$$

where C_4 is a constant obtained by integrating over $n-r$ extraneous variables. Note that $\Delta_1,\ldots,\Delta_r$ satisfy

$$k+\sum_{i=1}^{k}\Delta_i=d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\})>0, \qquad k=1,\ldots,r.$$

Hence Lemma 4.4 implies the conclusion in this case.

Case 2. $I<r$. In this case there are constants $a_1,\ldots,a_I$ so that

$$M_{\tau_{I+1}}=a_1M_{\tau_1}+\ldots+a_IM_{\tau_I}.$$

Then

$$\begin{aligned}L_{\tau_{I+1}}&=M_{\tau_{I+1}}+\theta_{\tau_{I+1}}\\&=a_1M_{\tau_1}+\ldots+a_IM_{\tau_I}+\theta_{\tau_{I+1}}\\&=a_1L_{\tau_1}+\ldots+a_IL_{\tau_I}+w,\end{aligned}$$

where $w=\theta_{\tau_{I+1}}-a_1\theta_{\tau_1}\ldots-a_I\theta_{\tau_I}$. Since $L_{\tau_{I+1}}$ is not a linear combination of $L_{\tau_1},\ldots,L_{\tau_I}$, if follows that $w\neq 0$. Thus we can choose a constant λ so that

$$|L_{\tau_{I+1}}|\leqq\left|\frac{w}{2}\right| \qquad \text{whenever } |L_{\tau_1}|\leqq\ldots\leqq|L_{\tau_I}|\leqq\lambda.$$

Since $|L_{\tau_I}|\leqq|L_{\tau_{I+1}}|\leqq\ldots\leqq|L_{\tau_r}|$ for $x\in E_\sigma^t$, there is a constant C_5 depending on λ and w so that

$$|L_{\tau_1}|^{\Delta_1}\ldots|L_{\tau_r}|^{\Delta_r}\leqq C_5|L_{\tau_1}|^{\Delta_1}\ldots|L_{\tau_I}|^{\Delta_I} \qquad \text{if } x\in E_\sigma^t, \quad |L_{\tau_I}|\leqq\lambda$$

and also a constant C_6 depending on λ so that

$$|L_{\tau_1}|^{\Delta_1}\ldots|L_{\tau_r}|^{\Delta_r}\leqq C_6|L_{\tau_1}|^{\Delta_1}\ldots|L_{\tau_{I-1}}|^{\Delta_{I-1}} \qquad \text{if } x\in E_\sigma^t, \quad |L_{\tau_I}|>\lambda.$$

Since $\{L_{\tau_1},\ldots,L_{\tau_I}\}$ and $\{L_{\tau_1},\ldots,L_{\tau_{I-1}}\}$ are strongly independent, the proof can be completed as in Case 1. □

Proof of Theorem 3.1. Suppose that the conditions of Theorem 3.1 hold. Let σ be a permutation of $\{1,\ldots,m\}$ and define I as in Lemma 4.4. The remark following Theorem 3.1 implies that $d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\})>0$, $k=1,\ldots,I$. Thus we can use Lemma 4.4 to conclude that $\int_{E_\sigma^t} P(x)dx<\infty$. Theorem 3.1 follows because U_t is the union over σ of the sets E_σ^t. □

5. Counting Powers

This section is devoted to "counting powers" in the function $p_\eta\colon \mathbb{R}^{2p}\to\mathbb{R}$ given by

$$\begin{aligned}p_\eta(x)=&|x_2+\ldots+x_{2p}|^{\eta-1}|x_2|^{\eta-1}|x_3|^{\eta-1}\ldots|x_{2p}|^{\eta-1}|x_1|^{-\alpha}|x_1+x_2|^{-\beta}|x_1+x_2+x_3|^{-\alpha}\\&\cdot|x_1+x_2+x_3+x_4|^{-\beta}\ldots|x_1+\ldots+x_{2p-1}|^{-\alpha}|x_1+\ldots+x_{2p}|^{-\beta},\end{aligned}$$

where $\alpha<1$, $\beta<1$ and $0<\eta<1$. The results are stated in Propositions 5.1 and 5.2. Introduce the set of linear functionals on $\mathbb{R}^{2p}$

$$T=\{x_2+\ldots+x_{2p}, x_2, x_3, \ldots, x_{2p}, x_1, x_1+x_2, \ldots, x_1+\ldots+x_{2p}\}.$$

For each $W\subset T$ we define the set $s\{W\}$ and the quantity $d(P_\eta, W)$ as in Sect. 3 and we say that W is an independent set if it is strongly independent. (Here W does not involve additive constants.)

Proposition 5.1. *Let $\alpha<1$, $\beta<1$ and let η satisfy $0<\eta<1$ and $n>(a+\beta)/2$. If $W\subset T$ is an independent set such that $|W|=2p-1$ and $W\subset\{x_2+\ldots+x_{2p}, x_2, x_3, \ldots, x_{2p}\}$, then $d(P_\eta, W)=2p\eta-1$.*

Proof. It is clear that if W satisfies the conditions of Proposition 5.1 then $s\{W\}=\{x_2+\ldots+x_{2p}, x_2, x_3, \ldots, x_{2p}\}$. Therefore

$$d(P_\eta, W)=(2p-1)+2p(\eta-1)=2p\eta-1. \quad \square$$

Proposition 5.2. *Let $\alpha<1$, $\beta<1$ and let η satisfy $0<\eta<1$ and $n>(\alpha+\beta)/2$. If $W\subset T$ is an independent set such that either $|W|\neq 2p-1$ or $W\not\subset\{x_2+\ldots+x_{2p}, x_2, x_3, \ldots, x_{2p}\}$, then $d(P_\eta, W)>0$.*

The rest of this section is devoted to the proof of Proposition 5.2.

In proving that proposition we can restrict ourselves to considering sets $W\subset T$ which do not contain $x_2+\ldots+x_{2p}$. To see this, assume that $x_2+\ldots+x_{2p}\in W$. Suppose first that the set $s\{W\}\setminus s\{W\setminus x_2+\ldots+x_{2p}\}$ contains some functional L other than $x_2+\ldots+x_{2p}$. Then we consider the set W' which is W with $x_2+\ldots+x_{2p}$ replaced by L, that is $W'=W\cup\{L\}\setminus\{x_2+\ldots+x_{2p}\}$. Clearly, $x_2+\ldots+x_{2p}\notin W'$. Furthermore, W' has the same span and cardinality as W. Therefore $d(P_\eta, W')=d(P_\eta, W)$. On the other hand, suppose that there is no such L. In this case we put $W'=W\setminus\{x_2+\ldots+x_{2p}\}$. We have $|W'|=|W|-1$ and $s\{W'\}=s\{W\}\setminus\{x_2+\ldots+x_{2p}\}$. Hence

$$d(P_\eta, W')=d(P_\eta, W)-1-(\eta-1)=d(P_\eta, W)-\eta<d(P_\eta, W).$$

Thus in either case there is a set W' which does not contain $x_2+\ldots+x_{2p}$ and satisfies $d(P_\eta, W')\leqq d(P_\eta, W)$. Hence we can assume that W does not contain $x_2+\ldots+x_{2p}$.

In proving Proposition 5.2 we can also restrict ourselves to sets $W\subset T$ which satisfy $\{x_k, x_1+\ldots+x_k\}\not\subset W$, $k=2,\ldots,2p$. For suppose that T does not satisfy this restriction. Let j be the largest k for which $\{x_k, x_1+\ldots+x_k\}\subset W$. Let $W'=W\cup\{x_1+\ldots+x_{j-1}\}\setminus\{x_1+\ldots+x_j\}$. Since the sets $\{x_j, x_1+\ldots+x_{j-1}\}$ and $\{x_j, x_1+\ldots+x_j\}$ have the same span and cardinality, it follows that $d(P_\eta, W')=d(P_\eta, W)$. It is clear that the largest value of k for which $\{x_k, x_1+\ldots+x_k\}\subset W'$ is at most $j-1$. After repeating this process at most $j-2$ more times we obtain a set W'' satisfying $d(P_\eta, W'')=d(P_\eta, W)$ and $\{x_k, x_1+\ldots+x_k\}\not\subset W'$, $k=2,\ldots,2p$. Thus we can restrict ourselves to sets W which do not contain both x_k and $x_1+\ldots+x_k$.

We will assume from now on that $W\subset T$ satisfies both of the above restrictions. To describe the sets W which we will be considering, it is helpful to think of the elements of $T\setminus\{x_2+\ldots+x_{2p}\}$ arranged in columns as follows:

$$x_1 \left| \begin{matrix} x_2 \\ x_1+x_2 \end{matrix} \right| \begin{matrix} x_3 \\ x_1+x_2+x_3 \end{matrix} \left| \dots \right| \begin{matrix} x_{2p} \\ x_1+\dots+x_{2p} \end{matrix} \Bigg| .$$

In the rest of this section we consider sets W which contain at most one element from each column. For any set $T' \subset T$ we say that T' contains the k^{th} column if $x_k \in T'$ or $x_1 + \dots + x_k \in T'$.

The proof of Proposition 5.2 involves three lemmas.

Lemma 5.3. *Suppose that W does not contain the k^{th} column. Then $s\{W\}$ does not contain the k^{th} column.*

Proof. We prove that neither x_k nor $x_1 + \dots + x_k$ is in $s\{W\}$. We distinguish two cases.

Case I. There is no $j > k$ such that $x_1 + \dots + x_j \in W$. In this case the conclusion of the lemma is clear since no element of W contains the summand x_k.

Case II. There exists $j > k$ such that $x_1 + \dots + x_j \in W$.

Suppose that j is the smallest index with this property. Then the only elements of W which contain the summand x_k are among $\{x_1 + \dots + x_j, x_1 + \dots + x_{j+1}, \dots, x_1 + \dots + x_{2p}\}$. Since $x_j \notin W$ these are also the only elements of W which contain the summand x_j. Thus in any linear combination of the elements of W the summands x_k and x_j appear with the same coefficient. Hence neither x_k nor $x_1 + \dots + x_k$ can be linear combinations of elements of W. This completes the proof of Lemma 5.3. □

We now partition W into blocks of contiguous columns. Any two blocks are separated by at least one column not in W. Formally, we will say that a set $B \subset W$ is a *block of columns*, if there exist $l_B < r_B$ such that

1) W contains neither column $l_B - 1$ nor column $r_B + 1$.
2) B contains column l_B through r_B and no other columns.

With this definition we obtain a partition $W = \bigcup_{j=1}^{n} B_j$, where each B_j is a block of columns. We will assume that B_j is to the left of B_{j+1} for each j.

Define the function $Q_\eta(x) = P_\eta(x) \cdot |x_2 + \dots + x_{2p}|^{1-\eta}$. It is clear that

$$d(P_\eta, W) = \begin{cases} d(Q_\eta, W), & \text{if } x_2 + \dots + x_{2p} \notin s\{W\} \\ n - 1 + d(Q_\eta, W) & \text{if } x_2 + \dots + x_{2p} \in s\{W\}. \end{cases}$$

Furthermore Lemma 5.3 implies that $d(Q_\eta, W) = \sum_{j=1}^{n} d(Q_\eta, B_j)$. Thus we have

$$d(P_\eta, W) = \begin{cases} \sum_{j=1}^{n} d(Q_\eta, B_j) & \text{if } x_2 + \dots + x_{2p} \notin s(W) \\ n - 1 + \sum_{j=1}^{n} d(Q_\eta, B_j) & \text{if } x_2 + \dots + x_{2p} \in s(W). \end{cases} \tag{5.1}$$

The next lemma is useful in determining the quantities $d(Q_\eta, B_j)$. A block of columns will be called *nonsimple* if it contains $x_1 + \dots + x_k$ for some $k \geq 1$.

Lemma 5.4. *Let B be a nonsimple block of columns. Put $l = l_B$ and $r = r_B$. Let m be the smallest k satisfying $x_1 + \dots + x_k \in B$.*

1) *If* $l \leqq j < m$, *then* $x_j \in s(B)$ *and* $x_1 + \ldots + x_j \notin s(B)$.

2) $x_m \notin s(B)$ *and* $x_1 + \ldots + x_m \in s(B)$.

3) *If* $m < j \leqq r$, *then* $x_j \in s(B)$ *and* $x_1 + \ldots + x_j \in s(B)$.

Proof. 1) Let $l \leqq j < m$. Since $j < m$ we have $\{x_l, x_{l+1}, \ldots, x_j\} \subset B$. Suppose that $x_1 + \ldots + x_j \in s(B)$. The identity $x_1 + \ldots + x_{l-1} = (x_1 + \ldots + x_j) - x_l - x_{l+1} - \ldots - x_j$ implies that $x_1 + \ldots + x_{l-1} \in s(B)$. This contradicts Lemma 5.3. We conclude that $x_1 + \ldots + x_j \notin s(B)$.

2) The definition of m implies that $x_1 + \ldots + x_m \in B$. Suppose that $x_m \in s(B)$. We have

$$x_1 + \ldots + x_{l-1} = (x_1 + \ldots + x_m) - x_l - x_{l+1} - \ldots - x_m,$$

again contradicting Lemma 5.3.

3) This is proven by induction. It is clear that if $x_1 + \ldots + x_j \in s(B)$ and B contains column $j+1$, then $\{x_{j+1}, x_1 + \ldots + x_{j+1}\} \subset s(B)$. To start the induction off, note that $x_1 + \ldots + x_m \in s(B)$ and B contains column $m+1$. This completes the proof of Lemma 5.4. □

If B is a simple block of columns, then $B \subset \{x_2, x_3, \ldots, x_{2p}\}$ and therefore

$$d(Q_\eta, B) = |B| + |B|(\eta - 1) = |B|\eta > 0. \tag{5.2}$$

To determine $d(Q_\eta, B)$ for a nonsimple block, we need to take into account the parities of the integers m and r introduced in the statement of Lemma 5.4. This is done in the next lemma. First define

$$\gamma_1 = (m-l)\eta + (r-m)\left[\eta - \frac{(\alpha+\beta)}{2}\right]$$

and

$$\gamma_2 = (m-l)\eta + (r-m+1)\left[\eta - \frac{(\alpha+\beta)}{2}\right].$$

Note that under the conditions of Proposition 5.2 we have $\gamma_1 \geqq 0$ and $\gamma_2 > 0$.

Lemma 5.5. *Suppose that the conditions of Proposition 5.2 hold. Let B be a nonsimple block of columns.*

1) *If m and r are both odd, then*

$$d(Q_\eta, B) = (1-\alpha) + \gamma_1 \geqq 1 - \alpha > 0.$$

2) *If m and r are both even, then*

$$d(Q_\eta, B) = (1-\beta) + \gamma_1 \geqq 1 - \beta > 0.$$

3) *If m and r have different parities, then*

$$d(Q_\eta, B) = (1-\eta) + \gamma_2 > 1 - \eta > 0.$$

Proof. Note that $d(Q_\eta, B)$ is equal to the cardinality of B plus the sum of the powers of all the elements of $s(B) \setminus \{x_2 + \ldots + x_{2p}\}$. The cardinality of B contributes $(r-l)+1$ to $d(Q_\eta, B)$.

According to Lemma 5.4, the set $s(B)\setminus\{x_2+\ldots+x_{2p}\}$ is equal to $W_1 \cup W_2$, where

$$W_1=\{x_l, x_{l+1}, \ldots, x_{m-1}, x_{m+1}, x_{m+2}, \ldots, x_r\}$$

and

$$W_2=\{x_1+\ldots+x_m, x_1+\ldots+x_{m+1}, \ldots, x_1+\ldots+x_r\}.$$

(When $m=1$ we let $W_1=\{x_2, \ldots, x_r\}$.)

Counting the powers associated with W_1 we obtain a contribution $(r-l)(\eta-1)=-(r-l)+(m-l)\eta+(r-m)\eta$.

Counting the powers associated with W_2 we obtain a contribution

$$-\alpha-\frac{(r-m)}{2}(\alpha+\beta) \quad \text{if } m,r \text{ are both odd}$$

$$-\beta-\frac{(r-m)}{2}(\alpha+\beta) \quad \text{if } m,r \text{ are both even}$$

$$-\frac{(r-m+1)}{2}(\alpha+\beta) \quad \text{if } m,r \text{ have different parities.}$$

Summing the appropriate contributions and using the inequalities $\alpha<1$, $\beta<1$, $\gamma_1 \geqq 0$ and $\gamma_2 \geqq 0$ we obtain the results of Lemma 5.5. □

Proof of Proposition 5.2. Suppose that the conditions of Proposition 5.2 hold and that the independent subset W of T also satisfies the restrictions described above. (Namely, W does not contain $x_2+\ldots+x_{2p}$ and $\{x_k, x_1+\ldots+x_k\} \not\subset W$, $k=2, \ldots, 2p$.) Relation (5.1), relation (5.2) and Lemma 5.5 imply that $d(P_\eta, W)>0$ if $x_2+\ldots+x_{2p} \notin s(W)$. To complete the proof, assume that $x_2+\ldots+x_{2p} \in s(W)$. This implies that $r_{B_n}=2p$ (where B_n is the rightmost block of W), because the summand x_{2p} appears only in the $2p^{\text{th}}$ column.

First we will show that B_n is nonsimple, that is, it contains $x_1+\ldots+x_k$ for some $k \geqq 1$. Put $l=l_{B_n}$. Put $l=1$, then $x_1 \in B_n$ and so B_n is nonsimple. If $l=2$ and B_n is simple, then $W=B_n=\{x_2, \ldots, x_{2p}\}$, contradicting the assumptions of the proposition. If $l>2$ and B_n is simple, then no element of W contains the summand x_{l-1}, contradicting the assumption that $x_2+\ldots+x_{2p} \in s(W)$. Thus B_n must be nonsimple.

Next we will show that $l_{B_1}=1$. Since B_n is nonsimple, Lemma 5.4 shows that $x_1+\ldots+x_{2p} \in s(W)$. Since we have assumed that $x_2+\ldots+x_{2p} \in s(W)$, it follows that $x_1 \in s(W)$. Thus we must have $l_{B_1}=1$ in order to avoid contradicting Lemma 5.3.

To complete the proof, we distinguish two cases, according to whether W consists of a single block or more than one block.

Case I. $n=1$. In this case we have only one block B_1 satisfying $l_{B_1}=m_{B_1}=1$ and $r_{B_1}=2p$. Lemma 5.5 implies that $d(Q_\eta, B_1)=1-\eta+\gamma_2$. According to (5.1),

$$d(P_\eta, W)=d(Q_\eta, B_1)+\eta-1=\gamma_2>0.$$

Case II. $n>1$. We again have $l_{B_1}=m_{B_1}=1$. Thus either Part 1 or Part 3 of Lemma 5.5 applies. Hence $d(Q_\eta, B_1) \geqq 1-\alpha$ or $d(Q_\eta, B_1) \geqq 1-\eta$.

Since $r_{B_n}=2p$ and B_n is nonsingular, either Part 2 or Part 3 of Lemma 5.5 applies to B_n. Thus $d(Q_\eta, B_n)\geqq 1-\beta$ or $d(Q_\eta, B_n)\geqq 1-\eta$.

The proof can now be completed as follows. According to (5.2) and Lemma 5.5, we have $d(Q_\eta, B_j)>0, j=1,\ldots,n$. Thus by (5.1),

$$\begin{aligned} d(P_\eta, W) &= \eta-1+\sum_{j=1}^{n} d(Q_\eta, B_j) \\ &\geqq n-1+d(Q_\eta, B_1)+d(Q_\eta, B_n). \end{aligned}$$

If $d(Q_\eta, B_1)\geqq 1-\eta$, then $d(P_\eta, W)\geqq d(Q_\eta, B_n)>0$. Similarly, $d(P_\eta, W)>0$ if $d(Q_\eta, B_n)\geqq 1-\eta$. Therefore we can assume that $d(Q_\eta, B_1)\geqq 1-\alpha$ and $d(Q_\eta, B_n)\geqq 1-\beta$. Then $d(P_\eta, W)\geqq \eta-1+(1-\alpha)+(1-\beta)=1-\eta+2[\eta-(\alpha+\beta)/2]>1-\eta>0$. This completes the proof of Proposition 5.2. □

6. Applications of Power Counting

In this section, we establish Propositions 6.1 and 6.2, which will be used in the proof of Theorem 1.

For each integer $N\geqq 1$ define the function

$$h_N(z)=\begin{cases} \min\left(\dfrac{1}{|z+2\pi|}, N\right) & -2\pi\leqq z\leqq -\pi \\ \min\left(\dfrac{1}{|z|}, N\right) & -\pi\leqq z\leqq \pi \\ \min\left(\dfrac{1}{|z-2\pi|}, N\right) & \pi\leqq z\leqq 2\pi \end{cases}$$

and the function $f_N\colon \mathbb{R}^{2p}\to\mathbb{R}$ by

$$\begin{aligned} f_N(y) &= h_N(y_1-y_{2p})h_N(y_2-y_1)h_N(y_3-y_2)\ldots h_N(y_{2p}-y_{2p-1}) \\ &\quad \cdot |y_1|^{-\alpha}|y_2|^{-\beta}|y_3|^{-\alpha}\ldots|y_{2p}|^{-\beta} \end{aligned}$$

where $\alpha<1$ and $\beta<1$. Given $t>0$ put $U_t=[-t,t]^{2p}$ and $V=\{y\in\mathbb{R}^{2p}\colon |y_1|\leqq \frac{1}{2}|y_2|\}$.

The following results are useful in studying the behavior of $\int_{U_\pi} f_N(y)\,dy$ as $N\to\infty$.

Proposition 6.1. *Let $\alpha<1$ and $\beta<1$.*

a) *If $\alpha+\beta\leqq 0$, then as $N\to\infty$,*

$$\int_{U_\pi\cap V} f_N(y)\,dy=O(N^\varepsilon)$$

for every $\varepsilon>0$.

b) *If $\alpha+\beta>0$, then as $N\to\infty$,*

$$\int_{U_\pi\cap V} f_N(y)\,dy=O(N^{p(\alpha+\beta)+\varepsilon})$$

for every $\varepsilon>0$.

Proposition 6.2. *Let* $\alpha<1$ *and* $\beta<1$.

a) *If* $p(\alpha+\beta)<1$, *then*

$$\lim_{t\to 0}\limsup_{N\to\infty}\frac{\int_{U_t} f_N(y)\,dy}{N}=0.$$

b) *If* $p(\alpha+\beta)\geqq 1$, *then as* $N\to\infty$

$$\int_{U_\pi} f_N(y)\,dy=O(N^{p(\alpha+\beta)+\varepsilon})$$

for every $\varepsilon>0$.

In order to prove Propositions 6.1 and 6.2 we need to put the problem into the framework described in Sect. 4. Choose η satisfying $0<\eta<1$. If $1/N\leqq|z|\leqq\pi$ then we have

$$h_N(z)=\frac{1}{|z|}\leqq\frac{1}{|z|}N^\eta|z|^\eta=N^\eta|z|^{\eta-1}.$$

If $|z|<1/N$ then

$$h_N(z)=N^\eta N^{1-\eta}\leqq N^\eta|z|^{\eta-1}.$$

Thus

$$h_N(z)\leqq N^\eta|z|^{\eta-1},\qquad -\pi\leqq z\leqq\pi,\ 0<\eta<1.$$

This implies $f_N(y)\leqq f_{N,\eta}(y)$, where $f_{N,\eta}(y)$ is defined as $f_N(y)$ with h_N replaced by

$$h_{N,\eta}(z)=\begin{cases}N^\eta|z+2\pi|^{\eta-1} & -2\pi\leqq z\leqq-\pi\\ N^\eta|z|^{\eta-1} & -\pi\leqq z\leqq\pi\\ N^\eta|z-2\pi|^{\eta-1} & \pi\leqq z\leqq 2\pi.\end{cases}$$

To study $\int_{U_t} f_{N,\eta}(y)\,dy$ we make the change of variable $x_1=y_1$, $x_k=y_k-y_{k-1}$, $k=2,\ldots,2p$. Thus we define

$$f'_N(x)=h_N(x_2+\ldots+x_{2p})h_N(x_2)\ldots h_N(x_{2p})$$
$$\cdot|x_1|^{-\alpha}|x_1+x_2|^{-\beta}|x_1+x_2+x_3|^{-\alpha}\ldots|x_1+\ldots+x_{2p}|^{-\beta}$$

and $f'_{N,\eta}(x)$ in the same way, with h_N replaced by $h_{N,\eta}$. Define the set U'_t so that $\int_{U'_t} f'_N(x)\,dx=\int_{U_t} f_N(y)\,dy$ and let $V'=\{x\colon |x_1|\leqq\frac{1}{2}|x_1+x_2|\}$.

Note that if $y\in U_\pi$ and $|y_k-y_{k-1}|\geqq\frac{3\pi}{2}$, then $|y_{k-1}|\geqq\pi/2$ and $|y_k|\geqq\pi/2$. Hence for $x\in U'_\pi$

$$|x_1+\ldots+x_{k-1}|\geqq\tfrac{\pi}{2}\quad\text{if } |x_k+2\pi|\leqq\tfrac{\pi}{2},\tag{6.1a}$$

$$|x_1+\ldots+x_{k-1}|\geqq\tfrac{\pi}{2}\quad\text{if } |x_k-2\pi|\leqq\tfrac{\pi}{2},\tag{6.1b}$$

$$|x_1+\ldots+x_k|\geqq\tfrac{\pi}{2}\quad\text{if } |x_k+2\pi|\leqq\tfrac{\pi}{2},\tag{6.2a}$$

and

$$|x_1+\ldots+x_k|\geqq\tfrac{\pi}{2}\quad\text{if } |x_k-2\pi|\leqq\tfrac{\pi}{2}.\tag{6.2b}$$

It is clear that if $y\in U_\pi\cap V$ then $|y_2-y_1|\leqq\frac{3}{2}|y_2|\leqq 3\pi/2$. Thus

$$|x_2|\leqq\frac{3\pi}{2}; \qquad x\in U'_\pi\cap V'. \tag{6.3}$$

In order to apply the lemmas of Sect. 4 introduce the functionals

$$\begin{aligned} M_1(x)&=x_2+\ldots+x_{2p}\\ M_2(x)&=x_2\\ &\vdots\\ M_{2p}(x)&=x_{2p}\\ M_{2p+1}(x)&=x_1\\ M_{2p+2}(x)&=x_1+x_2\\ &\vdots\\ M_{4p}(x)&=x_1+\ldots+x_{2p}.\end{aligned}$$

Choose $\{\theta_1,\ldots,\theta_{2p}\}\subset\{-2\pi,0,2\pi\}$ and put $\theta_{2p+1}=\theta_{2p+2}=\ldots=\theta_{4p}=0$. Define $L_j(x)=M_j(x)+\theta_j$, $j=1,\ldots,4p$. Let U''_t be the subset of U'_t on which $f'_{N,\eta}(x)=N^{2p\eta}P'_\eta(x)$, where

$$\begin{aligned} P'_\eta(x)=&|L_1|^{\eta-1}|L_2|^{\eta-1}\ldots|L_{2p}|^{\eta-1}\\ &\cdot|L_{2p+1}|^{-\alpha}|L_{2p+2}|^{-\beta}\ldots|L_{4p}|^{-\beta}.\end{aligned}$$

In the proofs of Propositions 6.1 and 6.2 we use the notation introduced in Sect. 3. For example $T=\{L_1,\ldots,L_{4p}\}$ and r is the rank of T.

Fixing a permutation $\sigma=(\sigma_1,\ldots,\sigma_{4p})$ of $\{1,\ldots,4p\}$ we define

$$E^t_\sigma=\{x\in U''_t: |L_{\sigma_1}(x)|\leqq\ldots\leqq|L_{\sigma_{4p}}(x)|\}$$

and, as in Sect. 4 construct a basis $\{L_{\tau_1},\ldots,L_{\tau_r}\}$ for T satisfying

$$|L_{\tau_1}|\leqq|L_{\tau_2}|\leqq\ldots\leqq|L_{\tau_r}|, \qquad x\in E^t_\sigma.$$

In proving Propositions 6.1 and 6.2 it suffices to show that the conclusions hold with U_t replaced by E^t_σ, V replaced by V' and $f_N(y)$ replaced by $f'_N(x)$ or $f'_{N,\eta}(x)$.

Proof of Proposition 6.1. Fix η satisfying $0<\eta<1$ and $\eta>(\alpha+\beta)/2$. Since $f'_{N,\eta}(x)=N^{2p\eta}P'_\eta(x)$, both parts of Proposition 6.1 will follow if we show

$$\int_{E^\pi_\sigma\cap V'} P'_\eta(x)\,dx<\infty. \tag{6.4}$$

To show (6.4) we distinguish two cases.

Case I. $\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\}\not\subset\{L_1,\ldots,L_{2p}\}$.

In this case we will show that in fact $\int_{E^\pi_\sigma} P'_\eta(x)\,dx<\infty$.

Subcase I.1. $\theta_1=\ldots=\theta_{2p}=0$.

In this case $P'_\eta(x)=P_\eta(x)$, where $P_\eta(x)$ is as defined in Sect. 5. Thus Proposition 5.2 and Lemma 4.4 imply $\int_{E^\pi_\sigma} P'_\eta(x)\,dx<\infty$.

Subcase I.2. $\theta_j \neq 0$ for some j.

Here some of the dimensions may be non-positive, but relations (6.1)-(6.3) will allow us to deal with that situation. If $d(P_\eta', \{L_{\tau_1}, \ldots, L_{\tau_k}\}) > 0$, $k = 1, \ldots, 2p$, then Lemma 4.4 implies that $\int_{E_\sigma^\pi} P_\eta'(x)\,dx < \infty$. Otherwise put $W = \{L_{\tau_1}, \ldots, L_{\tau_k}\}$, where k is the smallest index satisfying $d(P_\eta', \{L_{\tau_1}, \ldots, L_{\tau_k}\}) \leqq 0$. As in Sect. 5, $s(W)$ consists of a collection of blocks of columns, plus possibly L_1. Let B be a block whose contribution to $d(P_\eta', W)$ is non-positive. Then it is clear that $\theta_j \neq 0$ for some j satisfying $l_B \leqq j \leqq r_B$ and that $x_1 + \ldots + x_m \in s(B)$ for some m satisfying $l_B \leqq m \leqq r_B$. (For $d(P_\eta', B)$ would be positive if there were no such j (Lemma 5.5) or if there were no such m (direct computation).)

Let m be the smallest index with this property. We distinguish two cases, in both of which we will show that $P_\eta'(x)$ is at most a constant times $|L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_{k-1}}|^{\Delta_{k-1}}$, with Δ_i defined as in Lemma 4.2.

Subcase I.2.i. Some j satisfies $m < j \leqq r_B$, $L_j \in s(B)$ and $\theta_j \neq 0$.

Let j be the smallest index with this property. Then it is clear that $x_1 + \ldots + x_{j-1} \in s(B)$. By Lemma 4.2, $P_\eta'(x)$ is at most a constant times $|L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_r}|^{\Delta_r}$. Since L_j and $x_1 + \ldots + x_{j-1}$ are in $s\{L_{\tau_1}, \ldots, L_{\tau_k}\}$ and $|L_{\tau_1}| \leqq \ldots \leqq |L_{\tau_r}|$ on E_σ^π, Lemma 4.1 implies that there is a constant C such that if $x \in E_\sigma^\pi$

$$C|x_1 + \ldots + x_{j-1}| \leqq |L_{\tau_k}|$$

and

$$C|L_j| \leqq |L_{\tau_k}|.$$

Thus if $x \in E_\sigma^\pi$ and $|L_j| \geqq \pi/2$, L_{τ_k} is bounded away from 0. If $|L_j| \leqq \pi/2$, relation (6.1) implies that L_{τ_k} is bounded away from 0, since $L_j = x_2 \pm 2\pi$. So L_{τ_k} is bounded away from 0 on E_σ^π, which implies that $L_{\tau_{k+1}}, \ldots, L_{\tau_r}$ are bounded away from 0 also. It follows that $P_\eta'(x)$ is at most a constant times $|L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_{k-1}}|^{\Delta_{k-1}}$ on E_σ^π.

Subcase I.2.ii. No j satisfies $m < j \leqq r_B$, $L_j \in s(B)$ and $\theta_j \neq 0$.

Note that $x_m \notin s(B)$, for otherwise we would have $x_1 + \ldots + x_{m-1} \in s(B)$, contracting our choice of m. Since $L_m = x_m + \theta_m$ it follows that either $L_m \notin s(B)$ or $\theta_m \neq 0$. If $L_m \notin s(B)$, then

$$s(B) = \{L_{l_B}, \ldots, L_{m-1}, x_{m+1}, \ldots, x_{r_B}, x_1 + \ldots + x_m, x_1 + \ldots + x_{m+1}, \ldots, x_1 + \ldots + x_{r_B}\}.$$

We see that $d(P_\eta', B)$ would not change if $L_{l_B} = x_{l_B}, \ldots, L_{m-1} = x_{m-1}$, but with this change it becomes the dimension of the block described in Lemma 5.4, which is positive by Lemma 5.5, contradicting our assumption. Hence $L_m \in s(B)$ and $\theta_m \neq 0$. But then relation (6.2) allows us to argue as in Subcase I.2.i that $|L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_{k-1}}|^{\Delta_{k-1}}$ provides a majorant for $P_\eta'(x)$.

To complete the proof in Subcase I.2 it suffices to show that $|L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_{k-1}}|^{\Delta_{k-1}}$ is integrable on E_σ^π. But this follows from Lemma 4.4, since $d(P_\eta', \{L_{\tau_1}, \ldots, L_{\tau_j}\}) > 0$, $j = 1, \ldots, k-1$. This establishes that $\int_{E_\sigma^\pi} P_\eta'(x)\,dx < \infty$ in Subcase I.2.

Case II. $\{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\} \subset \{L_1, \ldots, L_{2p}\}$.

Subcase II.1. $\theta_1 = \theta_2 + \ldots + \theta_{2p}$.

In this case we have $s\{L_{\tau_1}, \ldots, L_{\tau_k}\} = \{L_{\tau_1}, \ldots, L_{\tau_k}\}$, $k = 1, \ldots, 2p-2$, and $s\{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\} = \{L_1, \ldots, L_{2p}\}$.

Suppose first that $\theta_2 \neq 0$, so that $L_2 = x_2 \pm 2\pi$. Then (6.3) implies that $|L_2| \geqq \pi/2$ on $E_\sigma^\pi \cap V'$. According to Lemma 4.2, $P_\eta'(x)$ is at most a constant times $|L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_{2p}}|^{\Delta_{2p}}$. Let $j < 2p$ be the integer satisfying $L_2 \in T_j$. Then Lemma 4.1 implies that $|L_{\tau_{2p}}| \geqq \ldots \geqq |L_j| \geqq \frac{1}{C_\sigma} |L_2| \geqq \frac{\pi}{2C_\sigma}$ for $x \in E_\sigma^\pi$. Hence $P_\eta'(x)$ is at most a constant times $|L_{\tau_1}|^{\Delta_1} \ldots |L_{\tau_{j-1}}|^{\Delta_{j-1}}$ for $x \in E_\sigma^\pi \cap V'$. Since $j-1 \leqq 2p-2$, $d(P_\eta', \{L_{\tau_1}, \ldots, L_{\tau_k}\}) > 0$, $k = 1, \ldots, j-1$. Therefore Lemma 4.4 implies that the integral of this product over $E_\sigma^\pi \cap V'$ is finite, establishing (6.4) when $\theta_2 \neq 0$.

Now suppose $\theta_2 = 0$, so that $L_2 = x_2$. Define

$$L_q = \{L_1, \ldots, L_{2p}\} \backslash \{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\}.$$

Then

$$T_k = \{L_{\tau_k}\}, \qquad k = 1, \ldots, 2p-2$$

and

$$T_{2p-1} = \{L_1, \ldots, L_{2p}\} \backslash \{L_{\tau_1}, \ldots, L_{\tau_{2p-2}}\} = \{L_{\tau_{2p-1}}, L_q\}.$$

The next step is to use this to establish

$$|L_{2p+1}| \leqq |L_q|, \qquad x \in E_\sigma^t \cap V'. \tag{6.5}$$

Since $L_q \in T_{2p-1}$, Lemma 4.1b implies that $|L_{\tau_{2p-1}}| \leqq |L_q|$ for $x \in E_\sigma^\pi$. Hence $|L_{\tau_1}| \leqq |L_{\tau_2}| \leqq \ldots \leqq |L_{\tau_{2p-1}}| \leqq |L_q|$ for $x \in E_\sigma^\pi$. Therefore on E_σ^π we have $L_q = \max\{|L_1|, |L_2|, \ldots, |L_{2p}|\}$. In particular $|L_2| \leqq |L_q|$ on E_σ^π. For $x \in V'$ we have $|L_{2p+1}| = |x_1| \leqq |x_2| = |L_2|$. These last two inequalities imply (6.5).

Since

$$P_\eta'(x) = |L_{\tau_1}|^{\eta-1} \ldots |L_{\tau_{2p-1}}|^{\eta-1} |L_q|^{\eta-1} |L_{2p+1}|^{-\alpha} |L_{2p+2}|^{-\beta} \ldots |L_{4p}|^{-\beta},$$

relation (6.5) implies that $P_\eta'(x) \leqq P_\eta''(x)$, $x \in E_\sigma^\pi \cap V'$, where

$$P_\eta''(x) = |L_{\tau_1}|^{\eta-1} \ldots |L_{\tau_{2p-1}}|^{\eta-1} |L_{2p+1}|^{-\alpha+\eta-1} |L_{2p+2}|^{-\beta} \ldots |L_{4p}|^{-\beta}.$$

Hence (6.4) will follow if we show

$$\int_{E_\sigma^\pi} P_\eta''(x)\, dx < \infty. \tag{6.6}$$

To show (6.6) we use Lemma 4.4. For $k \leqq 2p-2$ we have $s\{L_{\tau_1}, \ldots, L_{\tau_k}\} = \{L_{\tau_1}, \ldots, L_{\tau_k}\}$, from which it follows that

$$d(P_\eta'', \{L_{\tau_1}, \ldots, L_{\tau_k}\}) = k + k(\eta - 1) = k\eta > 0, \qquad k \leqq 2p-2.$$

Since $s\{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\} = \{L_1, \ldots, L_{2p}\}$ we have

$$d(P_\eta'', \{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\}) = (2p-1) + (2p-1)(\eta - 1) = (2p-1)\eta > 0.$$

Finally, if $\theta_1 = \ldots = \theta_{2p} = 0$, then

$$d(P_\eta'', \{L_{\tau_1}, \ldots, L_{\tau_{2p}}\}) = 2p + 2p(\eta - 1) - p\alpha - p\beta = 2p\left[\eta - \frac{(\alpha+\beta)}{2}\right] > 0.$$

On the other hand, if some $\theta_j \neq 0$ then $\{x_1 + \ldots + x_m, L_1, \ldots, L_{2p}\} \subset s\{L_{\tau_1}, \ldots, L_{\tau_{2p}}\}$ for some m and we can argue as in Subcase I.2 that P_η'' is at most a constant times $|L_{\tau_1}|^{\eta-1} \ldots |L_{\tau_{2p-1}}|^{\eta-1}$. Either way, (6.6) follows from Lemma 4.4, completing the proof in Subcase II.1.

Subcase II.2. $\theta_1 \neq \theta_2 + \ldots + \theta_{2p}$.

In this case we will show that $\int_{E_\sigma^\pi} P_\eta'(x)\,dx < \infty$. We have

$$s\{L_{\tau_1}, \ldots, L_{\tau_k}\} = \{L_{\tau_1}, \ldots, L_{\tau_k}\}, \qquad k = 1, \ldots, 2p-1.$$

Put $L_q = \{L_1, \ldots, L_{2p}\} \setminus \{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\}$.

Suppose first that $L_{\tau_{2p}} = L_q$. Then $s\{L_{\tau_1}, \ldots, L_{\tau_{2p}}\} = \{L_1, \ldots, L_{2p}\}$, so that $d\{L_{\tau_1}, \ldots, L_{\tau_k}\} = k(\eta - 1) + k = k\eta > 0$, $k = 1, \ldots, 2p$. Hence Lemma 4.4 implies that $\int_{E_\sigma^\pi} P_\eta'(x)\,dx < \infty$.

Now suppose that $L_{\tau_{2p}} \neq L_q$ so $L_{\tau_{2p}} = x_1 + \ldots + x_m$ for some $1 \leqq m \leqq 2p$. Since $\theta_j \neq 0$ for some j, we can argue as in Subcase I.2 that P_η' is at most a constant times $|L_{\tau_1}|^{\eta-1} \ldots |L_{\tau_{2p-1}}|^{\eta-1}$. Since $d(P_\eta', \{L_{\tau_1}, \ldots, L_{\tau_k}\}) > 0$, $k = 1, \ldots, 2p-1$, this product is integrable over E_σ^π, completing the proof of Proposition 6.1. □

Proof of Proposition 6.2. Let σ be a permutation of $\{1, \ldots, 4p\}$.

Case I. $\{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\} \not\subset \{L_1, \ldots, L_{2p}\}$.

As in Case I of the proof of Proposition 6.1, we show that $\int_{E_\sigma^\pi} P_\eta'(x)\,dx < \infty$ if $\eta > (\alpha + \beta)/2$. Since $f_{N,\eta}'(x) = N^{2p\eta} P_\eta'(x)$, both parts of Proposition 6.2 follow by choosing η appropriately.

Case II. $\{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\} \subset \{L_1, \ldots, L_{2p}\}$.

Subcase II.1. $\theta_1 = \theta_2 + \ldots + \theta_{2p}$.

Proof of Part a in Subcase II.1. We saw in the proof of Proposition 6.1 that

$$|L_{\tau_1}| \leqq |L_{\tau_2}| \leqq \ldots \leqq |L_{\tau_{2p-1}}| \leqq |L_q|, \qquad x \in E_\sigma^t, \tag{6.7}$$

where

$$L_q = \{L_1, \ldots, L_{2p}\} \setminus \{L_{\tau_1}, \ldots, L_{\tau_{2p-1}}\}.$$

Since $h_N(z) \leqq N$, we have

$$f_N'(x) \leqq N^{2p} |L_{2p+1}|^{-\alpha} |L_{2p+2}|^{-\beta} \ldots |L_{4p}|^{-\beta}, \qquad x \in \mathbb{R}^{2p}.$$

If $\theta_1 = \ldots = \theta_{2p} = 0$ then $T_{2p} = \{L_{2p+1}, \ldots, L_{4p}\}$ and Lemma 4.1 implies that $|L_{2p+1}|^{-\alpha} |L_{2p+2}|^{-\beta} \ldots |L_{4p}|^{-\beta}$ is at most a constant times $|L_{\tau_{2p}}|^{-p(\alpha+\beta)}$ for $x \in E_\sigma^\pi$. On the other hand if some $\theta_j \neq 0$ the fact that $L_{\tau_{2p}} = x_1 + \ldots + x_m$ for some m allows us to argue as in Subcase I.2.i that $L_{\tau_{2p}}$ is bounded away from 0 on E_σ^π, which implies that $|L_{2p+1}|^{-\alpha} \ldots |L_{4p}|^{-\beta}$ is bounded on E_σ^π. Thus this product is at most a constant times $M(L_{\tau_{2p}})$, where

$$M(z)=\max(|z|^{-p(\alpha+\beta)}, 1).$$

Hence there is a constant C so that

$$f_N'(x)\leqq CN^{2p}M(L_{\tau_{2p}}), \qquad x\in E_\sigma^\pi. \tag{6.8}$$

Define the sets

$$G_{N,0}=E_\sigma^t\cap\left\{\frac{1}{N}\leqq|L_{\tau_1}|\right\},$$

$$G_{N,j}=E_\sigma^t\cap\left\{|L_{\tau_j}|\leqq\frac{1}{N}\leqq|L_{\tau_{j+1}}|\right\}, \qquad j=1,\dots,2p-2$$

and

$$G_{N,2p-1}=E_\sigma^t\cap\left\{|L_{\tau_{2p-1}}|\leqq\frac{1}{N}\right\}.$$

Because of (6.7) it is clear that

$$E_\sigma^t=G_{N,0}\cup G_{N,1}\cup\dots\cup G_{N,2p-1}.$$

Thus it suffices to show that the conclusion of Part a holds with U_t replaced by $G_{N,j}$, $j=0,\dots,2p-1$. Define also the sets

$$K_{N,j}=\left\{|L_{\tau_k}|\leqq\frac{1}{N}, k=1,\dots,j\right\}$$
$$\cap\left\{\frac{1}{N}\leqq|L_{\tau_k}|, L_{\tau_k}\in A_t, k=j+1,\dots,2p-1\right\}\cap\{L_{\tau_{2p}}\in A_t\},$$

where $A_t=\bigcup_{k=1}^{4p}\{L_k(x)\colon x\in E_\sigma^t\}$. Note that the measure of A_t tends to 0 as $t\to 0$. In view of (6.7) we have $G_{N,j}\subset K_{N,j}$. To prove Part a we distinguish two subcases according to whether $j=2p-1$ or not.

Subcase II.1.i. $j=2p-1$. From (6.8) we conclude that $\int_{G_{N,2p-1}} f_N'(x)dx$ is at most a constant times

$$N^{2p}\int_{G_{N,2p-1}} M(L_{\tau_{2p}})\leqq N^{2p}\int_{K_{N,2p-1}} M(L_{\tau_{2p}}),$$

where on $K_{N,2p-1}$, we have $|L_{\tau_k}|\leqq 1/N$ for $k=1,\dots,2p-1$ and $L_{\tau_{2p}}\in A_t$. We see, on making the appropriate change of variable, that the right hand side is majorized by

$$N^{2p}\left[\int_{-1/N}^{1/N} dw\right]^{2p-1}\int_{A_t} M(z)dz.$$

Therefore

$$\limsup_N \frac{1}{N}\int_{G_{N,2p-1}} f_N'(x)dx$$

is at most a constant times $\int_{A_t} M(z)dz$, which implies the conclusion of Part a.

Subcase II.1.ii. $j<2p-1$. For $x\in G_{N,j}$ we have

$$h_N(L_{\tau_k})=N, \quad k=1,\dots,j,$$

$$h_N(L_{\tau_k})=|L_{\tau_k}|^{-1}, \quad k=j+1,\dots,2p-1,$$

and, by (6.7)

$$h_N(L_q)=|L_q|^{-1}\leqq|L_{\tau_{2p-1}}|^{-1}.$$

These facts in combination with (6.8) yield

$$f_N'(x)\leqq CN^j|L_{\tau_{j+1}}|^{-1}\dots|L_{\tau_{2p-2}}|^{-1}|L_{\tau_{2p-1}}|^{-2}M(L_{\tau_{2p}}), \qquad x\in G_{N,j}.$$

According to (6.7), $|L_{\tau_k}|\leqq|L_{\tau_{2p-1}}|$, $k=j+1,\dots,2p-1$, and thus $f_N'(x)$ is at most

$$CN^j\left\{\prod_{k=j+1}^{2p-1}|L_{\tau_k}|^{-1-\frac{1}{2p-1-j}}\right\}M(L_{\tau_{2p}})$$

for $x\in G_{N,j}$. Integrating this expression over $K_{N,j}$ we have at most a constant times

$$N^j\left(\int_{-1/N}^{1/N}dv\right)^j\left(\int_{\frac{1}{N}\leqq w,\,w\in A_t}w^{-\{1+\frac{1}{2p-1-j}\}}dw\right)^{2p-1-j}\int_{A_t}M(z)\,dz.$$

The first integral in brackets is $2N^{-1}$. The second is $O(N^{1/(2p-1-j)})$. So the whole expression is $O(N)\int_{A_t}M(z)\,dz$. This concludes the proof of Part a in Subcase II.1.

Proof of Part b in Subcase II.1. Fix $\varepsilon>0$. Under the conditions of Part b we can choose η satisfying $0<\eta<1$, $\eta>(\alpha+\beta)/2$ and $1<2p\eta<p(\alpha+\beta)+\varepsilon$. Thus it suffices to show that $\int_{E_\sigma^\pi}P_\eta'(x)\,dx<\infty$ under these conditions.

First suppose that $\theta_1=\theta_2=\dots=\theta_{2p}=0$. Then Propositions 5.1 and 5.2 imply that $d(P_\eta',W)>\min(2p\eta-1,0)$ for every strongly independent $W\subset T$. Since $2p\eta>1$, Theorem 3.1 implies the desired conclusion in this case.

On the other hand if some $\theta_j\neq0$, then some dimensions may be negative. However, since $2p\eta>1$, $d(P_\eta',W)>0$ whenever $W\subset\{L_1,\dots,L_{2p}\}$. Hence if k is the smallest index satisfying $d(P_\eta',\{L_{\tau_1},\dots,L_{\tau_k}\})\leqq0$, then we must have $x_1+\dots+x_m\in s\{L_{\tau_1},\dots,L_{\tau_k}\}$ for some m, and thus we can argue as Subcase I.2 of the proof of Proposition 6.1 that P_η' is at most a constant times $|L_{\tau_1}|^{\Delta_1}\dots|L_{\tau_{k-1}}|^{\Delta_{k-1}}$, so that Lemma 4.4 can be used to complete the proof of Part b in Subcase II.1.

Subcase II.2. $\theta_1\neq\theta_2+\dots+\theta_{2p}$.

As in Subcase II.2 of the proof of Proposition 6.1, we have

$$\int_{E_\sigma^\pi}P_\eta'(x)\,dx<\infty \qquad \text{if } \eta>(\alpha+\beta)/2.$$

Since $f_{N,\eta}'(x)=N^{2p\eta}P_\eta'(x)$, both parts of Proposition 6.2 follow by choosing η appropriately. □

7. Proof of Theorem 1

The proof requires a lemma in addition to Propositions 6.1 and 6.2. We use the notation introduced in Sect. 2 prior to the statement of Theorem 1. Fix $p \geqq 1$ and note that

$$\begin{aligned}\operatorname{Tr}(R_N A_N)^p &= \sum_{j_1=0}^{N-1} \cdots \sum_{j_{2p}=0}^{N-1} r_{j_1-j_2} a_{j_2-j_3} r_{j_3-j_4} \ldots a_{j_{2p}-j_1} \\ &= \sum_{j_1=0}^{N-1} \cdots \sum_{j_{2p}=0}^{N-1} \left(\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{i(j_1-j_2)y_1} e^{i(j_2-j_3)y_2} \ldots e^{i(j_{2p}-j_1)y_{2p}} \right. \\ &\qquad \left. \cdot f(y_1) g(y_2) f(y_3) \ldots g(y_{2p}) \, dy_1 \ldots dy_{2p} \right) \\ &= \int_{U_\pi} P_N(y) Q(y) \, dy, \end{aligned} \tag{7.1}$$

where

$$P_N(y) = \sum_{j_1=0}^{N-1} \cdots \sum_{j_{2p}=0}^{N-1} e^{i(j_1-j_2)y_1} e^{i(j_2-j_3)y_2} \ldots e^{i(j_{2p}-j_1)y_{2p}},$$

$$Q(y) = f(y_1) g(y_2) f(y_3) \ldots g(y_{2p}),$$

and

$$U_t = [-t, t]^{2p}.$$

To state the lemma, introduce the diagonal

$$D = \{ y \in U_\pi \colon y_1 = y_2 = \ldots = y_{2p} \}.$$

Let μ be the measure on U_π which is concentrated on D and satisfies $\mu\{y\colon a \leqq y_1 = y_2 = \ldots = y_{2p} \leqq b\} = b - a$ for all $-\pi \leqq a \leqq b \leqq \pi$. Thus μ is Lebesgue measure on D, normalized so that $\mu(D) = 2\pi$.

Lemma 7.1. *Define the measure* μ_N *on* U_π *by*

$$\mu_N(E) = \frac{1}{(2\pi)^{2p-1} N} \int_E P_N(y) \, dy, \qquad E \subset U_\pi.$$

Then μ_N *converges weakly to* μ *as* $N \to \infty$.

Proof. Since U_π is compact, it suffices to show that the Fourier coefficients of μ_N converge to those of μ. Fixing integers $n_1, n_2, \ldots, n_{2p}$, the corresponding Fourier coefficient of μ is

$$\begin{aligned}\int_{U_\pi} e^{i(n_1 y_1 + \ldots + n_{2p} y_{2p})} \, d\mu(y) &= \int_{-\pi}^{\pi} e^{i(n_1 + \ldots + n_{2p})x} \, dx \\ &= \begin{cases} 0 & \text{if } \sum_{j=1}^{2p} n_j \neq 0 \\ 2\pi & \text{if } \sum_{j=1}^{2p} n_j = 0. \end{cases}\end{aligned}$$

The corresponding Fourier coefficient of μ_N is

$$
\begin{aligned}
C_N &= C_N(n_1, n_2, \dots, n_{2p}) \\
&= \int_{U_\pi} e^{i[n_1 y_1 + \dots + n_{2p} y_{2p}]} d\mu_N(y) \\
&= \frac{1}{(2\pi)^{2p-1} N} \sum_{j_1=0}^{N-1} \dots \sum_{j_{2p}=0}^{N-1} \left\{ \int_{-\pi}^{\pi} e^{i[n_1 + j_1 - j_2] y_1} dy_1 \right. \\
&\quad \left. \cdot \int_{-\pi}^{\pi} e^{i[n_2 + j_2 - j_3] y_2} dy_2 \dots \int_{-\pi}^{\pi} e^{i[n_{2p} + j_{2p} - j_1] y_{2p}} dy_{2p} \right\}. \qquad (7.2)
\end{aligned}
$$

Fix $j_1, \dots, j_{2p}$. In order for the expression in braces to be nonzero we must have

$$
\begin{aligned}
n_1 &= -(j_1 - j_2) \\
n_2 &= -(j_2 - j_3) \\
&\vdots \\
n_{2p-1} &= -(j_{2p-1} - j_{2p}) \\
n_{2p} &= -(j_{2p} - j_1).
\end{aligned} \qquad (7.3)
$$

But then $n_1 + \dots + n_{2p} = 0$. Thus if $n_1 + \dots + n_{2p} \neq 0$ each of the summands in (7.2) is equal to 0. Therefore $C_N = 0$ if $n_1 + \dots + n_{2p} \neq 0$.

Suppose $n_1 + \dots + n_{2p} = 0$. Then each summand in (7.2) equals 0 or $(2\pi)^{2p}$. When the summand equals $(2\pi)^{2p}$, the indices $j_1, \dots, j_{2p}$ satisfy (7.3), which implies

$$
\begin{aligned}
j_2 &= j_1 + (n_1) \\
j_3 &= j_1 + (n_1 + n_2) \\
&\vdots \\
j_{2p} &= j_1 + (n_1 + \dots + n_{2p-1}).
\end{aligned} \qquad (7.4)
$$

Define

$$M = \max\{n_1 + \dots + n_k \colon k = 1, \dots, 2p-1\}, \qquad M^+ = \max(M, 0),$$

$$m = \min\{n_1 + \dots + n_k \colon k = 1, \dots, 2p-1\}, \quad \text{and} \quad m^+ = \max(-m, 0).$$

Fix j_1 satisfying $0 \leq j_1 \leq N-1$ and determine $j_2, \dots, j_{2p}$ according to (7.4). In order for the inequalities $0 \leq j_k \leq N-1$, $k = 2, \dots, 2p$ to be satisfied we must have $j_1 \leq N - 1 - M$ and $j_1 \geq m$. Thus the sum in (7.2) is equal to

$$\sum_{j_1 = m^+}^{N-1-M^+} (2\pi)^{2p} = (N - M^+ - m^+)(2\pi)^{2p}.$$

Therefore

$$C_N = \frac{2\pi(N - M^+ - m^+)}{N},$$

which tends to 2π as $N \to \infty$. This completes the proof of Lemma 7.1. □

Proof of Theorem 1. We must evaluate the asymptotic behavior of

$$\int_{U_\pi} P_N(y) Q(y) dy.$$

Introduce the sets

$$W_k=\left\{y\in\mathbb{R}^{2p}\colon |y_k|\leqq\frac{|y_{k+1}|}{2}\right\},\qquad k=1,\dots,2p-1,$$

$$W_{2p}=\left\{y\in\mathbb{R}^{2p}\colon |y_{2p}|\leqq\frac{|y_1|}{2}\right\},$$

and

$$W=W_1\cup W_2\cup\dots\cup W_{2p}.$$

We shall divide the domain of integration U_π into three parts as follows. Let

$$E_t=U_\pi\setminus\{W\cup U_t\},$$

$$F_t=U_t\setminus W,$$

and

$$G=U_\pi\cap W.$$

For each $0<t\leqq\pi$, the sets E_t, F_t and G are disjoint and satisfy $U_\pi=E_t\cup F_t\cup G$.

According to (7.1), Part a of the theorem will be proven if we show that $p(\alpha+\beta)<1$ implies

$$\lim_{N\to\infty}\frac{\int_{E_t}P_NQ}{N}=(2\pi)^{2p-1}\int_{t\leqq|z|\leqq\pi}[f(z)g(z)]^p\,dz,\qquad 0<t\leqq 1,\tag{7.5}$$

$$\lim_{t\to 0}\limsup_{N\to\infty}\frac{\int_{F_t}P_NQ}{N}=0,\tag{7.6}$$

and

$$\lim_{N\to\infty}\frac{\int_{G}P_NQ}{N}=0.\tag{7.7}$$

To prove (7.6) it is enough to show that when $p(\alpha+\beta)<1$

$$\lim_{t\to 0}\limsup_{N\to\infty}\frac{\int_{U_t}|P_NQ|}{N}=0.\tag{7.8}$$

Since $G=\bigcup_{k=1}^{2p}[W_k\cap U_\pi]$, relation (7.7) will hold, if $p(\alpha+\beta)<1$ implies

$$\lim_{N\to\infty}\frac{\int_{U_t\cap W_k}|P_NQ|}{N}=0,\qquad k=1,\dots,2p.\tag{7.9}$$

From the definitions of P_N and Q it is clear that

$$\int_{U_\pi\cap W_1}|P_NQ|=\int_{U_\pi\cap W_3}|P_NQ|=\dots=\int_{U_\pi\cap W_{2p-1}}|P_NQ|$$

and

$$\int_{U_\pi\cap W_2}|P_NQ|=\int_{U_\pi\cap W_4}|P_NQ|=\dots=\int_{U_\pi\cap W_{2p}}|P_NQ|.$$

Because of the symmetry between α and β in the hypotheses of the theorem, it is clear that if we prove that $p(\alpha+\beta)<1$ implies

$$\lim_{N\to\infty} \frac{\int\limits_{U_\pi \cap W_1} |P_N Q|}{N} = 0, \tag{7.10}$$

we will have also established

$$\lim_{N\to\infty} \frac{\int\limits_{U_\pi \cap W_2} |P_N Q|}{N} = 0.$$

Thus (7.9) will follow from (7.10).

In conclusion, Part a of the theorem will be proven if we show that $p(\alpha+\beta)<1$ implies (7.5), (7.8) and (7.10).

To prove Part b, we must show that for $p(\alpha+\beta)\geqq 1$

$$\int\limits_{U_\pi} |P_N Q| = o(N^{p(\alpha+\beta)+\varepsilon}) \quad \text{for every } \varepsilon>0. \tag{7.11}$$

We start with relation (7.5) and show that it holds in fact for all real values of α and β. We begin by showing that Q is bounded on E_t. Let $y \in E_t$. Since E_t is in the complement of U_t, there is some k such that $|y_k|>t$. Since E_t is also in the complement of W_j, $j=1,\dots,2p$, we have $|y_j|>|y_{j+1}/2|$, $j=1,\dots,2p-1$ and $y_{2p}>y_1/2$. Thus we have

$$|y_{k+1}|>\frac{|y_{k+2}|}{2}>\frac{|y_{k+3}|}{4}>\dots>\frac{|y_{2p}|}{2^{2p-k}}>\frac{|y_1|}{2^{2p-k+1}}>\dots>\frac{|y_k|}{2^{2p-1}}>\frac{t}{2^{2p-1}}.$$

Therefore $|y_j|>t/2^{2p-1}$, $j=1,\dots,2p$, for $y\in E_t$. Hence Q is bounded on E_t. Since $E_t \cap D = \{y\colon y_1=\dots=y_{2p},\ t\leqq|y_1|\leqq\pi\}$, relation (7.5) follows from Lemma 7.1.

Before proving (7.8), (7.10) and (7.11) we need to obtain majorants for P_N and Q. We have

$$\begin{aligned} P_N(y) &= \left(\sum_{j_1=0}^{N-1} e^{i(y_1-y_{2p})j_1}\right)\left(\sum_{j_2=0}^{N-1} e^{i(y_2-y_1)j_2}\right)\dots\left(\sum_{j_{2p}=0}^{N-1} e^{i(y_{2p}-y_{2p-1})j_{2p}}\right)\\ &= h_N^*(y_1-y_{2p})h_N^*(y_2-y_1)\dots h_N^*(y_{2p}-y_{2p-1}), \end{aligned}$$

where

$$h_N^*(z)=\sum_{j=0}^{N-1} e^{izj}.$$

Since $h_N^*(z)=(1-e^{iNx})/(1-e^{ix})$ for $z\neq 0$, $|1-e^{iNz}|\leqq 2$ and $|1-e^{iz}|\geqq|z|/2$ for $|z|\leqq\pi$, we obtain $h_N^*(z)\leqq 4|z|^{-1}$ for $|z|\leqq\pi$. For $\pi\leqq z\leqq 2\pi$ this implies $h_N^*(z)=h_N^*(z-2\pi)\leqq 4|z-2\pi|^{-1}$. For $-2\pi\leqq z\leqq-\pi$ we have $h_N^*(z)=h_N^*(z+2\pi)\leqq 4|z+2\pi|^{-1}$. Since $|h_N^*(z)|\leqq N$, these inequalities imply that $|h_N^*(z)|\leqq 4h_N(z)$, $-2\pi\leqq z\leqq 2\pi$, where $h_N(z)$ is as defined at the beginning of Sect. 6.

Thus $|P_N(y)|$ is at most 4^{2p} times

$$h_N(y_1-y_{2p})h_N(y_2-y_1)\dots h_N(y_{2p}-y_{2p-1}).$$

For fixed $\delta>0$ let $\alpha_0=\alpha+\delta$ and $\beta_0=\beta+\delta$. It is clear that under the hypotheses of the theorem, $Q(y)$ is at most a constant times

$$Q'(y)=|y_1|^{-\alpha_0}|y_2|^{-\beta_0}|y_3|^{-\alpha_0}\dots|y_{2p}|^{-\beta_0}.$$

Thus, the proof of the theorem can be completed by showing that (7.8), (7.10) and (7.11) hold with the integrand $P_N Q$ replaced by $f_N(y)$, defined as in Sect. 6. We can now apply Propositions 6.1 and 6.2. Assume first that $p(\alpha+\beta)\geqq 1$. Choose $\delta>0$. Then $p(\alpha_0+\beta_0)>1$. Therefore Part b of Proposition 6.2 implies that

$$\int_{U_\pi} f(y)\,dy=O(N^{p(\alpha_0+\beta_0)+\varepsilon})=O(N^{p(\alpha+\beta)+2p\delta+\varepsilon}).$$

Since δ can be made arbitrarily small, (7.11) follows.

Now suppose $p(\alpha+\beta)<1$. To prove (7.8), choose $\delta>0$ such that $p(\alpha_0+\beta_0)<1$. Then (7.8) follows from Part a of Proposition 6.2. To prove (7.10) we consider two cases. If $\alpha+\beta<0$ choose $\delta>0$ such that $\alpha_0+\beta_0<0$. Then (7.10) is a consequence of Part a of Proposition 6.1. If $\alpha+\beta\geqq 0$, choose δ such that $p(\alpha_0+\beta_0)<1$ and use Part b of Proposition 6.1. This completes the proof of Theorem 1. □

8. Proof of Theorem 4

Lemma 8.1. *If the conditions of Theorem 2 are satisfied, then*

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}E|x_N'A_Nx_N-\tilde{x}_N'A_N\tilde{x}_N|=0.$$

Proof. The beginning of the proof follows Walker (1964). Note

$$\begin{aligned}x_N'A_Nx_N-\tilde{x}_N'A_N\tilde{x}_N&=\sum_{j=1}^N\sum_{k=1}^N a_{j-k}(\bar{X}_N^2-X_j\bar{X}_N-X_k\bar{X}_N)\\&=\bar{X}_N^2\int_{-\pi}^{\pi}g(x)\sum_{j=1}^N\sum_{k=1}^N e^{i(j-k)x}\,dx\\&\quad-2\bar{X}_N\int_{-\pi}^{\pi}g(x)\sum_{j=1}^N\sum_{k=1}^N X_je^{i(j-k)x}\,dx\\&=:F_N-G_N.\end{aligned}$$

We consider $E|F_N|$ first. We have

$$\begin{aligned}E\bar{X}_N^2&=\frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N r_{j-k}=\frac{1}{N^2}\int_{-\pi}^{\pi}f(x)\sum_{j=1}^N\sum_{k=1}^N e^{i(j-k)x}\,dx\\&=\frac{1}{N^2}\int_{-\pi}^{\pi}f(x)h_N^*(x)h_N^*(-x)\,dx,\end{aligned}$$

which is at most a constant times

$$\frac{1}{N^2}\int_{-\pi}^{\pi}|x|^{-\alpha-\delta}h_N(x)h_N(-x)\,dx \leqq N^{2\eta-2}\int_{-\pi}^{\pi}|x|^{-\alpha-\delta+2\eta-2}\,dx,$$

where $\delta>0$, $0<\eta<1$, $h_N^*(x)=\sum_{j=1}^{N-1} e^{ixj}$ and $h_N(x)\leqq h_{N,\eta}(x)$ as at the beginning of Sect. 6. Choose δ so that $\alpha+2\delta<1$ and put $\eta=(1+\alpha+2\delta)/2$. Then the last integral is finite, so $E\bar{X}_N^2=O(N^{\alpha-1+2\delta})$.

A similar argument shows that

$$\int_{-\pi}^{\pi} g(x)\sum_{j=1}^{N}\sum_{k=1}^{N} e^{i(j-k)x}\,dx=O(N^{\beta+1+2\delta}).$$

Hence

$$E|F_N|=o(N^{\frac{1}{2}}).$$

Next we consider

$$(E|G_N|)^2\leqq 4E\bar{X}_N^2 E\left\{\int_{-\pi}^{\pi} g(x)\sum_{j=1}^{N}\sum_{k=1}^{N} X_j e^{i(j-k)x}\,dx\right\}^2.$$

The second expectation above is equal to

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x)g(y)f(z)\sum_{k_1=0}^{N-1}\cdots\sum_{k_4=0}^{N-1} \cdot e^{i(k_1-k_2)x}e^{i(k_3-k_4)y}e^{i(k_1-k_3)z}\,dx\,dy\,dz,$$

which is at most a constant times

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|x|^{-\beta-\delta}|y|^{-\beta-\delta}|z|^{-\alpha-\delta}h_N(x+z)h_N(-x)h_N(y-z)h_N(-y)\,dx\,dy\,dz.$$

Put $\eta_1=\beta+2\delta$ and $\eta_2=(1+\alpha)/2$. We have $h_N(-x)\leqq N^{\eta_1}h_{N,\eta_1}(-x)$,

$$h_N(-y)\leqq N^{\eta_1}h_{N,\eta_1}(-y),\qquad h_N(x+z)\leqq N^{\eta_2}h_{N,\eta_2}(x+z)$$

and $h_N(y-z)\leqq N^{\eta_2}h_{N,\eta_2}(y-z)$.

In order for $|x+z|$ to exceed $3\pi/2$ we must have $|x|\geqq\pi/2$ and $|z|\geqq\pi/2$, in which case the integrand is majorized by $N^{2\eta_1+2\eta_2}$ times

$$|y|^{-\beta-\delta+\eta_1-1}|x+z+\theta_1|^{\eta_2-1}|y-z+\theta_2|^{\eta_2-1},$$

where $\{\theta_1,\theta_2\}\subset\{-2\pi,0,2\pi\}$. This product is clearly integrable for any choice of θ_1 and θ_2.

In order for $|y-z|$ to exceed $3\pi/2$ we must have $|y|\geqq\pi/2$ and $|z|\geqq\pi/2$ in which case the integrand is majorized by $N^{2\eta_1+2\eta_2}$ times

$$|x|^{-\beta-\delta+\eta_1-1}|x+z+\theta_3|^{\eta_2-1}|y-z+\theta_4|^{\eta_2-1}$$

for some $\theta_3,\ \theta_4\subset\{-2\pi,0,2\pi\}$. This product is also integrable.

If neither of the above cases holds, then the integrand is majorized by $N^{2\eta_1+2\eta_2}$ times

$$|x|^{-\beta-\delta+\eta_1-1}|y|^{-\beta-\delta+\eta_1-1}|z|^{-\alpha-\delta}|x+z|^{\eta_2-1}|y-z|^{\eta_2-1}.$$

Using Theorem 3.1, it is easily checked that this product is integrable. Thus we conclude that

$$(E|G_N|)^2 \leqq 4E\bar{X}_N^2 O(N^{2\eta_1+2\eta_2}) = O(N^{2\alpha+2\beta+4\delta}).$$

Since $\alpha+\beta<1/2$, we see that $E|G_N|=o(N^{\frac{1}{2}})$, completing the proof of Lemma 8.1. □

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