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Probability Theory and Related Fields

Best-possible Bounds for the Distribution of a Sum – a Problem of Kolmogorov

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Summary. Recently, in answer to a question of Kolmogorov, G.D. Makarov obtained best-possible bounds for the distribution function of the sum $X+Y$ of two random variables, X and Y, whose individual distribution functions, F_X and F_Y, are fixed. We show that these bounds follow directly from an inequality which has been known for some time. The techniques we employ, which are based on copulas and their properties, yield an insightful proof of the fact that these bounds are best-possible, settle the question of equality, and are computationally manageable. Furthermore, they extend to binary operations other than addition and to higher dimensions.

1. Introduction

In a recent paper [3], G.D. Makarov, in response to a question and conjecture of A.N. Kolmogorov, solved the following problem: Let X and Y be random variables with respective distribution functions F and G. Find functions $\underline{F}$ and $\bar{F}$ such that, for all z in $\mathbb{R}$,

$$\underline{F}(z) = \inf P(X+Y<z) \tag{1.1}$$

and

$$\bar{F}(z) = \sup P(X+Y<z), \tag{1.2}$$

where the infimum and supremum are taken over all possible joint distribution functions H having the margins F and G.

In this paper we will show that the bounds $\underline{F}$ and $\bar{F}$ – which Makarov obtains via a cumbersome, ad hoc argument – are just a special case of a basic inequality which has been known for some years [4, 6]. The techniques used to establish this inequality yield an insightful proof of the fact that these bounds are best-possible and a necessary and sufficient condition for equality as well. They also allow us to determine $\underline{F}(z)$ and $\bar{F}(z)$ explicitly in a number of special cases. Furthermore, the basic inequality holds, not only for addition, but also for a large class of functions $L: \mathbb{R}^2 \to \mathbb{R}$. Thus we can immediately answer Kolmogorov's question for many other binary operations on random variables. Lastly, we indicate how these results extend to finite collections of random variables.

2. Preliminaries

In the sequel, we let Δ denote the set of all one-dimensional probability distribution functions (d.f.'s) that are non-defective at $-\infty$ and left-continuous on the real line $\mathbb{R}$. Specifically,

$$\Delta = \{F \mid \mathrm{Dom}\, F = [-\infty, \infty], \mathrm{Ran}\, F \subseteq [0, 1], F \text{ is left-continuous and non-decreasing on } \mathbb{R}, F(+\infty) = 1, \text{ and } F(-\infty) = 0 = \lim_{x \to -\infty} F(x)\}. \tag{2.1}$$

The elements of Δ are partially ordered via

$$F \leqq G \quad \text{iff} \quad F(x) \leqq G(x) \qquad \text{for all } x \text{ in } \mathbb{R}. \tag{2.2}$$

For any a in $(-\infty, \infty]$ the unit step at a is the d.f. ε_a defined by

$$\varepsilon_a(x) = \begin{cases} 0, & x \leqq a, \\ 1, & x > a, \end{cases} \qquad -\infty < a < \infty, \tag{2.3}$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0, & -\infty \leqq x < \infty, \\ 1, & x = \infty. \end{cases}$$

We denote the d.f. of a real random variable (r.v.) X either by F_X or by $df(X)$.

The key to our development is the notion of a copula, first introduced by A. Sklar in 1959 [7] (see also [6], Chap. 6).

Definition 2.1. A (two-dimensional) copula is a mapping C from the unit square $[0, 1] \times [0, 1]$ onto the unit interval $[0, 1]$ satisfying the conditions:

(a) $C(a, 0) = C(0, a) = 0$ and $C(a, 1) = C(1, a) = a$, for all a in $[0, 1]$.

(b) $C(a_2, b_2) - C(a_1, b_2) - C(a_2, b_1) + C(a_1, b_1) \geqq 0$, for all a_1, a_2, b_1, b_2 in $[0, 1]$ such that $a_1 \leqq a_2$, $b_1 \leqq b_2$.

It is readily verified that any copula C is non-decreasing in each place, is continuous, and satisfies

$$W(a, b) \leqq C(a, b) \leqq M(a, b), \tag{2.4}$$

for all (a, b) in $[0, 1] \times [0, 1]$, where W and M are the copulas given by

$$W(a, b) = \mathrm{Max}\,(a + b - 1, 0), \tag{2.5}$$

$$M(a, b) = \mathrm{Min}\,(a, b). \tag{2.6}$$

Henceforth we denote the set of all copulas by $\mathscr{C}$.

Copulas were so-named because they link multidimensional d.f.'s to their one-dimensional margins. They therefore provide a natural setting for the study of questions dealing with properties of d.f.'s with fixed margins (the present paper being a case in point). The exact connection is given by the following basic result, due to A. Sklar [7] (see also [5]).

Theorem 2.1. *Let H be a two-dimensional d.f. with margins F and G. Then there is a copula C such that*

$$H(u,v)=C(F(u),G(v)), \tag{2.7}$$

for all u, v in $\mathbb{R}$. In the other direction, for any F, G in Δ and any copula C, the function H defined by (2.7) *is a two-dimensional d.f. with margins F and G.*

If F and G are continuous on $[-\infty, \infty]$, then C is unique; otherwise, C is uniquely determined on $(\operatorname{Ran} F)\times(\operatorname{Ran} G)$.

If X and Y are real r.v.'s with $df(X)=F$, $df(Y)=G$ and joint d.f. H, then we call any copula C that satisfies (2.7) an XY-copula. Thus, for example, X and Y are independent if and only if Π is an XY-copula, where

$$\Pi(a,b)=ab. \tag{2.8}$$

Note also that combining (2.4) and (2.7) yields the familiar Fréchet bounds for H.

Now, for any C in $\mathscr{C}$ and any F, G in Δ, let $\sigma_C(F,G)$ be the function in Δ defined by

$$\sigma_C(F,G)(-\infty)=0, \qquad \sigma_C(F,G)(\infty)=1.$$

and

$$\sigma_C(F,G)(x)=\iint\limits_{u+v<x} dC(F(u),G(v)), \qquad \text{for } -\infty<x<\infty. \tag{2.9}$$

It is well-known that if X and Y are r.v.'s with $df(X)=F$, $df(Y)=G$, and if C is an XY-copula, then

$$df(X+Y)=\sigma_C(F,G). \tag{2.10}$$

Thus, for fixed F, G in Δ, the sets

$$\{\sigma_C(F,G)\,|\,C \text{ in } \mathscr{C}\}$$

and

$$\{df(X+Y)\,|\,df(X)=F, df(Y)=G\}$$

are coextensive; and, clearly, for any fixed C in $\mathscr{C}$, σ_C is a binary operation on Δ.

In order to obtain bounds on σ_C, we need to introduce two additional families of binary operations on Δ. These are the families τ_C and ρ_C which, for any C in $\mathscr{C}$, are defined for all F, G in Δ and x in $[-\infty, \infty]$ via

$$\tau_C(F,G)(x)=\sup_{u+v=x} C(F(u),G(v)) \tag{2.11}$$

and

$$\rho_C(F,G)(x)=\inf_{u+v=x} \tilde{C}(F(u),G(v)), \tag{2.12}$$

respectively, where

$$\bar{C}(a,b)=a+b-C(a,b). \tag{2.13}$$

The operations σ_C, τ_C, ρ_C have been studied extensively (see [6] for details and references to the literature). For our immediate purposes we need the fact that for any copula C,

$$\tau_W \leqq \tau_C \leqq \sigma_C \leqq \rho_C \leqq \rho_W, \tag{2.14}$$

where each inequality holds for all F, G in Δ and all x in $[-\infty, \infty]$. These inequalities were established in [4] (see also [6], Theorem 7.55) for d.f.'s having the value 0 at $x=0$. The elementary arguments given there extend without difficulty to all of Δ, since (2.4) and Fig. 1 make it clear that, for any given copula C and any pair of points (u_1, v_1), (u_2, v_2) on the line $u+v=x$, we have

$$\begin{aligned}
W(F(u_1), G(v_1)) &\leqq C(F(u_1), G(v_1)) = \iint_A dC(F(u), G(v)) \\
&\leqq \sigma_C(F, G)(x) \leqq \iint_B dC(F(u), G(v)) \\
&= F(u_2) + G(v_2) - C(F(u_2), G(v_2)) \\
&= \bar{C}(F(u_2), G(v_2)) \leqq \bar{W}(F(u_2), G(v_2)).
\end{aligned}$$

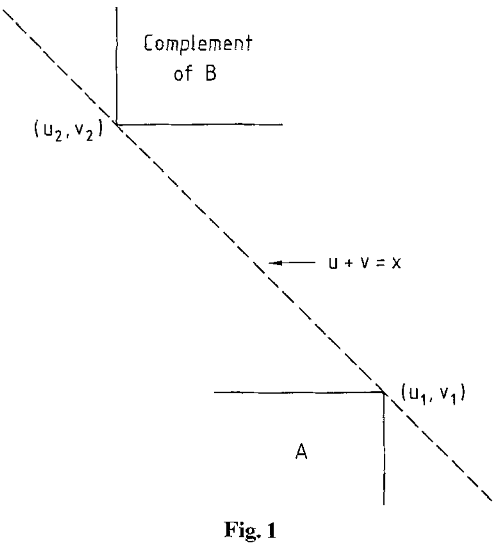

Fig. 1

3. Bounds for $df\,(X+Y)$

From (2.10) and (2.14), we immediately obtain

Theorem 3.1. *Let X and Y be r.v.'s with d.f.'s F_X and F_Y, respectively. Then*

$$\tau_W(F_X, F_Y) \leqq df(X+Y) \leqq \rho_W(F_X, F_Y). \tag{3.1}$$

In other words, the functions $\underline{F}$ and $\bar{F}$ in (1.1) and (1.2) are given, respectively, by

$$\underline{F}=\tau_W(F_X, F_Y) \quad \text{and} \quad \bar{F}=\rho_W(F_X, F_Y).$$

In [3] Makarov obtains (3.1), not directly in terms of the d.f.'s involved, but rather in terms of their quasi-inverses. The left-continuous quasi-inverse of any d.f. F in Δ is the function F^* defined on [0, 1] by

$$F^*(a)=\inf\{x \mid F(x) \geqq a\}.$$

The transition from our formulation to Makarov's, and vice versa, is an immediate consequence of a basic duality theorem which was established in [2] (see also [6], Sect. 7.7]). In the special case of the operations τ_W and ρ_W this theorem states that

$$\tau_W(F, G)^*(a)= \inf_{W(s,t)=a} [F^*(s)+G^*(t)]$$

and

$$\rho_W(F, G)^*(a)= \sup_{\tilde{W}(s,t)=a} [F^*(s)+G^*(t)].$$

Thus (3.1) is equivalent to

$$\inf_{W(s,t)=a} [F_X^*(s)+F_Y^*(t)] \leqq [df(X+Y)]^*(a) \leqq \sup_{\tilde{W}(s,t)=a} [F_X^*(s)+F_Y^*(t)],$$

which, apart from a slight difference in notation, is Makarov's result.

Next, a straightforward extension of the argument used to establish Corollary 2 of Theorem 9 in [4] yields the fact that

$$\sigma_C(F, G)=\tau_W(F, G)$$

if and only if $F=\varepsilon_a$ for some $a>-\infty$ or $G=\varepsilon_b$ for some $b>-\infty$; and similarly

$$\sigma_C(F, G)=\rho_W(F, G)$$

under the same conditions. From this it follows that, viewed as an inequality among all functions in Δ, (3.1) cannot be improved. But more is true: (3.1) cannot be improved for any pair of functions in Δ, i.e., the bounds are pointwise best-possible. More precisely, we have

Theorem 3.2. *Let F and G be any d.f.'s in Δ and x any point in $(-\infty, \infty)$. Then:*

(i) *There exists a copula C_t, dependent only on the value t of $\tau_W(F, G)$ at x, such that*

$$\sigma_{C_t}(F, G)(x)=\tau_W(F, G)(x)=t. \tag{3.2}$$

(ii) *There exists a copula C_r, dependent only on the value r of $\rho_W(F, G)(x+)$, such that*

$$\sigma_{C_r}(F, G)(x+)=\rho_W(F, G)(x+)=r. \tag{3.3}$$

Proof. For the given x, let A_x and B_x be the regions of the extended plane above and below the line $u+v=x$, i.e., let

$$A_x = \{(u, v) \mid u+v > x\}$$

and

$$B_x = \{(u, v) \mid u+v < x\}.$$

To establish (i), we will show that if C_t is the copula defined by

$$C_t(a, b) = \begin{cases} \operatorname{Max}(a+b-1, t), & (a, b) \text{ in } [t, 1] \times [t, 1], \\ M(a, b), & \text{otherwise}, \end{cases} \tag{3.4}$$

then $\sigma_{C_t}(F, G)(x) = t$. To this end, first note that if $\tau_W(F, G)(x) = 1$ then, in view of (2.14), $\sigma_C(F, G)(x) = 1$ for any copula C. Thus it suffices to show that for $0 \leqq t < 1$,

$$\sigma_{C_t}(F, G)(x) = \iint_{B_x} dC_t(F(u), G(v)) \leqq t. \tag{3.5}$$

For any (u, v) in $\bar{B}_x$, the closure of B_x, we have

$$W(F(u), G(v)) \leqq \tau_W(F, G)(u+v) \leqq \tau_W(F, G)(x) = t,$$

so that $F(u) + G(v) - 1 \leqq t$. Consequently,

$$C_t(F(u), G(v)) = \operatorname{Min}(F(u), G(v), t), \quad \text{for } (u, v) \text{ in } \bar{B}_x. \tag{3.6}$$

In particular, $C_0(F(u), G(v)) = 0$ for all (u, v) in $\bar{B}_x$, whence $\sigma_{C_0}(F, G)(x) = 0$, so that (3.2) holds for $t = 0$.

Now suppose $0 < t < 1$, and let

$$u_0 = \sup\{u \mid (F(u) < t\}.$$

Since $\lim_{u \to -\infty} F(u) = 0$, we have $u_0 > -\infty$. If $u_0 = \infty$, then $F(u) < t$ for all finite u. But since $\lim_{u \to -\infty} G(u) = 0$, there is a finite u' such that $G(x-u') < 1$. Thus, for $u \leqq u'$, we have

$$F(u) + G(x-u) - 1 \leqq F(u) \leqq F(u') < t,$$

and for $u \geqq u'$, we have

$$F(u) + G(x-u) - 1 \leqq t + G(x-u') - 1 < t.$$

Thus $\tau_W(F, G)(x) \leqq \operatorname{Max}(F(u'), t + G(x-u') - 1) < t$, which is a contradiction. Consequently, u_0 is finite.

Next, we show that $G(v) \geqq t$ whenever $v > x - u_0$. Suppose, to the contrary, that there exists a $v' > x - u_0$ such that $G(v') < t$. Since $x - v' < u_0$, we have $F(x-v') < t$. Thus, for $u \leqq x - v'$,

$$F(u) + G(x-u) - 1 \leqq F(u) \leqq F(x-v') < t,$$

and for $u \geqq x - v'$,

$$F(u) + G(x-u) - 1 \leqq G(x-u) \leqq G(v') < t,$$

and again, $\tau_W(F, G)(x) < t$, which is a contradiction.

It follows that $F(u) < t$ for $u < u_0$, whence by left-continuity $F(u_0) \leqq t$; that $F(u) \geqq t$ for $u > u_0$; and that $G(v) \geqq t$ for $v > x - u_0$. Combining these facts with

(3.4) yields (see Fig. 2) that for (u, v) in $\bar{B}_x$,

$$C_t(F(u), G(v)) = \begin{cases} F(u), & u < u_0,\ v > x - u_0, \\ \mathrm{Min}\,(F(u), G(v)), & u \leqq u_0,\ v \leqq x - u_0, \\ \mathrm{Min}\,(G(v), t), & u > u_0. \end{cases} \tag{3.7}$$

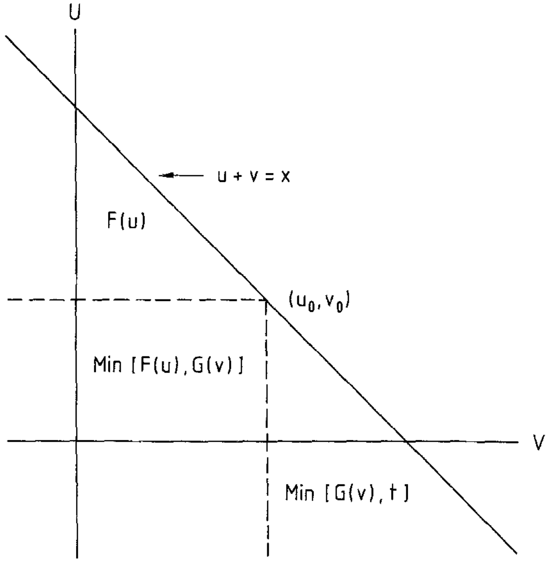

Fig. 2

To evaluate $\sigma_{C_t}(F, G)(x)$, choose $\delta > 0$; let R_1, R_2, R_3 be the rectangles given by

$$R_1 = [-\infty, u_0 - \delta] \times [v_0, v_0 + \delta],$$
$$R_2 = [-\infty, u_0] \times [-\infty, v_0],$$
$$R_3 = [u_0, u_0 + \delta] \times [-\infty, v_0 - \delta];$$

let R_4, R_5 be the sectors given by

$$R_4 = B_x \cap \{(u, v) \mid v > v_0 + \delta\},$$
$$R_5 = B_x \cap \{(u, v) \mid u > u_0 + \delta\};$$

and, for $k = 1, 2, \ldots, 5$, let $I(R_k)$ denote the contribution of the region R_k to the integral in (3.5). Then, using (3.7) (again see Fig. 2), we obtain

$$I(R_1) = F(u_0 - \delta) - \mathrm{Min}\,(F(u_0 - \delta), G(v_0)),$$
$$I(R_2) = \mathrm{Min}\,(F(u_0), G(v_0)),$$
$$I(R_3) = \mathrm{Min}\,(G(v_0 - \delta), t) - \mathrm{Min}\,(F(u_0), G(v_0 - \delta)),$$
$$I(R_4) = I(R_5) = 0.$$

Hence, since F and G are left-continuous and $F(u_0) \leqq t$, we have

$$\begin{aligned}\sigma_{C_t}(F, G)(x) &= \lim_{\delta \downarrow 0} [I(R_1) + I(R_2) + I(R_3)] \\ &= F(u_0) + \text{Min}\,(G(v_0), t) - \text{Min}\,(F(u_0), G(v_0)) \\ &= \begin{cases} t, & \text{if } G(v_0) > t, \\ \text{Max}\,(F(u_0), G(v_0)) \leqq t, & \text{if } G(v_0) \leqq t, \end{cases}\end{aligned}$$

and (3.5) holds. This proves (i).

Note. When F is continuous, the above argument simplifies and, in particular, in this case $I(R_1) = I(R_3) = 0$.

To establish (ii), we will show that if C_r is the copula defined by

$$C_r(a, b) = \begin{cases} \text{Max}\,(a+b-r, 0), & (a, b) \text{ in } [0, r] \times [0, r], \\ M(a, b), & \text{otherwise}, \end{cases} \tag{3.8}$$

then $\sigma_{C_r}(F, G)(x+) = r$. Again, first note that if $\rho_W(F, G)(x+) = 0$ then, since $0 \leqq \sigma_C(F, G)(x+\delta) \leqq \rho_W(F, G)(x+\delta)$ for any C in $\mathscr{C}$ and any $\delta > 0$, we have $\sigma_{C_r}(F, G)(x+) = 0$. Thus, in view of (2.14), it suffices to show that for $0 < r \leqq 1$,

$$\sigma_{C_r}(F, G)(x + 2\delta) \geqq r, \qquad \text{for all } \delta > 0.$$

For any (u, v) in A_x, we have

$$\bar{W}(F(u), G(v)) \geqq \rho_W(F, G)(u+v) \geqq \rho_W(F, G)(x+) = r > 0, \tag{3.9}$$

so that, since $\bar{W}(a, b) = \text{Min}\,(a+b, 1)$, we have $F(u) + G(v) - r \geqq 0$. Consequently, for (u, v) in A_x,

$$C_r(F(u), G(v)) = \text{Min}\,(F(u), G(v), F(u) + G(v) - r). \tag{3.10}$$

If $r = 1$, then $C_1(F(u), G(v)) = F(u) + G(v) - 1 \geqq 0$, for all (u, v) in A_x. Since both F and G are non-defective at $-\infty$, it follows that they, and consequently $\sigma_{C_1}(F, G)$, are also non-defective at ∞. Thus, for any $\delta > 0$,

$$\sigma_{C_1}(F, G)(x+\delta) = 1 - \iint_{\bar{A}_{x+\delta}} dC_1(F(u), G(v)). \tag{3.11}$$

But the contribution to the integral in (3.11) from any rectangle whose vertices are in A_x is identically 0. Hence $\sigma_{C_1}(F, G)(x+) = 1 = \rho_{C_1}(F, G)(x+)$.

Now suppose $0 < r < 1$, and let

$$u_0 = \inf \{u \mid F(u) > r\}.$$

Then arguments entirely similar to those used to establish (i) yield that u_0 is finite, that $G(v) \leqq r$ whenever $v < x - u_0$, that $F(u) > r$ for $u > u_0$, and that $F(u) \leqq r$ for $u \leqq u_0$. Thus, using (3.10), for (u, v) in A_x we have

$$C_r(F(u), G(v)) = \begin{cases} G(v), & u > u_0,\, v \leqq x - u_0, \\ \text{Min}\,(F(u), G(v)), & u > u_0,\, v > x - u_0, \\ \text{Min}\,(F(u), F(u) + G(v) - r), & u \leqq u_0. \end{cases} \tag{3.12}$$

To evaluate $\sigma_{C_r}(F, G)(x+)$, choose $\delta > 0$; let S_1, S_2, S_3 be the rectangles given by

$$S_1=[u_0, u_0+\delta]\times[v_0+\delta, \infty],$$
$$S_2=[u_0+\delta, \infty]\times[v_0+\delta, \infty],$$
$$S_3=[u_0+\delta, \infty]\times[v_0, v_0+\delta];$$

let S_4 and S_5 be the sectors given by

$$S_4=A_{x+2\delta}\cap\{(u,v)\,|\,v>v_0+\delta\},$$
$$S_5=A_{x+2\delta}\cap\{(u,v)\,|\,u>u_0+\delta\};$$

and let

$$I(S)=\iint_S dC_r(F(u), G(v)).$$

Then it follows from (3.12) that

$$\begin{aligned}I(S_1)&=F(u_0+\delta)-F(u_0)+\mathrm{Min}\,(F(u_0), F(u_0)+G(v_0+\delta)-r)\\&\quad-\mathrm{Min}\,(F(u_0+\delta), G(v_0+\delta)),\\I(S_2)&=1-F(u_0+\delta)-G(v_0+\delta)+\mathrm{Min}\,(F(u_0+\delta), G(v_0+\delta)),\\I(S_3)&=G(v_0+\delta)-\mathrm{Min}\,(F(u_0+\delta), G(v_0+\delta)),\\I(S_4)&=I(S_5)=0.\end{aligned}$$

Adding yields

$$\begin{aligned}I(\bar{A}_{x+2\delta})\leqq 1-F(u_0)&+\mathrm{Min}\,(F(u_0), F(u_0)+G(v_0+\delta)-r)\\&-\mathrm{Min}\,(F(u_0+\delta), G(v_0+\delta)).\end{aligned}$$

Since $F(u_0+\delta)\geqq r$, we have that

$$I(\bar{A}_{x+2\delta})\leqq\begin{cases}1-r, & \text{if } G(v_0+\delta)<r,\\ 1-\mathrm{Min}\,(F(u_0+\delta), G(v_0+\delta))\leqq 1-r. & \text{if } G(v_0+\delta)\geqq r.\end{cases}$$

Consequently,

$$\sigma_{C_r}(F, G)(x+2\delta)=I(B_{x+2\delta})=1-I(\bar{A}_{x+2\delta})\geqq r,$$

which, on letting δ decrease to 0, yields (3.3) and completes the proof of Theorem 3.2.

We conclude this section with several remarks.

(1) The statement (ii) of Theorem 3.2 cannot be strengthened to $\sigma_{C_r}(F, G)(x) = \rho_W(F, G)(x)$, not even when $\rho_W(F, G)$ is continuous at x. To see this, let F be the uniform d.f. on $[0, 1]$. Then $\rho_W(F, F)=F$; but from (3.8) it follows that for any x in $(0, 1)$, $C_x(F(u), F(x-u))=0$ for all u, whence $\sigma_{C_x}(F, F)(x)=0<x = \rho_W(F, F)(x)=\rho_W(F, F)(x+)$.

(2) The crucial property of the copula C_t which is used in the proof of (i) is the fact that when $a+b-1=t$, then C_t, considered as a joint d.f., assigns the mass t to any rectangle of the form $[0, a]\times[0, b]$. It follows that C_t is not unique. For example, the copula C_t' defined by

$$C_t'(a, b)=\begin{cases}ab/t, & (a, b) \text{ in } [0, t]\times[0, t],\\ C_t(a, b), & \text{otherwise},\end{cases}$$

would do just as well; indeed, with C_t' the calculations that yield (3.2) remain unchanged. Similar remarks apply to the copula C_r used in the proof of (ii).

(3) The fact that the results of Theorem 3.2 are best-possible is closely related to the fact that the binary operations τ_W and ρ_W are not derivable from any binary operation on random variables, in the following sense: There are no Borel-measurable functions β and γ such that, for all pairs of random variables X, Y defined on a common probability space, $\tau_W(df(X), df(Y)) = df(\beta(X, Y))$ or $\rho_W(df(X), df(Y)) = df(\gamma(X, Y))$ (see [5] and [6], Sect. 7.6).

4. Examples

In a number of cases of special interest, the bounds for $df(X+Y)$ given in (3.1) can be determined explicitly.

(1) For $-\infty < r < s < \infty$, let U_{rs} denote the uniform distribution on $[r, s]$, i.e., let

$$U_{rs}(t) = \begin{cases} 0, & t \leqq r, \\ (t-r)/(s-r), & r \leqq t \leqq s, \\ 1, & s \leqq t. \end{cases}$$

Then, as shown by C. Alsina in [1],

$$\tau_W(U_{ab}, U_{cd})(x) = U_{\mathrm{Min}(a+d,\, b+c),\, b+d},$$

and

$$\rho_W(U_{ab}, U_{cd})(x) = U_{a+c,\, \mathrm{Max}(a+d,\, b+c)}.$$

(2) For any $\theta > 0$, let E_θ denote the exponential d.f. with parameter θ, i.e., let

$$E_\theta(x) = \begin{cases} 0, & x \leqq 0, \\ 1 - e^{-x/\theta}, & x \geqq 0. \end{cases}$$

Then, using the method of Lagrange multipliers to determine the extrema of $E_\alpha(u) + E_\beta(v)$ under the constraint $u + v = x$, we find that

$$\tau_W(E_\alpha, E_\beta)(x) = E_{\alpha+\beta}(x-k),$$

where $k = (\alpha+\beta)\log(\alpha+\beta) - \alpha\log\alpha - \beta\log\beta$, and that

$$\rho_W(E_\alpha, E_\beta)(x) = E_{\mathrm{Max}(\alpha,\beta)}(x).$$

(3) Let N_i denote the normal d.f. with mean μ_i and variance σ_i^2, i.e., let

$$N_i(x) = \Phi\left(\frac{x - \mu_i}{\sigma_i}\right),$$

where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2}\, ds.$$

Then, after some calculations similar to but more involved than those in (2), we find:

(i) If $\sigma_1^2=\sigma_2^2=\sigma^2$, then

$$\tau_W(N_1, N_2)(x)=\begin{cases}0, & x\leqq\mu_1+\mu_2,\\ 2\Phi\left(\dfrac{x-\mu_1-\mu_2}{2\sigma}\right)-1, & x\geqq\mu_1+\mu_2,\end{cases}$$

and

$$\rho_W(N_1, N_2)(x)=\begin{cases}2\Phi\left(\dfrac{x-\mu_1-\mu_2}{2\sigma}\right), & x\leqq\mu_1+\mu_2,\\ 1, & x\geqq\mu_1+\mu_2.\end{cases}$$

(ii) If $\sigma_1^2\neq\sigma_2^2$, then

$$\tau_W(N_1, N_2)(x)=\Phi\left(\frac{-\sigma_1 s-\sigma_2 t}{\sigma_2^2-\sigma_1^2}\right)+\Phi\left(\frac{\sigma_2 s-\sigma_1 t}{\sigma_2^2-\sigma_1^2}\right)-1,$$

and

$$\rho_W(N_1, N_2)(x)=\Phi\left(\frac{-\sigma_1 s+\sigma_2 t}{\sigma_2^2-\sigma_1^2}\right)+\Phi\left(\frac{\sigma_2 s+\sigma_1 t}{\sigma_2^2-\sigma_1^2}\right),$$

where $s=x-\mu_1-\mu_2$ and $t=[s^2+2(\sigma_2^2-\sigma_1^2)\log(\sigma_2/\sigma_1)]^{1/2}$.

(4) Let A_i denote the Cauchy distribution with location parameter α_i and scale parameter β_i, i.e., let

$$A_i(x)=\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x-\alpha_i}{\beta_i}\right).$$

Then calculations similar to those in (3) yield:

(i) If $\beta_1=\beta_2=\beta$, then

$$\tau_W(A_1, A_2)(x)=\begin{cases}0, & x\leqq\alpha_1+\alpha_2,\\ \dfrac{2}{\pi}\arctan\left(\dfrac{x-\alpha_1-\alpha_2}{2\beta}\right), & x>\alpha_1+\alpha_2,\end{cases}$$

and

$$\rho_W(A_1, A_2)(x)=\begin{cases}1+\dfrac{2}{\pi}\arctan\left(\dfrac{x-\alpha_1-\alpha_2}{2\beta}\right), & x\leqq\alpha_1+\alpha_2,\\ 1, & x>\alpha_1+\alpha_2.\end{cases}$$

(ii) If $\beta_1\neq\beta_2$, then

$$\tau_W(A_1, A_2)(x)=\frac{1}{\pi}\left[\arctan\left(\frac{-s+\beta_2 t}{\beta_2-\beta_1}\right)+\arctan\left(\frac{s-\beta_1 t}{\beta_2-\beta_1}\right)\right]$$

and

$$\rho_W(A_1, A_2)(x) = 1 + \frac{1}{\pi}\left[\arctan\left(\frac{-s-\beta_2 t}{\beta_2-\beta_1}\right) + \arctan\left(\frac{s+\beta_1 t}{\beta_2-\beta_1}\right)\right]$$

where $s = x - \alpha_1 - \alpha_2$ and $t = \{[s^2 - (\beta_2 - \beta_1)(\beta_2^2 - \beta_1^2)]/\beta_1 \beta_2\}^{1/2}$.

5. Extensions

Let $\mathscr{L}$ be the class of all functions L from $[-\infty, \infty] \times [-\infty, \infty]$ onto $[-\infty, \infty]$ that are non-decreasing in each place and continuous, except possibly at the points $(\infty, -\infty)$ and $(-\infty, \infty)$.

The basic operations τ_C, ρ_C and σ_C may be extended in the sense that the operation of addition which appears in (2.9), (2.11) and (2.12) may be replaced by any L in $\mathscr{L}$. (See [4] and [6], Chap. 7.) This yields new families of binary operations $\tau_{C,L}$, $\rho_{C,L}$ and $\sigma_{C,L}$ on Δ, which are defined via

$$\tau_{C,L}(F, G)(x) = \sup_{L(u,v)=x} C(F(u), G(v)),$$

$$\rho_{C,L}(F, G)(x) = \inf_{L(u,v)=x} \tilde{C}(F(u), G(v)),$$

and

$$\sigma_{C,L}(F, G)(x) = \iint_{L(u,v)<x} dC(F(u), G(v)).$$

Again, if X and Y are r.v.'s with d.f.'s F_X and F_Y, respectively, and if C is an XY-copula, then $L(X, Y)$ is also a r.v. and

$$df(L(X, Y)) = \sigma_{C,L}(F_X, F_Y).$$

Furthermore, the basic inequality (2.14) immediately extends to

$$\tau_{W,L} \leqq \tau_{C,L} \leqq \sigma_{C,L} \leqq \rho_{C,L} \leqq \rho_{W,L}, \tag{5.1}$$

and thus we have

Theorem 5.1. *Let X and Y be r.v.'s with d.f.'s F_X and F_Y, respectively. Then, for any L in $\mathscr{L}$,*

$$\tau_{W,L}(F_X, F_Y) \leqq df(L(X, Y)) \leqq \rho_{W,L}(F_X, F_Y). \tag{5.2}$$

The bounds in (5.2) are best-possible in the sense that equality holds throughout (5.2) whenever one of F_X, F_Y is a unit step function. Whether or not they are pointwise best-possible remains to be determined.

When $[0, \infty]$ is closed under L, then (5.2) is an inequality for non-negative r.v.'s (more precisely, for d.f.'s whose support is $[0, \infty]$). In particular, if $P(x, y) = xy$ we have that

$$\tau_{W,P}(F_X, F_Y) \leqq df(XY) \leqq \rho_{W,P}(F_X, F_Y).$$

for any pair of non-negative r.v.'s X and Y.

The basic inequality (3.1) admits a simple extension to sums of any finite number of r.v.'s. From (3.1) and the fact that τ_W and ρ_W preserve the ordering

(2.2) of Δ, we have

$$\tau_W(\tau_W(F_X, F_Y), F_Z) \leqq \tau_W(F_{X+Y}, F_Z) \leqq F_{(X+Y)+Z} \leqq \rho_W(F_{X+Y}, F_Z) \leqq \rho_W(\rho_W(F_X, F_Y), F_Z),$$

for any r.v.'s X, Y, Z; and similarly,

$$\tau_W(F_X, \tau_W(F_Y, F_Z)) \leqq F_{X+(Y+Z)} \leqq \rho_W(F_X, \rho_W(F_Y, F_Z)).$$

In view of the fact that the operations τ_W and ρ_W are associative ([6], Chap. 7], we may write

$$\tau_W(F_X, F_Y, F_Z) \leqq df(X+Y+Z) \leqq \rho_W(F_X, F_Y, F_Z)$$

and, more generally,

$$\tau_W(F_{X_1}, \ldots, F_{X_n}) \leqq df\left(\sum_{k=1}^{n} X_k\right) \leqq \rho_W(F_{X_1}, \ldots, F_{X_n}),$$

for any r.v.'s $X_1, \ldots, X_n$.

Finally, upon replacing addition by any associative L in $\mathscr{L}$, the above argument yields

$$\tau_{W,L}(F_{X_1}, \ldots, F_{X_n}) \leqq df[L(X_1, \ldots, X_n)] \leqq \rho_{W,L}(F_{X_1}, \ldots, F_{X_n}).$$

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