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EXACT REGULAR MODELS FOR STATIC RELATIVISTIC STARS*

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We solve the Einstein equations for the case of a static, spherically symmetric distribution of a perfect fluid. We propose a method yielding a broad class of exact solutions. We investigate the question of how to obtain a fortiori *regular and equilibrium solutions. We find a new exact solution in a rather simple form and show that it describes a neutron star with mass limit of 0.33 solar masses.*

In order to construct an analytical model for a relativistic star, the model must be based on an exact solution of the gravitational equations which is a regular equilibrium solution. As a rule, the regular equilibrium solution is stable if the parameters of this solution do not reach the critical values. Despite the fact that the number of exact solutions is rather high, only a few of them are suitable for describing relativistic stars. Therefore methods for obtaining exact solutions of the Einstein equations with prespecified properties are of great interest.

The idea of the method of constructing exact solutions with specified properties was briefly outlined in [1]; in this paper, we consider this method in detail.

By the term "relativistic star" we will mean a static, spherically symmetric distribution of matter with finite mass M, confined within a sphere of radius R with vacuum outside the sphere. The ratio of the gravitational radius $R_g = 2MG/c^2$ to the radius of the body R is less than unity but markedly different from zero; therefore we will consider the effects of general relativity theory. We need to find the interior solution of the Einstein equation

$$R_{\alpha\beta} - g_{\alpha\beta}R/2 = -8\pi Gc^{-4}T_{\alpha\beta}, \tag{1}$$

joined on the surface of the sphere with the exterior Schwarzschild solution. The minus sign on the right-hand side of (1) is connected with the definition of the Ricci tensor as the contraction of the curvature tensor on the first and last indices; R is the trace of the Ricci tensor; $T_{\alpha\beta}$ is the energy-momentum tensor; the quantity $8\pi Gc^{-4}$, called the Einstein gravitational constant, is designated by the letter $\varkappa$ in the following.

Let us describe the matter by the energy-momentum tensor of a perfect fluid

$$T_{\alpha\beta} = (\mu + p)\, u_\alpha u_\beta - p g_{\alpha\beta}, \tag{2}$$

where μ is the energy density; p is the pressure; the 4-velocity $u^\alpha = dx^\alpha/ds$; $g_{\alpha\beta}$ is the metric tensor corresponding to the metric

$$ds^2 = F du^2 + 2L du dr - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

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here $F = g_{00}(r)$, $L = g_{01}(r)$, $r = x^1$, $\theta = x^2$, $\varphi = x^3$ are the spherical coordinates; instead of the conventional time t we use the time delay $u = t - r$. In this paper, we use a geometric (relativistic) system of units in which the speed of light and the Newtonian gravitational constant are equal to unity ($c = G = 1$), therefore the Einstein constant is $\varkappa = 8\pi$.

1. METHOD FOR OBTAINING THE EXACT SOLUTIONS

If we use the convenient symbols $\varepsilon = F/L^2$, $y = F'/F$, then the Einstein equation is reduced to three expressions:

$$\varkappa\mu(r) = -\varepsilon'/r + (1-\varepsilon)/r^2; \tag{4}$$

$$\varkappa p(r) = \varepsilon y/r - (1-\varepsilon)/r^2; \tag{5}$$

$$r(2+ry)\varepsilon' + \varepsilon[r^2(2y'+y^2) - 2(2+ry)] = -4, \tag{6}$$

the prime (′) indicates the derivative with respect to r.

The important relation $T^{\alpha\beta}{}_{;\beta} = 0$ is a consequence of Eq. (1); if we rewrite it in explicit form, then we obtain the equation

$$p' + (\mu+p)y/2 = 0, \tag{7}$$

called the hydrostatic equilibrium equation, which we find useful in the next section.

Equation (6) is a linear nonhomogeneous equation which is first- order relative to the function ε, and consequently if the function y is assumed to be known then ε can be expressed in general form in terms of the quadrature

$$(2+ry)^2\varepsilon = r^2 e^f [C - 4\int(2+ry)r^{-2}e^{-f}dr], \tag{8}$$

where $f(r) = \int(2 - ry)(2 + ry)^{-1}y\,dr$; C is a constant.

A broad class of exact solutions is obtained if we require that

$$F = a(1+br^{2n})^m, \tag{9}$$

where a, b, n and m are constants. Using the symbol $x = r^2$ and substituting (9) into (8), we arrive at the expression

$$\varepsilon = x(1+bx^n)^{2-m}h^\beta[C - \int(1+bx^n)^{m-1}h^{-1-\beta}x^{-2}dx], \tag{10}$$

where $h = 1 + (1 + mn)bx^n$; $\beta = 2(m - mn - 1)/(1 + mn)$.

Choosing the values m and n, we can express the integral in (10) in terms of elementary functions. At least four such cases exist when the integrand is reduced to: 1) a rational fraction; 2) a linear fractional irrational expression; 3) a binomial differential (the Chebyshev case); 4) a quadratic irrational expression (the Euler case).

This method yields a large number of exact solutions but some of them are singular at the center and some do not satisfy the equilibrium and smoothness conditions. We note that the hydrostatic equilibrium equation (7) is automatically satisfied.

2. METHOD FOR CONSTRUCTING REGULAR MODELS

First of all, let us list the requirements which are usually imposed on the exact solutions considered as models for relativistic stars.

From the physical meaning of the g_{00} component of the metric tensor it follows that $0 < F < 1$; from this follows the restriction on the gradient $F' \geq 0$, where the equals sign holds at the center of the star.

From the hydrostatic equilibrium equation we obtain the following restriction: $p' = -(\mu + p)y/2 \leq 0$. Here we assume the condition $\mu \geq 0$ and $p \geq 0$, obvious for ordinary matter.

The condition $\mu' \leq 0$ for the static case is equivalent to the equilibrium condition. In order to understand this, we need to draw on a concept such as the adiabatic speed of sound in matter $dp/d\mu = v^2$, $0 \leq v \leq 1$. For statics, $dp/d\mu = p'/\mu'$, from which also follows the restriction on the energy density gradient.

Let us also recall the important energy dominance condition $p/\mu \leq 1$, although for ordinary matter we require the stricter condition $p/\mu \leq 1/3$, where the equality holds in the so-called ultrarelativistic limit, which makes a perfect fluid similar to equilibrium radiation and leads to loss of stability.

Now let us use the listed requirements to construct solutions which are *a fortiori* regular and equilibrium solutions. To do this, let us expand the functions in Eqs. (4)-(7) in a power series about zero. We find that such properties of the solutions as regularity and equilibrium character are determined by the coefficients for the lowest powers. Thus for a solution to be regular at the center and to be an equilibrium solution, its power expansion should have the following form:

$$\mu = \mu_0 - \mu_2 r^2 + \mu_s r^s + \ldots; \tag{11}$$

$$p = p_0 - p_2 r^2 + p_4 r^4 + \ldots; \tag{12}$$

$$\varepsilon = 1 - (\varkappa\mu_0/3) r^2 + (\varkappa\mu_2/5) r^4 - (\varkappa\mu_s/s+3) r^{s+2} + \ldots; \tag{13}$$

$$F = a(1 + b_2 r^2 + b_4 r^4 + \ldots), \tag{14}$$

here $s > 2$, and the highest powers may be nonintegral, i.e., the expansions (11)-(14) generally speaking are not necessarily power series.

Substituting (11)-(14) into (5) and (7), and comparing the coefficients for identical powers, we obtain a relation between the coefficients:

$$6b_2 = 3\varkappa p_0 + \varkappa\mu_0; \tag{15}$$

$$2p_2 = b_2(\mu_0 + p_0); \tag{16}$$

$$4b_4 = b_2(\varkappa p_0 + \varkappa\mu_0) - (\varkappa p_2 + \varkappa\mu_2/5); \tag{17}$$

$$4p_4 + 2b_4(\mu_0 + p_0) = b_2(\mu_2 + 3p_2), \tag{18}$$

where b_2, p_2 and μ_2 are positive. We also note the following: if $p_4 = 0$, then $b_4 \neq 0$; if $b_4 = 0$, then $p_4 \neq 0$ and in this case $p_2/\mu_2 = v_0^2 = 1/5$, where v_0 is the speed of sound at the center of the star.

Since the parameters m and n from (9) can be selected even before calculation of the quadrature in (10) while the coefficients in (15)-(18) depend on these parameters, we can talk about constructing exact solutions with prespecified properties. Such a procedure ensures construction of a model suitable for describing a relativistic state.

By this method we found that exact regular solutions are obtained for $F = a(1 + br^2)^m$, where m is an integer; it is important that the signs of m and b match. For natural m and $m = -3$, all the solutions are equivalent. We note that the cases $m = 1$ and $m = 2$ are already known as the fourth Tolman solution [2] and the Adler solution [3]. A rather simple model of a relativistic star is obtained for $m = 3$, since here the exact solution seems rather laconic. Let us consider this model in more detail.

Let us introduce new symbols: $x = br^2$; $g = (1 + x)^{-1}$; $z = (1 + 4x)^{-1/2}$. Substituting $F = a(1 + x)^3$ into (10) (taking into account the new symbol for x), we obtain

$$\varepsilon = g[1 - x/2 - Cxz]. \tag{19}$$

At the boundary of the sphere r = R, the expressions found should match the exterior Schwarzschild solution, which in the coordinates of (3) looks like F = 1 − 2M/R, L = 1.

The three constants a, b, C can be expressed in terms of the two independent variables M and R using three conditions: continuity of the function F at the boundary; continuity of the derivative of F, which is equivalent to the requirement p(R) = 0; continuity of the function L, which is equivalent to the requirement $\varepsilon(R) = F(R)$. We note that the derivative of the function L at the boundary loses continuity; this is connected with the condition $\mu(R) \neq 0$. As a result of matching the solutions, we obtain the following expressions:

$$a = (1-7\omega/6)^3(1-\omega)^{-2}; \tag{20}$$

$$bR^2 = \omega/(6-7\omega); \tag{21}$$

$$C = (6-3\omega)^{1/2}(6-7\omega)^{-1/2}[9/2-6\omega], \tag{22}$$

here $\omega = 2M/R$.

Substituting F and ε into (4) and (5) allows us to write the density and pressure as

$$\mu/\mu_0 = (9/2+3C)^{-1}[g^2(2-x-2Cxz)+g(5/2+3Cz-4Cxz^3)], \tag{23}$$

$$p/p_0 = (9/2-C)^{-1}[6g^2(1-x/2-Cxz)-g(3/2+Cz)], \tag{24}$$

where $\varkappa\mu_0 = 3b(3/2 + C)$, $\kappa p_0 = b(9/2 - C)$ are the density and the pressure at the center of the star.

Let $p_0/\mu_0 = \kappa$; then $C = (9/2)(1-\kappa)(1+3\kappa)$ and from the restriction $0 < \kappa < 1/3$ it follows that $3/2 < C < 9/2$. At the same time, we found an expression for the physically important quantities μ_0 and p_0 in terms of M and R. We should note that when constructing relativistic models of stars, often the quantities μ_0 and κ are used as the independent variables since they can be determined from physical considerations, in contrast to the mass and the radius which are astronomically observable (measurable) quantities.

CONCLUSION

In conclusion, let us discuss the stability of the models. The method used here allows us to estimate the adiabatic index at the center of the star

$$\gamma_0 = v_0^2(1+1/\kappa), \tag{25}$$

where the square of the speed of sound at the center of the star (for any m) is equal to

$$v_0^2 = (3m/5)(1+\kappa)[m+2+3\kappa(2-m)]^{-1}. \tag{26}$$

For m = 3, γ_0 decreases with an increase in κ, but the minimum for γ_0 is found in the region $\kappa > 1/3$. For $\kappa \cong 1/3$, we have $\gamma_0 \cong 2.4$; i.e., the stability margin is rather large. In the Adler solution (m = 2) for $\kappa = 1/3$, we have $\gamma_0 = 5/3$. The solution for m = 1 seems to be the least stable because even for $\kappa = 0.18$, we have $\gamma_0 < 4/3$. With an increase in m, the minimum for γ_0 is shifted toward the region of lower κ, but γ_0 itself increases. For example, for m = 9, the minimum is realized for $\kappa = 0.21$ but $\gamma_0 = 5.7$. The given estimates show the qualitative behavior of the models as the parameters change. In fact, the models lose stability before κ reaches the value 1/3.

If we connect p_0 and μ_0 with the equation of state of a degenerate Fermi gas for neutrons, then the calculations show that the total mass of the star with an increase in μ_0 reaches the maximum value M_{max}, and then decreases. This means a loss in stability for $\mu_0 > \mu_{max}$, i.e., a neutron star does not have a mass $M > M_{max}$. With an increase in M, the radius of the star decreases down to R_{min}.

For $m=3$ $M_{max}=0{,}33$ M_S $R_{min}=5$ km, $\lg(\mu_{max})=15{,}6$;
For $m=2$ $M_{max}=0{,}34$ M_S, $R_{min}=3{,}6$ km, $\lg(\mu_{max})=15{,}75$;
For $m=1$ $M_{max}=0{,}42$ M_S, $R_{min}=3{,}4$ km, $\lg(\mu_{max})=15{,}9$,

here M_S is the mass of the sun, μ_{max} is expressed in g/cm^3. The result for m = 1 (the Tolman solution) was first obtained in [4]; in that paper, the method for investigation of the stability of the exact solutions is described, using the equation of state for a Fermi gas at the center of the star. We should note that the solution for m = 3 proved to be the least stable, although the critical value is $\kappa_3 = 0.09$, $\kappa_2 = 0.11$ and $\kappa_1 = 0.12$.

The use of large or fractional values of m requires numerical modeling; in this case, the relations (15)-(18) are very useful. Since r = 0, a singular point exists for the difference scheme; it is convenient to expand the functions in a power series at such points, if of course there are no singularities there from the analytical standpoint.

REFERENCES

1. N. N. Paklin, in: Abstracts, Eighth Russian Gravitational Conference: Theoretical and Experimental Problems in Gravitation [in Russian], Moscow (1993), p. 137.
2. R. C. Tolman, Phys. Rev., **55**, 364 (1939).
3. R. J. Adler, J. Math. Phys., **15**, 727 (1974).
4. J. R. Oppenheimer and G. Volkoff, Phys. Rev., **55**, 374 (1939).