JOURNAL OF MATERIALS SCIENCE 8 (1973) 352–362

Theory of multiple fracture of fibrous composites

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The theoretical stress-strain behaviour of a composite with a brittle matrix in which the fibre-matrix bond remains intact after the matrix has cracked, is described. From a consideration of the maximum shear stress at the fibre-matrix interface, the extent of fibre debonding and the crack spacing in a partially debonded composite are derived. The energetics of cracking and the conditions leading to an enhanced matrix failure strain are then discussed and, finally, the crack spacing expected in composites containing fibres isotropically arranged in two or in three dimensions is derived for the case of very thin and hence very flexible fibres.

1. Introduction

In a recent paper Aveston, Cooper, and Kelly [1] introduced a number of simple ideas which have application to understanding the stress-strain curve and type of cracking found in reinforced cements, plasters and other brittle materials. The theory applies to a matrix with a small failure strain containing fibres with a much larger failure strain and accounts quite well for the practical results obtained in a number of systems, e.g. for glass reinforced cement [2] and for gypsum reinforced with pvc or glass [3, 4].

However, the theory as at present developed has a number of shortcomings. These are: (1) the two components are regarded as being unbonded in the sense that there is assumed to be no connection between the elastic displacements in the fibres and in the matrix, and (2) the fibres are all aligned parallel to one another and to the direction in which the composite is strained.

It is the purpose of this paper to remove these two limitations. The second can be removed for the case in which the fibre may be regarded as perfectly flexible, and this is the case which we treat. The first limitation is very difficult to remove in an exact fashion since the problem involves cracking of the matrix, which is by definition a non-elastic process. Here we present an approximate elastic analysis which should apply when the shear rigidity of the matrix is much less than that of the fibres, and this enables us to make some semi-quantitative estimates of the shear stress at the matrix fibre interface close to the ends of the fibres. We compare the elastic treatment with that for the unbonded case, previously reported [1].

The properties we attempt to predict are the density of cracks in the matrix as a function of fibre diameter, elastic properties and volume fraction, the stress-strain curve of the composite and the conditions that should lead to complete suppression of cracks in the matrix at its normal failure strain. These we treat in turn in Sections 4, 5 and 8 for aligned fibres. Following this we deduce results for the non-aligned fibres.

2. Load transfer between fibres and matrix

When long strong fibres of breaking *strain* ϵ_{fu} much larger than the breaking strain of the matrix ϵ_{mu} are incorporated into a fibrous composite, then cracks in the matrix are restrained from opening by the presence of the fibres. Consider aligned fibres and stress applied parallel to these. If the matrix fails at a stress σ_{mu}, the load carried by the matrix, which is $\sigma_{mu} V_m$ per unit area of cross-section, is then thrown onto the fibres and provided the breaking stress of the fibres σ_{fu} satisfies the inequality

$$\sigma_{fu} V_f \geqslant \sigma_{mu} V_m + \sigma' V_f \qquad (1)$$

where σ' (equal to $E_f \epsilon_{mu}$ for an elastic matrix) is the load borne by the fibres when the matrix breaks, then the fibres will not fail and may be further extended, leading to multiple cracking of the matrix.

After the first crack is formed we can consider either of two limiting conditions to apply. In the first of these, considered in detail by Aveston *et al* [1] the matrix and fibres are considered to be "unbonded" in the sense that there is no connection between the elastic dislpacements in the two components, and providing a limiting shear stress τ' is exceeded, the fibres may be drawn through the matrix or the matrix may slide over the fibre. The second limiting case, which we shall treat in this paper, is where the matrix remains bonded to the fibre after it has cracked and, apart from the deformation exactly at the crack surface, remains linearly elastic.

For both cases the fundamental equation governing load transfer between fibres and matrix is obtained from a simple force balance and is, for discrete fibres of radius r in a continuous matrix

$$\frac{dF}{dy} = \frac{2V_f \tau}{r} \tag{2}$$

where dF is the load transferred from fibre to matrix in distance dy and τ is the shear stress acting at the interface.

3. Unbonded case

For the unbonded case τ is a constant equal to τ' and, after the appearance of the first crack in the matrix, continued extension of the specimen will lead to the matrix being traversed by a set of parallel cracks spaced between x' and $2x'$ apart where x' is obtained from Equation 2 by integrating and setting $F = \sigma_{mu} V_m$ to obtain

$$x' = \frac{V_m}{V_f} \frac{\sigma_{mu} r}{2\tau'} \tag{3}$$

If the deformation of the fibres is fully elastic then the additional stress thrown upon the fibres produces a maximum additional strain in the fibres of $\alpha\epsilon_{mu}$ where $\alpha = (E_m V_m)/(E_f V_f)$. The minimum additional strain in the fibres is zero if the crack spacing is $2x'$ and $(\alpha\epsilon_{mu})/2$ if the spacing is x'. The additional strain in the composite, which is equal to the average additional strain in the fibres, thus varies between $(\alpha\epsilon_{mu})/2$ and $3/4(\alpha\epsilon_{mu})$ when cracking is complete, provided the breaking strain of the matrix is single-valued. Aveston *et al* [1] derive these results in detail. It is worth noting in passing that the additional strain which occurs during cracking is independent of the total number of cracks.

The mean crack spacing will be closer to x' than to $2x'$. Further extension of the composite after cracking is complete results in the fibres being stretched further and slipping through the blocks of matrix which can take no further share of the load, so Young's modulus of the specimen will then be equal to $E_f V_f$.

4. Bonded case

For the bonded case τ in Equation 2 is not independent of y. After the first crack has occurred in the matrix an additional stress

$$\Delta\sigma_0 = \frac{\sigma_a}{V_f} - \epsilon_{mu} E_f \tag{4}$$

where σ_a is the applied stress, is placed upon the fibres. This *additional stress* has its maximum value $\Delta\sigma_0$ at the plane of the matrix crack and decays with distance from the crack surface. Appendix I contains an approximate elastic solution of the problem.

It is found that

$$\Delta\sigma = \Delta\sigma_0 \exp - \sqrt{\phi}\, y \tag{5}$$

where

$$\phi^{\frac{1}{2}} = \left(\frac{2G_m E_c}{E_f E_m V_m}\right)^{\frac{1}{2}} \frac{1}{r[\ln(R/r)]^{\frac{1}{2}}} \tag{6}$$

where R is the radial distance from the centre of the fibre at which the displacement in the matrix is equal to the average displacement in the matrix (see Appendix 1). The shear stress at the interface between the fibres and matrix is given by

$$\tau = \frac{r}{2} \Delta\sigma_0 \sqrt{\phi} \exp(-\sqrt{\phi}\, y)\,. \tag{7}$$

The shear stress at the interface is, in this case, dependent upon the value of $\Delta\sigma_0$ and on the elastic constants and decays rapidly with distance y from the crack surface. Since $\phi^{\frac{1}{2}}$ depends inversely on the radius of the fibre, τ is independent of fibre size.

From Equations 7 and 2 we can now find the load F transferred to the matrix in any distance l from the crack surface. Substituting Equation 7 into Equation 2 and integrating, we have

$$F = V_f \Delta\sigma_0[1 - \exp(-\sqrt{\phi}\, l)] \tag{8}$$

Clearly, if the value of $\Delta\sigma_0$ is just that due to the breaking of the matrix, namely $\sigma_{mu}(V_m/V_f)$, the matrix will never break again and only a

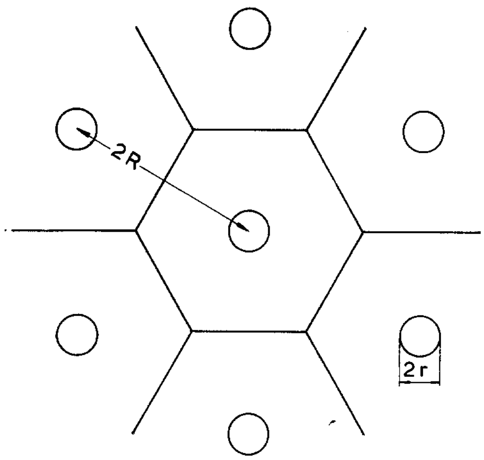

Figure 1 Fibre geometry assumed in the derivation of H.

single crack will result. A small increase in $\triangle\sigma_0$ will, however, result in further cracking into blocks of length between l and $2l$, where from Equation 8, setting $F = \sigma_{mu} V_m$, we find

$$l = -\frac{1}{\sqrt{\phi}} \ln\left\{1 - \frac{\sigma_{mu} V_m}{\Delta\sigma_0 V_f}\right\}; \quad \Delta\sigma_0 \geqslant \frac{\sigma_{mu} V_m}{V_f} \tag{9}$$

When $\Delta\sigma_0$ is larger than a few times $\sigma_{mu}(V_m/V_f)$, Equation 9 reduces to

$$l = \frac{1}{\sqrt{\phi}} \frac{\sigma_{mu} V_m}{\Delta\sigma_0 V_f}. \tag{10}$$

This equation is identical to Equation 3 with τ' replaced by $(\Delta\sigma_0/2)\sqrt{\phi}\, r$, which is shown in Section 6 to be equal to the maximum shear stress at the interface between the fibre and matrix.

Values of ϕ depend upon the value assumed for R in Equation 6. We will take R as one-half the centre-to-centre separation of the fibres, namely $r[\pi/(2\sqrt{3}V_f)]^{\frac{1}{2}}$ for the hexagonal array shown in Fig. 1. The values of $\ln(R/r)$ then vary with volume fraction as shown in Table I.

TABLE I Values of $\ln[\pi/(2\sqrt{3}V_f)]^{\frac{1}{2}} = \psi$

V_f	0.01	0.05	0.1	0.2	0.3	0.4
ψ	2.25	1.45	1.09	0.76	0.54	0.42

5. Stress-strain curve

From Equation 9 we can deduce the stress-strain curve for the elastically bonded case. The additional displacement for the two sides of a single crack is

$$\Delta V' = 2\int_0^{l/2} \frac{\Delta\sigma}{E_f} \mathrm{d}y = 2\frac{\Delta\sigma_0}{E_f\sqrt{\phi}}\left\{1 - \left(1 - \frac{\sigma_{mu} V_m}{\Delta\sigma_0 V_f}\right)^{\frac{1}{2}}\right\}, \tag{11}$$

using Equations 5 and 9. The mean additional strain due to $1/l$ cracks per unit length, as a result of an additional fibre stress $\Delta\sigma_0$, is then $\Delta V'$ times $(1/l)$ or

$$\Delta\epsilon = \frac{-2\Delta\sigma_0}{E_f}\left\{\frac{1 - (1 - \sigma_{mu} V_m/\Delta\sigma_0 V_f)^{\frac{1}{2}}}{\ln(1 - \sigma_{mu} V_m/\Delta\sigma_0 V_f)}\right\} \tag{12}$$

which reduces to $\Delta\sigma_0/E_f$ as $\Delta\sigma_0$ becomes very large and to zero when $\Delta\sigma_0 = \sigma_{mu} V_m/V_f$, as it should. It is noteworthy that Equation 12 is independent of the value of ϕ and so, as in the unbonded case, the additional strain due to cracking is independent of the density of cracks.

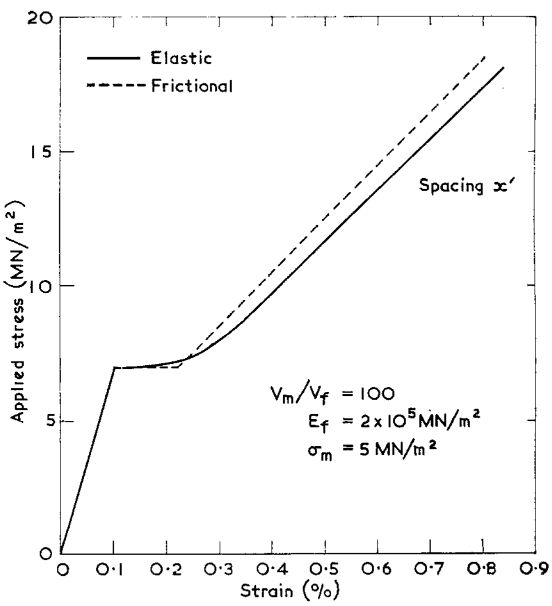

Figure 2 Idealized stress-strain curve for Portland cement reinforced by 1% by volume of long steel fibres. Broken curve assumes fibres are completely unbonded, and the full curve that there is complete elastic continuity between fibre and matrix.

Fig. 2 compares the stress-strain curves deduced for the bonded and unbonded cases for a specimen with the properties of 1% (by volume) of steel in Portland cement. For the sake of comparison the minimum value of crack spacing is taken in both cases. In the bonded case a smoothly rising stress-strain curve occurs without the additional assumption of a variable matrix strength, an assumption which must be made to account for such a rise in the unbonded case. In practice the curves would probably not be sufficiently different to differentiate between the two cases. However, on unloading from any strain greater than that necessary to produce the first crack, the curve will return to the origin if the fibre/matrix bond remains intact. In contrast the specimen will retain some permanent set either if the bonding is initially frictional (our debonded case) or if debonding takes place, after the appearance of the first crack but before the specimen is unloaded.

6. Occurrence of debonding

The elastically bonded case leads, according to Equation 9, to a continuous decrease in the crack spacing without limit, provided $\Delta\sigma_0$ may be increased without limit. Such will not occur in practice since increase in $\Delta\sigma_0$ results in an increase in τ and we expect the value of τ to be limited by the shear strength of the interface, τ_u. From Equation 13a of Appendix 1 the maximum value of τ occurs at the crack surface ($y = 0$) and is given by

$$\tau_{\max} = \frac{\Delta\sigma_0}{2}\left(\frac{2E_c\,G_m}{\psi E_f\,E_m\,V_m}\right)^{\frac{1}{2}} = \frac{\Delta\sigma_0}{2}\,r\,\phi^{\frac{1}{2}} \quad (13)$$

putting $\ln(R/r) = \psi$, where the suffix c refers to the composite.

The condition for elastic continuity between the fibre and matrix to be maintained after the formation of the first crack, may be taken to be that the additional stress thrown onto the fibres should be less than that required to produce a shear stress at the crack, $\tau_{\max}$, which is greater than τ_u. Setting $\Delta\sigma_0 = (\sigma_{mu}\,V_m)/V_f$ and $\tau_{\max} = \tau_u$ in Equation 13, this condition may be written

$$\frac{\sigma_{mu}\,V_m}{V_f} \leqslant 2\tau_u\left[\frac{\psi E_f\,E_m\,V_m}{2E_c\,G_m}\right]^{\frac{1}{2}} \quad (14)$$

Putting $\tau_u = n\,\sigma_{mu}$ and $2G_m = E_m$, Equation 14 reduces to

$$\frac{E_f}{E_c} \geqslant \frac{V_m}{4n^2\,V_f^2\,\psi} \quad (15)$$

A plot of $[V_m/(4\,\psi\,V_f^2)]$ is made in Fig. 3 for small values of V_f. Clearly if $n = 1$ the values of (E_f/E_c) required are very large indeed and usually unattainable in practice so that the expected shear strength of the interface will be exceeded, and some debonding must occur.

For reinforced cement and $n = 1$ the inequality is obeyed at fibre volume fractions of

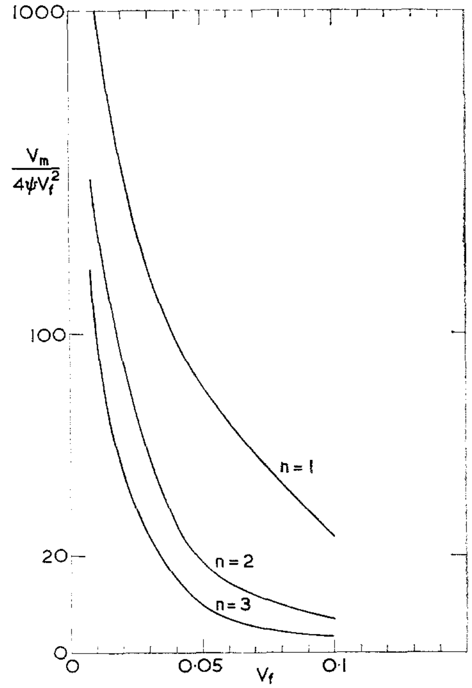

Figure 3 Plot of $V_m/(4\,\psi\,V_f^2)$ versus V_f.

$V_f = 0.36$ for carbon, $V_f = 0.38$ for steel and, for glass, $V_f = 0.50$. These values are outside the range of practically interesting cements for constructional purposes. The compressive strength of cement is of the order of ten times greater than the tensile strength and so we might assume that τ_u may in this case be greater than σ_{mu}. For steel wire-reinforced cement the critical volume fraction for Equation 15 to be obeyed would then be reduced to 0.09 for $n = 2$ and 0.02 for $n = 4$. The occurrence of debonding therefore depends sensitively upon the ratio of τ_u to σ_{mu}, particularly for materials in which τ_u is greater than σ_{mu}.

7. Partial debonding

The calculations of Section 6 indicate that in many systems the appearance of the first crack in the matrix will lead to some debonding at the interface between fibre and matrix. We therefore consider the case in which fibres and matrix are debonded for a certain length and thereafter elastically bonded.

After the formation of the first crack the length of fibre over which debonding occurs will depend on the value of τ', the limiting shear stress at the fibre-matrix interface *after* debonding. The additional fibre stress at a debonded distance l' from the crack is simply

$$\Delta\sigma_0' = \Delta\sigma_0 - \frac{2\pi r\, l'\, \tau'}{\pi r^2} = \Delta\sigma_0 - \frac{2l'\tau'}{r} \qquad (16)$$

and debonding will continue until l' is sufficient to reduce $\Delta\sigma_0'$ so that $\tau_{max} \leqslant \tau_u$. Substituting the value of $\Delta\sigma_0'$ from Equation 16 for $\Delta\sigma_0$ in Equation 13 and putting $\tau_{max} = \tau_u$ we obtain for the debonded length

$$\frac{l'}{r} = \frac{\Delta\sigma_0}{2\tau'} - \frac{\tau_u}{\tau'}\frac{\phi^{-\frac{1}{2}}}{r} \qquad (17)$$

The debonded length (l'/r) must always be less than (x'/r) in Equation 3 because x' represents the length necessary to transfer the breaking load into the matrix at an unbonded interface. There is therefore an upper limit to $\Delta\sigma_0$ obtained by equating l' and x' which is

$$\Delta\sigma_{0max} = \sigma_{mu}\frac{V_m}{V_f} + \frac{2\tau_u}{r\sqrt{\phi}} \qquad (18)$$

for which some part of the interface remains elastically bonded. Since $\Delta\sigma_0$ cannot be less than the first term on the right hand side of Equation 18, the second term represents the range of values of $\Delta\sigma_0$ for which partial debonding occurs.

To find the crack spacing when the interface is part bonded and part debonded we note that the total load per unit area of composite transferred by friction at the surface of $N = V_f/\pi r^2$ fibres over distance l' is $(2l'\tau' V_f)/r$ and so for the matrix to crack again the additional load

$$F = \sigma_{mu} V_m - \frac{2l'\tau'}{r} V_f \qquad (19)$$

must be transferred elastically over a length l of bonded fibre where

$$l = \frac{-1}{\sqrt{\phi}} \ln\left(\frac{r\sqrt{\phi}\,(\Delta\sigma_0 V_f - \sigma_{mu} V_m)}{2\tau_u V_f}\right) \qquad (20)$$

This result is obtained by substituting l' from Equation 17 into Equation 19 and the result for F in place of $\sigma_{mu} V_m$ together with $\Delta\sigma_0'$ from Equation 16 in place of $\Delta\sigma_0$ in Equation 9. The minimum crack spacing L is equal to $(l' + l)$ and is then

$$L = \frac{r\Delta\sigma_0}{2\tau'} - \frac{1}{\sqrt{\phi}}\left(\frac{\tau_u}{\tau'} + \ln\left[\frac{r\sqrt{\phi}(\Delta\sigma_0 - \sigma_{mu} V_m/V_f)}{2\tau_u}\right]\right) \qquad (21)$$

using Equation 17 for l' and Equation 20 for l.

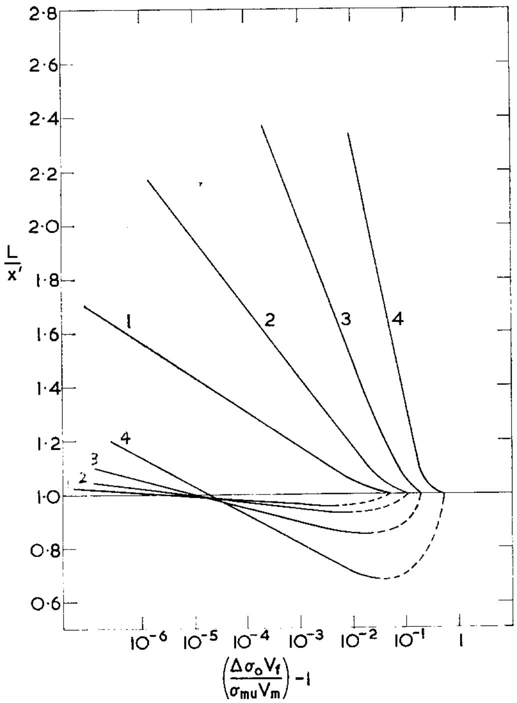

Figure 4 Crack spacing for partial debonding L, relative to that for the unbonded case x' as a function of the fractional increase in fibre stress over that occurring at the first crack, for cement reinforced with
5% polypropylene, $\tau_u = \sigma_{mu}$, $r\sqrt{\phi} = 0.54$, curve 1
1% steel, $\tau_u = \sigma_{mu}$, $r\sqrt{\phi} = 5.65$, curve 2
1% steel, $\tau_u = 2\sigma_{mu}$, $r\sqrt{\phi} = 5.65$, curve 3
5% steel, $\tau_u = \sigma_{mu}$, $r\sqrt{\phi} = 4.12$, curve 4.
Upper curves are for $\tau_u = \tau'$ and lower curves for $\tau_u = 10\tau'$.

In Fig. 4 the crack spacing under conditions of partial debonding calculated from Equation 21, is compared with the debonded crack spacing x' given by Equation 3 for a range of fibre reinforced cements. Practically useful volume fractions of other reinforcements such as glass will have values of $r\sqrt{\phi}$ within the range covered by these examples. For the maximum value of τ' equal to τ_u the load transferred to the matrix will increase linearly over distance l' according to Equation 2, and then exponentially over the remaining distance l with a rate that decreases continuously from the linear value. The crack spacing therefore remains greater than x' until debonding is complete at the value of $\Delta\sigma_0 = \Delta\sigma_{0max}$ given by Equation 18. For $\tau' < \tau_u$, dF/dy increases at the end of the debonded region so that the crack spacing may become less than x' and pass through a minimum before $\Delta\sigma_0 =$

$\Delta\sigma_{0\max}$ If $\Delta\sigma_0$ is set equal to $\Delta\sigma_{0\max}$ in Equation 21 by substitution from Equation 18, then L equals x' as it should. The crack spacing cannot, of course, increase after the minimum is reached and in practice the crack spacing will remain constant (over the region shown by the broken lines) as the debonded length increases to x'.

With the exception of the lower portion of curve 4 in Fig. 4, which corresponds to a rather high volume fraction of stiff fibres and an extreme value of τ_u/τ' the minimum crack spacing is either equal to x' (upper curves) or at least 85% of x' (lower curves). For practical purposes, therefore, the simple theory should suffice to describe the behaviour of fibre-reinforced cement unless the fibre-matrix bond strength in shear, τ_u, is much greater than σ_{mu}.

8. Energetics

We assume that a crack cannot form in the matrix unless the work done by the applied stress is greater than the increase in elastic strain energy of the composite plus the fracture surface work of the matrix per unit area of cross section of the composite. For the fully elastic body considered in Section 4 the work done by the applied stress is precisely twice the increase in strain energy and therefore to find the condition for the formation of the first crack, knowing the load on the composite, we merely need to calculate the increase in length of the fibres when a crack forms in the matrix. This increase in length is due to an increase in load carried by the fibres at the crack of $\Delta\sigma_0 = (\sigma_{mu} V_m)/V_f$ so that after failure the length of the specimen increases by

$$\delta l = 2\int_0^\infty \frac{\Delta\sigma}{E_f}\,dy = \frac{2}{E_f}\int_0^\infty \Delta\sigma_0 \exp - \phi^{\frac{1}{2}}\, y\, dy = \frac{2\Delta\sigma_0\,\phi^{-\frac{1}{2}}}{E_f} = 2\alpha\,\epsilon_{mu}\,\phi^{-\frac{1}{2}} \quad \ldots\ldots (22)$$

using Equation 5.

The work done by the applied stress per unit area of cross section is $E_c\,\epsilon_{mu}\,\delta l$ so that a crack cannot form unless

$$E_c\,\epsilon_{mu}^2\,\alpha\phi^{-\frac{1}{2}} \geqslant 2\gamma_m V_m \qquad (23)$$

where γ_m is the fracture surface work of the matrix. Writing Equation 23 as an equality we have

$$\epsilon_{mu}^2 = \frac{2\gamma_m V_m \phi^{\frac{1}{2}}}{\alpha E_c} = \frac{2\gamma_m V_m}{r\alpha E_c}\left(\frac{2G_m E_c}{\psi E_f E_m V_m}\right)^{\frac{1}{2}}. \quad \ldots\ldots (24)$$

Clearly for a small r, ϵ_{mu} can become very large and if its value from Equation 24 becomes larger than that of the unreinforced matrix there will be complete suppression of cracking at the expected failure strain of the matrix.

Aveston *et al* [1] have considered the energetics of formation of the first crack in the matrix for the unbonded case. It is of interest to compare the results for the two cases. A simple way to do this is to write Equation 21 of Aveston *et al* in the form

$$\epsilon^2_{mud} = \frac{6\gamma_m V_m}{\alpha E_c x'} \qquad (25)$$

where ϵ_{mud} is the (enhanced) failure strain of the completely debonded composite. Then from Equations 24, 3 and 25 and the condition for the fibre-matrix bond to remain intact after the formation of the first crack (Equation 15 with $n = 1$) we get

$$\frac{\epsilon_{mu}^2}{\epsilon_{mud}^2} = \frac{\sigma_{mu}}{3\tau'} \qquad (26)$$

and so the failure strain will be greater for the elastic case, provided $\tau' < \sigma_{mu}/3$. τ' must be less than τ_u. Section 6 showed that τ_u must generally be greater than σ_{mu} for the elastic case to apply. It follows that there will be a large difference between the enhanced failure strain of the matrix in the two cases of elastic interface and debonded interface only if

$$\tau' \ll \sigma_{mu}/3 \quad \text{and} \quad \tau_u > \sigma_{mu} \qquad (27)$$

Our present knowledge of the quantities τ', τ_u and σ_{mu} for specific systems of matrix and fibre is too scanty to carry the discussion further.

9. Non-parallel fibres

We have shown that during cracking of the matrix the fibre matrix interface is expected to debond, and that the limiting crack spacing for large strains of the composite correspond to the case when the fibre matrix bond is entirely frictional. The problem remains of what modifications to the debonded case are needed when the fibres are randomly arranged in direction either in a plane or in three dimensions. This problem can be resolved into two distinct parts: the first is to calculate the number of fibres that actually bridge a crack, and the second is to determine the distance in the direction of the applied stress over which the fibres bridging the crack transfer

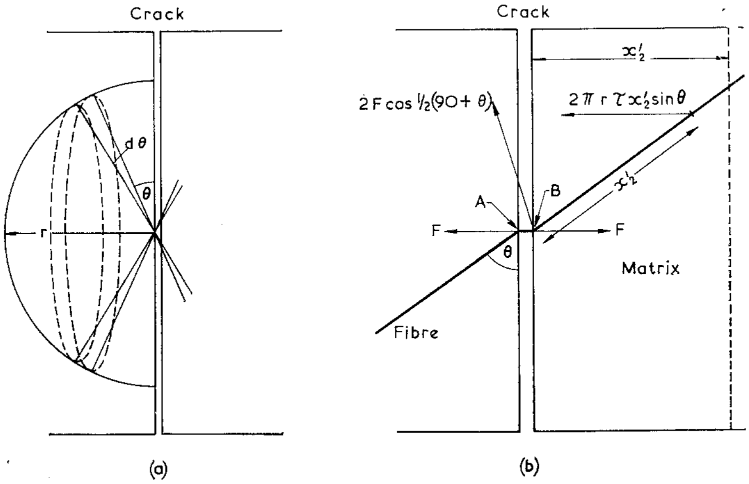

Figure 5 (a) Diagram showing that probability of fibre crossing a crack at angle θ in a three dimensionally random composite increases with decrease in θ. (b) Geometry assumed for a fibre crossing a crack.

sufficient load to cause further cracking of the matrix.

If the fibres are aligned, the number crossing unit area of a crack is equal to $V_f/\pi r^2$ irrespective of whether the fibres are continuous or discontinuous. When the fibres are random in a plane it is shown in Appendix 2 that this number is reduced to $2/\pi(V_f/\pi r^2)$ and for a completely random arrangement in three dimensions to $\frac{1}{2}(V_f/\pi r^2)$.

To obtain the crack spacing we assume that the lengths of fibre bridging the crack are normal to the crack face, as this is the only geometry that enables two points in the matrix, A and B (Fig. 5b), that were coincident before cracking, to remain adjacent to each other as the crack opens and thus allow the matrix to remain intact. The assumption will apply best for thin and, hence, very flexible fibres. In practice there must be some crumbling of the matrix in order that a fibre of diameter comparable with the crack width can bend through an angle $(90 - \theta)$. In addition, the fibre must either undergo some plastic deformation or be very weak in shear parallel to its axis so as to reduce the large bending stresses which would otherwise fracture the fibre at A or B when the stress along AB is much less than the normal fibre strength [5].

Neglecting these effects, the fibres may be regarded as passing over a pulley at A and B so that the load transferred by each fibre back into the matrix on either side of the crack is the sum of the reaction of the "pulley" at A which operates close to the crack surface, and the load transferred by frictional forces between the fibre and matrix. We are grateful to Mr D. K. Hale for pointing out to us the existence of this pulley.

10. Fibres random in a plane

When the matrix cracks, an additional stress equal to $(\pi/2)(\sigma_{mu} V_m)/V_f$ is placed upon each fibre. Since each fibre is considered flexible all fibres increase in length elastically by an amount

$$\frac{1}{2}\left(\frac{\pi}{2}\right)\frac{\sigma_{mu}\, V_m\, 2x'}{V_f\, E_f} = \frac{\pi}{2}\,\alpha\,\epsilon_{mu}\, x' \cdot \quad (28)$$

Call x_2', $= (\pi/2)\,x'$, the distance measured along the fibre over which the fibres shed this additional load due to the action of a shear stress τ' at the fibre-matrix interface. At distances greater than x_2' from the crack the strain in fibre and matrix will be the same. Each fibre therefore exerts a force on the matrix, $2\pi\, r\, \tau'\, x_2' \sin\theta$, normal to the crack, where θ is the angle between the fibre and the crack surface. The total force per unit area of crack exerted by all fibres is equal to the force per fibre times the number of fibres at the particular angle summed for all angles, i.e.

$$F_1 = \int_0^{\pi/2} 2\pi r\, \tau'\, x_2' \sin\theta \,\frac{N \sin\theta\, d\theta}{\pi/2}$$
$$= \pi r\, \tau'\, x_2'\, N \quad (29)$$

see Appendix 2.

In addition to the frictional force there is a "pulley" force. The force per fibre resolved normal to the crack due to the pulley force is

$$\pi r^2 \left(\frac{\pi}{2}\right) \frac{\sigma_{mu} V_m}{V_f} (1 - \sin\theta)$$

(see Fig. 5b). Hence the total pulley force which equals the force per fibre times the number of fibres is, per unit area of the crack,

$$F_2 = \frac{\pi^2 r^2}{2} \frac{\sigma_m V_m}{V_f} \int_0^{\pi/2} (1 - \sin\theta) \frac{N \sin\theta}{\pi/2} \, d\theta$$

$$= N \pi r^2 \left(\frac{\sigma_m V_m}{V_f}\right) (1 - \pi/4) \cdot \quad (30)$$

The matrix will crack again at a distance x_2' from the crack such that the sum of the forces $(F_1 + F_2)$ given by Equations 29 and 30 respectively, add to $\sigma_{mu} V_m$, yielding

$$x_2' = \frac{\pi}{2} \left(\frac{V_m}{V_f}\right) \frac{\sigma_{mu} r}{2\tau'} = \frac{\pi}{2} x' \quad (31)$$

This result (Equation 31) shows that the matrix cracks again at a distance from the first crack of precisely that distance measured along the fibre over which the fibre transfers the additional load to the matrix. This "coincidence" arises because the sum of the forces F_1 and F_2 per fibre is equal to $(\pi/2)\, \pi r^2 (\sigma_{mu} V_m/V_f)$ and hence independent of the orientation of the fibre.

11. Fibres random in three dimensions

The procedure is the same as for two dimensions but the number of fibres crossing unit area of crack and running in directions between θ and $(\theta + d\theta)$ to the crack face is $N \cos\theta \, d\theta$. The final result is

$$x_3' = \frac{V_m}{V_f} \frac{\sigma_{mu} r}{\tau'} = 2x' \cdot \quad (32)$$

Comparing Equations 31 and 32 with Equation 3, we see that for the debonded case, with fibres sufficiently flexible for bonding and shearing forces of the type considered by Hing and Groves [6] to be neglected, that the minimum crack spacing is predicted to be increased from that obtained with aligned fibres to about 50% greater for fibres in a planar mat and by a factor of two for fibres running in three dimensions.

12. Conclusions and discussion

When the matrix of a continuous fibre-reinforced composite fails at a lower strain than the fibres, multiple cracking of the matrix will always result as long as the fibres are strong enough to withstand the additional load. If the fibres are discontinuous and fibre debonding occurs, the increased loading of the fibres resulting from the transfer of load from the matrix will put the interface in tension and so may lead to a low value of τ' and pull-out of the fibres. Multiple cracking is most easily observed with resin matrices when the matrix is cracked by decreasing the temperature of a composite with $\alpha_m > \alpha_f$ (α is the linear thermal expansion coefficient) so that the matrix contracts around the fibre and a hoop stress is produced which puts the interface between the fibre and matrix into normal compression. These experimental conditions were used by Cooper and Sillwood [7].

Multiple cracking will also be observed when there is a strong mechanical bond between the fibre and matrix so that the bond strength depends ultimately on the shear strength of the matrix. If such a mechanical bond can accommodate a small amount of shear strain without a drastic loss of strength, as appears to be the case with cement, multiple cracking will still be observed even though the fibres have debonded, and the situation will be equivalent to the frictional bond treated previously.

For the purely elastic case to apply, two conditions are necessary: the ratio τ_u/σ_{mu} and V_f must be sufficiently high for inequality 15 to be satisfied and the fibre must be sufficiently tough for the stress concentration at the tip of the crack in the matrix not to cause failure of the fibre in the plane of the crack (which is sometimes observed experimentally when brittle fibres are well bonded to a brittle matrix). If these conditions are fulfilled the crack spacing as given by Equation 9 will be very close and the crack opening correspondingly small. Normally, however, as the applied stress is increased above that necessary to produce the first crack, partial debonding will eventually occur and the ultimate crack spacing, given by Equation 20 will depend on the ratio τ_u/τ'. In practice, τ_u will often be limited to the order of σ_{mu} and the crack spacing will then be close to that given by Equation 3 for all reasonable values of τ'/τ_u.

For the debonded case the situation is essentially the same whether the fibres are aligned or are in a random configuration, either in a plane or in three dimensions. The only difference is that the number of cracks per unit length is either $(2/\pi)$ or $\frac{1}{2}$ that of an aligned composite, respectively. Perhaps of greater significance is the fact that if the fibres can accommodate the bending strains incurred when a fibre crosses a crack at an acute angle, the strength of a brittle matrix composite will be related to that of the aligned material by the same factors. These are much greater than the values of $\frac{1}{3}$ and $\frac{1}{6}$ often suggested in the literature on the basis of an extension of Cox's elastic analysis [8] to the strength of composites, and are also greater than the efficiency factors recently proposed by Allen [9] and Laws [10]. The work of fracture derived from fibre pull-out will be an equally large proportion of that attained with an aligned composite, and as there will be a further gain from the work of bending the fibres through an angle, as recently suggested by Hing and Groves [6], there would appear to be little to be gained with a cross-ply, or even a fully aligned brittle-matrix composite, compared with a random two-dimensional arrangement, if the aim of reinforcement is solely to increase strength and toughness of a brittle matrix without a need to influence the elastic modulus.

The principal limitation to the theory presented in Sections 4 to 8 is the neglect of any tensile stresses in the matrix in the radial and tangential directions with respect to the fibre surface. These arise for the simple form taken for the equation of equilibrium. It follows that our expression for the maximum shear stress at the fibre matrix interface is only approximate; it depends on E_f and on G_m. Some dependence on the Poisson ratios of fibre and matrix must arise and this cannot be treated with the present elastic solution. Nevertheless the analysis we believe has value since using it the physical principles may be explored which will govern the behaviour in the elastic case and in the partially debonded case.

We also note that a rigorous elastic analysis – only possible at present by numerical methods and for particular values of the elastic constants [11] – for the case of a broken fibre in a continuous matrix yields a value for the shear stres at the interface which agrees with the shear lag analysis at distances from the end of the fibre greater than 0.4 of a fibre radius. The radial stresses at the interface are found to be very much less than the shear stresses.

The treatment for the effects of non-aligned fibres apply in the limit of very thin fibres. In this approximation the ratio of crack spacing for aligned fibres, for fibres arranged in a planar mat, and for fibres in a three-dimensional felt, of $1:(\pi/2):2$ will apply both in the debonded case and in the fully elastic case, if in the latter the crack spacings for the aligned case, planar mat and felt were all made at the same value of $\Delta\sigma$. This follows by comparing Equations 3 and 10 if the expression for τ_{max} in Equation 13 is substituted into Equations 10. The fully elastic and debonded cases are the same if τ' and τ_{max} are interchanged.

The small difference between the predictions for aligned and non-aligned fibres accounts for the fact that Aveston *et al* [1] obtained quite good agreement between the simple theory developed for an aligned composite and experimental results for composites containing randomly arranged fibres.

Acknowledgements
This work forms part of the research programme of the National Physical Laboratory. We are grateful to Messrs D. K. Hale, J. Sillwood and G. D. Sims for helpful comments on the manuscript.

Appendix 1
A. Crack spacing; elastic case
Consider an aligned fibrous composite stretched in tension parallel to the fibres and let the matrix crack at a strain ϵ_{mu}. When the tensile strain ϵ_c of the composite reaches ϵ_{mu} a crack appears in the matrix as in Fig. 6a. We idealize the situation as in Fig. 6b and suppose elastic continuity maintained at the interface between fibre and matrix. We calculate the stress in the fibre by a modified shear lag analysis.

We assume that the tensile strain in the fibre and matrix are equal for large values of y and that the stress in the fibre is then $E_f\,\epsilon_{mu}$. If $\sigma_f(y)$ is the stress in the fibre at distance y from the crack we put

$$\Delta\sigma = \sigma_f - E_f\,\epsilon_{mu} \tag{1a}$$

Then we assume

$$\frac{d\Delta\sigma}{dy} = H(V_f - V_m) \tag{2a}$$

where H is a constant and V_f and V_m are the y-components of elastic displacement in fibre and matrix respectively. Then differentiating Equation 2a we have

$$\frac{d^2\Delta\sigma}{dy^2} = H\left\{\frac{\sigma_f}{E_f} - \frac{dV_m}{dy}\right\} \tag{3a}$$

We will assume

$$\frac{dV_m}{dy} = \frac{d\bar{V}_m}{dv} = \bar{\epsilon}_m \tag{4a}$$

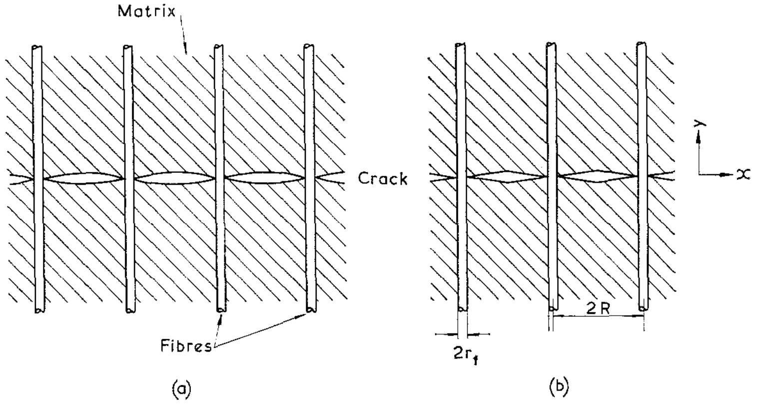

Figure 6 Matrix cracking with elastic continuity between fibres and matrix (a) and as assumed in the shear lag analysis (b).

where $\bar{\epsilon}_m$ is the average strain in the matrix. Since the tensile load supported by all cross sections of the composite must be the same, we have

$$\bar{\epsilon}_m = \frac{1}{E_m V_m} (E_c \epsilon_{mu} - \sigma_f V_f) \cdot \qquad (5a)$$

Substituting Equation 5a into 4a and Equation 4a into 3a and using Equation 1a we have

$$\frac{d^2 \Delta\sigma}{dy^2} = \phi \Delta\sigma \qquad (6a)$$

where

$$\phi = \frac{HE_c}{E_f E_m V_m}$$

The general solution of Equation 6a subject to the boundary condition $\Delta\sigma = 0$ at large y is

$$\Delta\sigma = \Delta\sigma_0 \exp - \sqrt{\phi} y \qquad (7a)$$

where $\Delta\sigma_0$ is the difference between the stress in the fibre at the surface of the crack and that in the fibre a long way from the crack.

To proceed further we need to estimate H. For the cross-sectional geometry shown in Fig. 1 we assume that the condition of stress equilibrium in the matrix is given simply by

$$\frac{\partial \tau_{ry}}{\partial r} + \frac{\tau_{ry}}{r} = 0 \qquad (8a)$$

of which a solution is $\tau_{ry} r = $ const, where τ_{ry} is the shear stress on planes parallel to the fibres at radial distance r. The constant is equal to $- \tau_i r_f$ where τ_i is the shear stress at the interface and r_f the radius of the fibre. From this solution to Equation 8a we have

$$\tau_{ry} r = G_m \frac{dV}{dr} r = - \tau_i r_f$$

where G_m is the shear modulus of the matrix, and so integrating to find V we have

$$(V_f - V_m) = + \frac{\tau_i r_f}{G_m} \ln \left(\frac{R}{r_f}\right) \qquad (9a)$$

where R is the radial distance from the centre of the fibre at which the displacement in the matrix is equal to the average displacement in the matrix. We now note that by a simple force balance the shear stress at the interface is related to the rate of change of stress in the fibre by

$$\tau_i = - \frac{r_f}{2} \frac{d\Delta\sigma}{dy} \qquad (10a)$$

Substituting for $(d\Delta\sigma)/dy$ from Equation 2a into 10a and then eliminating $(V_f - V_m)$ between Equations 10a and 9a we have

$$H = \frac{2G_m}{r^2_f \ln R/r_f} \qquad (11a)$$

and so

$$\phi^{\frac{1}{2}} = \left(\frac{2G_m E_c}{E_f E_m V_m}\right)^{\frac{1}{2}} \frac{1}{r[\ln (R/r)]^{\frac{1}{2}}} \qquad (12a)$$

writing r for the radius of the fibre in place of r_f in Equation 11a.

If we substitute for $\Delta\sigma$ in Equation 10a from 7a we have

$$\tau_i = \frac{r}{2} \Delta\sigma_0 \sqrt{\phi} \exp(- \sqrt{\phi}\, y) \qquad (13a)$$

Appendix 2

If there are N aligned fibres per unit volume, the number crossing a plane at angle θ to the fibres per unit area of that plane is $N \sin \theta$. If all angles θ between 0 and $\pi/2$ are equally probable then the number of fibres crossing any plane at angle between θ and $(\theta + d\theta)$ to the plane, per unit area of the plane, is

$$N \sin\theta \frac{d\theta}{(\pi/2)}$$

Hence the total number of fibres crossing unit area of any plane is

$$\int_0^{\pi/2} N \frac{2}{\pi} \sin\theta \, d\theta = \frac{2N}{\pi} \cdot \qquad (14a)$$

If the fibre directions are randomized in three dimensions the number of fibres per unit volume lying at angles between θ and $(\theta + d\theta)$ to any direction is $N \cos\theta \, d\theta$ (Fig. 5a) so the number of fibres crossing any plane per unit area of that plane is

$$\int_0^{\pi/2} N \cos\theta \sin\theta \, d\theta = \frac{N}{2} \cdot \qquad (15a)$$

In all cases

$$N = \frac{V_f}{\pi r^2} \qquad (16a)$$

where V_f is the volume fraction and r the radius of a fibre.

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Received 4 July and accepted 5 September 1972.