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# Some Theoretical Considerations of Fibre Pull-Out from an Elastic Matrix

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Previous theoretical work on fibre pull-out from an elastic matrix is briefly discussed and its relation to this present work is indicated. The form of the distribution of shear stress and that of load along the fibre length is determined and its dependence on the elastic properties and fibre length is shown. The theory has been developed to account for the debonding of fibres from the matrix. The maximum fibre load necessary to cause complete debonding and subsequent pull-out is determined and the dependence of the maximum shear stress on the effective embedded fibre length is shown to affect the shear strength calculated from a pull-out test.

## 1. Introduction

With the development of brittle-matrix composite materials the importance of the contribution of fibre pull-out to the strength of the composite has grown; the extent of this fibre pull-out is influenced by the development of an interfacial bond between the fibre and matrix. In comparison with the extensive literature on composites in general very few papers, a few of which are cited by Greszczuk [1], deal specifically with the fibre-matrix interface. A theoretical analysis of stresses near a discontinuity in a filament-reinforced composite metal has been made by Dow [2] and one of his theoretical model configurations is close to a pull-out configuration. However, his equation for the shear stresses along the fibre-matrix interface is not equivalent to that derived by Greszczuk [1], who considered fibre pull-out, since the boundary conditions are not identical. In the former case the fibre is effectively stressed at both ends but in the latter the embedded end of the fibre carries no load, the load having been entirely transferred to the matrix at this point.

Very little other theoretical work on the shear stresses developed during fibre pull-out has been done. Watstein [3] measured experimentally the shear stresses developed when extracting iron rods from cement. Consequently the plastic-matrix theory which assumes a uniform shear stress along the length of the embedded fibre has been applied, as a first approximation, to fibre pull-out from an elastic matrix in cases where the embedded fibre length is short. De Vekey and Majumdar [4] employed this approach, pointing out that the calculated values of the shear strength of a glass fibre/cement interface were likely to be low.

If the matrix is elastic the shear stress is no longer uniform and the load transferred between the fibre and the matrix does not change uniformly with length along the fibre. The distribution of shear stress and that of load along the fibre length depend on the elastic properties of the fibre and the matrix and the length of the embedded fibre. A determination of these functions for shear stress and load is presented here, the former being directly equivalent to the equation derived by Greszczuk [1], the sole difference being the end of the fibre from which distance along the fibre is measured. The theory has been further developed to include debonding. The value of the maximum shear stress developed relative to the shear strength of the fibre-matrix interface determines whether debonding and pull-out will occur. Based on this criterion the variation in the load necessary to produce fibre pull-out and in the maximum shear stress developed with change in the effective embedded fibre length has been determined.

## 2. Distribution of Load and Shear Stress along the Fibre Length

### 2.1. Distribution of Load along the Fibre Length

Consider a single fibre embedded in a matrix to a

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length $l/2$. Let the $x$ direction be parallel to the fibre length and the embedded end of the fibre be at $x = 0$, see fig. 1.

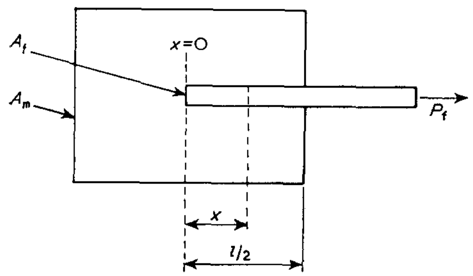


*Figure 1* Geometry of pull-out test.

If an axial load $P_f$ is applied to the fibre then

$$dP = C\tau(x)\,dx \tag{1}$$

where $dP$ is the change in the load $P$ at $x$, between $x$ and $x + dx$ along the fibre, $\tau(x)$ is the function for the shear stress in terms of $x$ (the length along the fibre) and $C$ is the circumference of the fibre in contact with the matrix.

If we assume that $\tau \propto (u - v)$, i.e. $\tau = K(u - v)$ where $K$ is some constant, $u$ is the virtual displacement in the direction of the fibre at a point in the fibre a distance $x$ from the embedded end if the matrix had the same elastic properties as the fibre and $v$ is the virtual displacement of the matrix at the same point, if the fibre was replaced by the matrix. Then we have

$$dP = CK(u - v)\,dx \tag{2}$$

or

$$\frac{dP}{dx} = H(u - v) \tag{3}$$

where $H = CK$, a constant.

Differentiating we obtain

$$\frac{d^2P}{dx^2} = H\left(\frac{du}{dx} - \frac{dv}{dx}\right) = H(e_f - e_m) \tag{4}$$

where $e_f$ and $e_m$ are the virtual fibre and matrix strains respectively at the point $x$. Thus we can write

$$\frac{d^2P}{dx^2} = HP\left(\frac{1}{A_fE_f} - \frac{1}{A_mE_m}\right) = \frac{HP}{R} \tag{5}$$

where

$$\frac{1}{R} = \frac{1}{A_fE_f} = \frac{1}{A_mE_m}$$

$A_f$ and $A_m$ are the cross-sectional areas of the fibre and matrix respectively and $E_f$ and $E_m$ are the elastic moduli of the fibre and matrix respectively.

We can solve the differential equation

$$\frac{d^2P}{dx^2} - aP = 0 \tag{6}$$

where

$$a = \frac{H}{R}$$

by transformation and obtain

$$P = P_f\,\frac{\sinh\sqrt{a}x}{\sinh\sqrt{a}l/2} \tag{7}$$

The sinh function is approximately linear for small values and exponential at large values; consequently the distribution of the relative load $P/P_f$ is affected by the length of the embedded fibre $l/2$. The ratio $\sinh\sqrt{a}x/\sinh\sqrt{a}\,l/2$ for $\sqrt{a}\,l/2$ equal to 1 and 10 is shown in fig. 2. Clearly when $l/2$ is sufficiently small that $\sqrt{a}\,l/2 = 1$, given $a \geqslant 1$, then the build-up in the load along the fibre length from the embedded end is approximately linear. This is not the case for a long embedded fibre, e.g. when $\sqrt{a}\,l/2 = 10$.

## 2.2. Distribution of the Shear Stress along the Fibre Length

Previously we have defined $\tau = K(u - v)$ thus we can write

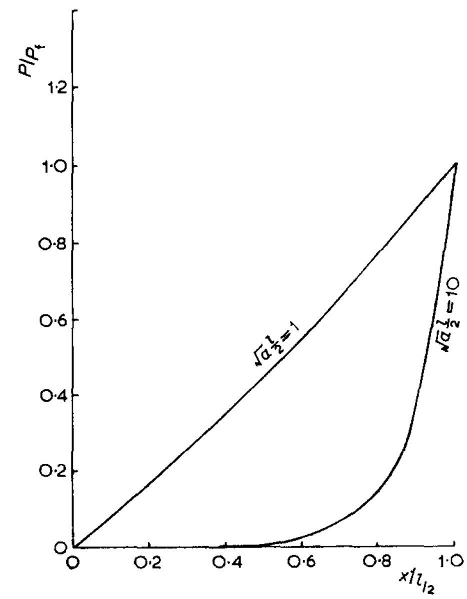


*Figure 2* Variation of fibre load with length along embedded fibre.

$$\frac{d\tau}{dx} = \frac{KP}{R} \tag{8}$$

and integrating we obtain

$$\tau = \frac{KP_f}{R\sqrt{a}} \frac{\cosh\sqrt{a}x}{\sinh\sqrt{a}\,l/2} \tag{9}$$

The maximum shear stress occurs when $\cosh\sqrt{a}x$ is a maximum, i.e. $x = l/2$, at the point where the fibre enters the matrix, and is given by

$$\tau_{max} = \frac{KP_f}{R\sqrt{a}} \coth\sqrt{a}\,l/2 \tag{10}$$

For an infinitely long fibre

$$\tau_{max} = \tau_{max}{}^{\infty} = \frac{KP_f}{R\sqrt{a}} \tag{11}$$

since $\cosh\sqrt{a}\,l/2$ tends towards $\sinh\sqrt{a}\,l/2$ as $\sqrt{a}\,l/2$ tends towards infinity. Thus we can write

$$\tau = \tau_{max}{}^{\infty} \frac{\cosh\sqrt{a}x}{\sinh\sqrt{a}\,l/2} \tag{12}$$

The distribution of the relative shear stress along the embedded fibre length is affected by the embedded fibre length for a given $a$. For short fibres $\cosh\sqrt{a}x$ can be either greater than or less than $\sinh\sqrt{a}\,l/2$ and $\tau$ can be greater than or less than $\tau_{max}{}^{\infty}$, see fig. 3. For long fibres $\cosh\sqrt{a}x$ is always less than $\sinh\sqrt{a}\,l/2$ and $\tau/\tau_{max}{}^{\infty}$ varies between zero (approximate since at $x = 0$,

$\cosh\sqrt{a}x/\sinh\sqrt{a}\,l/2 = 1/\sinh\sqrt{a}\,l/2$ where $\sinh\sqrt{a}\,l/2 \gg 1$) and unity.

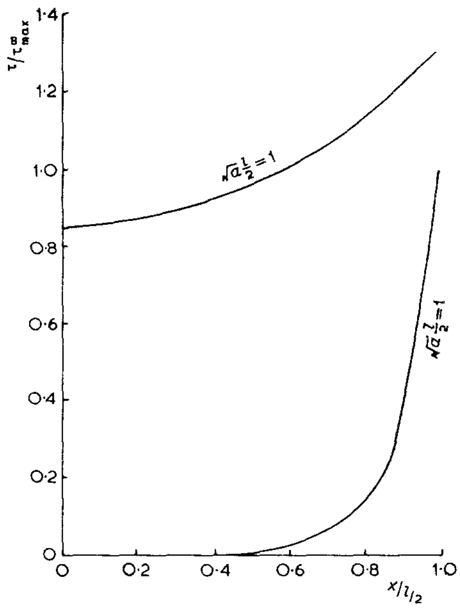


*Figure 3* Variation of shear stress with length along embedded fibre.

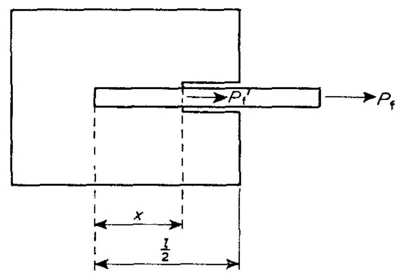


*Figure 4* Debonded fibre configuration.

## 3. Load to Pull-out and the Determination of the Shear Strength

### 3.1. The Variation of the Fibre Load to Pull-out with Embedded Fibre Length

For an embedded fibre loaded to $P_f$ the shear stress at the point where the fibre enters the matrix is given by equation 10 above.

$$\tau_{max} = \frac{KP_f}{R\sqrt{a}} \coth\sqrt{a}\,l/2 \tag{10}$$

This is the *maximum* shear stress along the embedded fibre length. If the load $P_f$ is such that $\tau_{max}$ equals $\tau_s$, the shear strength of the interface, then the fibre will debond from the matrix at the point where the fibre enters the matrix. Whether the fibre continues to debond at a constant load $P_f$ or whether an increase in load is necessary depends on a number of factors.

Consider an embedded fibre of length $l/2$, debonded from the free end up to a length $(l/2 - x)$ into the matrix, see fig. 4, under the load $P_f$. At the bonded/debonded interface the load in the fibre $P_f'$ is given by

$$P_f' = P_f - \tau_i C(l/2 - x) \tag{13}$$

where $\tau_i$, the interfacial shear strength due to friction, is assumed to be constant over the debonded region. The shear stress at this point is given by

$$\tau = \frac{KP_f'}{R\sqrt{a}} \coth\sqrt{a}x \tag{14}$$

If as the debonded length increases (i.e. $x$ decreases) this expression is always equal to $\tau_s$, the fibre will continue to debond. This will occur if the decrease in the term $P_f'$ is compensated by the increase in the term $\coth\sqrt{a}x$ as $x$ decreases. Thus we can write

$$P_f' = P_f - \tau_i C(l/2 - x) = \tau_s \frac{R\sqrt{a}}{K} \tanh \sqrt{a}x \quad (15)$$

Differentiating we obtain

$$\frac{dP_f}{dx} = \frac{\tau_s R a}{K} \operatorname{sech}^2 \sqrt{a}x - \tau_i C \quad (16)$$

and the maximum value of $P_f$ occurs when $dP_f/dx = 0$, i.e. when

$$x = x_{max} = \frac{1}{\sqrt{a}} \cosh^{-1} \sqrt{\frac{\tau_s R a}{\tau_i C K}} = \frac{1}{\sqrt{a}} \cosh^{-1} \sqrt{\frac{\tau_s}{\tau_i}} \quad \ldots\ldots (17)$$

At this point debonding continues without any further increase in $P_f$ and the failure of the bond is catastrophic. Clearly the stage at which debonding becomes catastrophic is dependent on the ratio $\tau_s/\tau_i$. When $\tau_s/\tau_i \geqslant \cosh^2 \sqrt{a}\, l/2$ then $x_{max} = l/2$ and the debonding process is catastrophic immediately it commences. If $\tau_s/\tau_i < \cosh^2 \sqrt{a}\, l/2$ a further increase in $P_f$ is necessary for debonding to continue.

The maximum load on the fibre required to achieve complete debonding and pull-out is given by

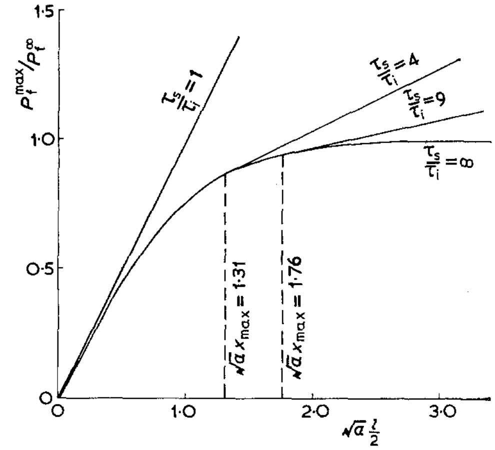


*Figure 5* Variation of maximum fibre load with embedded fibre length factor for various friction conditions.

value $\tau_i C\, l/2$ and continue to fall as the fibre is withdrawn from the matrix.

Greszczuk [1] considers only the immediate catastrophic failure of the interfacial bond and assumes that all the fibre load is transferred to the matrix by shear forces with no frictional

$$\begin{cases} P_f^{max} = \dfrac{\tau_s R\sqrt{a}}{K} \tanh \sqrt{a}\, l/2 & l/2 \leqslant x_{max} \quad (18a) \\ P_f^{max} = \dfrac{\tau_s R\sqrt{a}}{K} \tanh \sqrt{a}\, x_{max} + \tau_i C(l/2 - x_{max}) & l/2 > x_{max} \quad (18b) \end{cases}$$

or alternatively

$$\begin{cases} P_f^{max} = P_f^{\infty} \tanh \sqrt{a}\, l/2 & (19a) \\ P_f^{max} = P_f^{\infty}\left[\tanh \sqrt{a}x_{max} + \dfrac{\tau_i}{\tau_s}\sqrt{a}(l/2 - x_{max})\right] & (19b) \end{cases}$$

where $P_f^{\infty}$, the load required to debond an infinitely long fibre with no frictional forces present, is given by

$$P_f^{\infty} = \frac{\tau_s R\sqrt{a}}{K} \quad (20)$$

The variation in the load required to achieve complete debonding with the embedded fibre length factor $\sqrt{a}\, l/2$ is shown in fig. 5 (by plotting the ratio $P_f^{max}/P_f^{\infty}$ for various ratios of $\tau_s/\tau_i$). It is assumed that $P_f^{max}$ is less than the breaking load of the fibre and pull-out, not fibre fracture, occurs. Once debonding has been completed and pull-out has commenced the load necessary to continue pull-out will fall to a forces present. This requires that $\tau_i = 0$, so that from equation 19 above the maximum load to pull-out being given by the expression

$$P_{f0}^{max} = P_f^{\infty} \tanh \sqrt{a}\, l/2 \quad (21)$$

This is identical with the $\tau_s/\tau_i = \infty$ plot in fig. 5 (i.e. $\tau_i = 0$) and is a particular case of the more general equation, for $P_f^{max}$ (equation 19), when $l/2 \leqslant x_{max}$ and catastrophic failure always occurs.

### 3.2. The Variation in the Maximum Shear Stress with Effective Embedded Fibre Length and the Determination of the Shear Strength from a Pull-out Test

The maximum shear stress along the embedded fibre length has previously been determined and is given by equation 10.

$$\tau_{\max} = \frac{KP_{\mathrm{f}}}{R\sqrt{a}} \coth \sqrt{a}\, l/2 \qquad (10)$$

This can be written as

$$\tau_{\max} = \tau_{\max}{}^{\infty} \coth \sqrt{a}\, l/2 \qquad (22)$$

where $\tau_{\max}{}^{\infty}$, the shear stress at the free end of an infinitely long embedded fibre, is given by

$$\tau_{\max}{}^{\infty} = \frac{KP_{\mathrm{f}}}{R\sqrt{a}} \qquad (23)$$

If the fibre has debonded to an effective bonded length $l'/2$ then

$$\tau_{\max} = \frac{KP_{\mathrm{f}}'}{R\sqrt{a}} \coth \sqrt{a}\, l'/2 \qquad (24)$$

Thus we can write the general equation

$$\tau_{\max} = \tau_{\max}{}^{\infty} \left[1 - \frac{\tau_{\mathrm{i}}\sqrt{a}}{2\tau_{\max}{}^{\infty}} (l - l')\right] \coth \sqrt{a}l'/2 \qquad \ldots\ldots (25)$$

The variation in the maximum shear stress with effective embedded fibre length is shown in fig. 6 (by plotting $\tau_{\max}/\tau_{\max}{}^{\infty}$ against the effective embedded fibre length factor $\sqrt{a}\, l'/2$). This has been done for various values of the ratio $n = \tau_{\mathrm{i}}\sqrt{a}\, l/2\tau_{\infty}{}^{\max}$ where $n$ is dependent only upon the relative value of the interfacial shear strength due to friction along the unbonded region of the fibre for a constant original embedded fibre length. It is clear that either for very short fibres or for long fibres after substantial debonding has shortened the effective embedded length the shear stress at the point where the fibre "enters" the matrix can be very large compared to the infinite fibre value for low friction conditions. At imminent catastrophic failure and pull-out

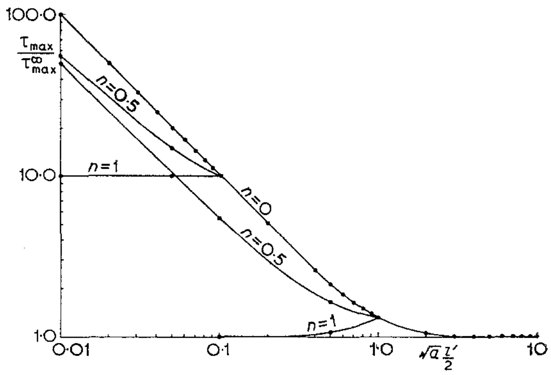


*Figure 6* Variation of maximum shear stress with effective embedded fibre length for various friction conditions.

$$\left\{\begin{array}{l} P_{\mathrm{f}}{}^{\max} = \tau_{\mathrm{s}} \dfrac{R\sqrt{a}}{K} \tanh \sqrt{a}\, l/2 \ldots [l/2 \leqslant x_{\max}] \\[2ex] P_{\mathrm{f}}{}^{\max} = \tau_{\mathrm{s}} \dfrac{R\sqrt{a}}{K} \tanh \sqrt{a}x_{\max} + \tau_{\mathrm{i}} \dfrac{Ra}{K} (l/2 - x_{\max}) \ldots [l/2 > x_{\max}] \end{array}\right\} \qquad (26)$$

If the maximum load $P_{\mathrm{f}}{}^{\max}$ is determined experimentally for various embedded fibre lengths then curves of the form indicated in fig. 5 may be obtained by plotting $P_{\mathrm{f}}{}^{\max}$ against $\sqrt{a}\, l/2$. At long embedded lengths i.e. $l/2 \gg x_{\max}$ the gradient $\Delta$ may be determined and put equal to $\tau_{\mathrm{i}} R\sqrt{a}/K$. The discontinuity in the curve at $l/2 = x_{\max}$ indicates the value of $x_{\max}$.

By substituting in the expression

$$x_{\max} = \frac{1}{\sqrt{a}} \cosh^{-1} \sqrt{\frac{\tau_{\mathrm{s}}}{\tau_{\mathrm{i}}}} \qquad (17)$$

the shear strength may be determined from

$$\tau_{\mathrm{s}} = \frac{\Delta\sqrt{a}}{C} \cosh^2 x_{\max}\sqrt{a} \qquad (27)$$

since $\Delta$, $C$ and $x_{\max}$ are known and $\sqrt{a}$ is estimated from $a = H/R$ with $H$ given by $H = 2\pi G_{\mathrm{m}}/\log_{\mathrm{e}}(r_1/r_0)$ where $G_{\mathrm{m}}$ is the shear modulus of the matrix and $r_1/r_0$ is the ratio of inter-fibre distance to fibre radius (Cox [5]). In a single fibre pull-out test the ratio $r_1/r_0$ is equal to the ratio between matrix and fibre radii and is determined by the geometry of the pull-out test.

If the experimental results indicate a curve for which there is no discontinuity with a linear part at long embedded lengths then $\tau_{\mathrm{i}}$ is very small, $\tau_{\mathrm{s}}/\tau_{\mathrm{i}}$ tends to $\infty$ and $l/2 < x_{\max}$. The shear strength may then be determined from the asymptote of the $P_{\mathrm{f}}{}^{\max}$ against $\sqrt{a}\, l/2$ curve since now by equation 26

$$P_{\mathrm{f}}{}^{\max} = \tau_{\mathrm{s}} \frac{R\sqrt{a}}{K} \tanh \sqrt{a}\, l/2 \qquad (26)$$

and as $\sqrt{a}\, l/2$ tends to $\infty$ then $\tanh \sqrt{a}\, l/2$ tends to 1. Thus

$$\tau_s = P_f^{\infty} \frac{\sqrt{a}}{C} \tag{28}$$

An alternative approach used by Greszczuk [1] applicable only to the latter case above is to examine the trend of $P_f^{max}$ at short embedded fibre lengths and extrapolate to zero embedded fibre length. Again we have by equation 26

$$P_f^{max} = \tau_s \frac{R\sqrt{a}}{K} \tanh \sqrt{a}\, l/2 \tag{26}$$

Now as $\sqrt{a}\, l/2$ tends to 0 then $\tanh \sqrt{a}\, l/2$ tends to $\sqrt{a}\, l/2$. Thus

$$\tau_s \text{ tends to } \frac{P_f^{max}}{Cl/2}$$

i.e.

$$\tau_s \text{ tends to } \tau_{av}$$

where $\tau_{av}$ is the average shear stress obtained by dividing the fibre load by the total interfacial area. This approach is only useful if frictional forces play little part in the fibre pull-out mechanism.

## 4. Conclusions

The effect of the embedded fibre length and the elastic properties on the shear stress and load distribution along the fibre length are shown to be marked. For a given value of $a$, determined by the elastic properties and geometry of the pull-out test, the difference in these distributions for long and short fibres can be highly significant. It has been shown that the maximum fibre load necessary to cause complete debonding and eventual pull-out is dependent on the length of the embedded fibre and the ratio between the shear strength and the frictional "shear strength" of the fibre matrix interface. Further, the development of debonding of the fibre from the matrix can have a marked effect on the maximum shear stress developed at the interface. Consequently, if frictional forces play any part in the pull-out mechanism, it is essential to differentiate clearly between catastrophic and non-catastrophic debonding in order to determine the true shear strength of the interface.

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