Z. Wahrscheinlichkeitstheorie verw. Gebiete
63, 297–348 (1983)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Symmetry Breaking and Random Waves for Magnetic Systems on a Circle

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Summary. The asymptotic behavior of certain magnetic systems on the circle $\mathbb{T} \doteq \mathbb{R}$ (mod 1) is studied by techniques of functional integration. An arbitrary d-body interaction is allowed, $d \in \{2, 3, \ldots\}$. We call these systems circle models. First, the specific free energy for such a system is evaluated as the supremum of a functional on $L_2(\mathbb{T})$. We refer to this functional as the free energy functional. Second, a global and a local law of large num-

* Alfred P. Sloan Research Fellow. Research supported by National Science Foundation Grant MCS 80-02149 and by a grant from Sonderforschungsbereich 123, Institut für Angewandte Mathematik, Universität Heidelberg

bers for the spin random variables of the system are deduced for a subclass of interactions. These laws of large numbers depend crucially upon the functions in $L_2(\mathbb{T})$ at which the free energy functional achieves its maximum. We prove that for any suitably normalized, positive, two-body interaction (ferromagnetic), the circle model behaves in the thermodynamic limit precisely like a Curie-Weiss model. On the other hand, we have examples of non-positive, two-body interactions (antiferromagnetic) for which the local law of large numbers implies the following behavior. In the thermodynamic limit, for sufficiently low temperatures, the local averages of the spin random variables are described by a random wave on $\mathbb{T}$.

Our methods of proof depend on the fact that in the thermodynamic limit, quantities of interest in the magnetic system can be expressed in terms of certain stochastic processes taking values in $L_2(\mathbb{T})$. The asymptotic behavior of the magnetic system follows from large deviation results for these stochastic processes and from a precise description of the set of functions in $L_2(\mathbb{T})$ at which the free energy functional achieves its maximum.

I. Introduction

For each $n \in \{1, 2, \ldots\}$ we define a magnetic system on the sites $\{\alpha/n;\ \alpha = 1, 2, \ldots, n\}$ of the circle $\mathbb{T} \doteq \mathbb{R} \pmod 1$. We refer to the system as the circle model. The circle model allows an arbitrary d-body interaction, $d \in \{2, 3, \ldots\}$. However, in order to simplify this introduction, we now consider only the case $d=2$. The general case is discussed in Sect. II.

Let $\{X_\alpha^{(n)};\ \alpha = 1, 2, \ldots, n\}$ denote the random variables which measure the magnetic spins at the sites $\{\alpha/n\}$. The joint distribution of the $\{X_\alpha^{(n)}\}$ is defined to be the probability measure

$$dP_{n,\beta}(x_1, \ldots, x_n) \doteq \frac{\exp\left[\frac{\beta}{2n} \sum_{\alpha_1, \alpha_2 = 1}^{n} J\left(\frac{\alpha_1}{n}, \frac{\alpha_2}{n}\right) x_{\alpha_1} x_{\alpha_2}\right] \prod_{\alpha=1}^{n} d\rho(x_\alpha)}{Z_{n,\beta}}, \tag{1.1}$$

where $Z_{n,\beta}$ is the normalization constant

$$Z_{n,\beta} \doteq \int_{\mathbb{R}^n} \exp\left[\frac{\beta}{2n} \sum_{\alpha_1, \alpha_2 = 1}^{n} J\left(\frac{\alpha_1}{n}, \frac{\alpha_2}{n}\right) x_{\alpha_1} x_{\alpha_2}\right] \prod_{\alpha=1}^{n} d\rho(x_\alpha). \tag{1.2}$$

In (1.1)–(1.2), $\beta > 0$ is proportional to the inverse absolute temperature; J is a continuous mapping from $\mathbb{T} \times \mathbb{T} \to \mathbb{R}$ and is called the interaction function; ρ is a Borel probability measure on $\mathbb{R}$ with bounded support, but not a one-point mass. The class of such measures is denoted by $\mathscr{M}_b$. We call ρ the single spin measure. The most important example of $\rho \in \mathscr{M}_b$ is the Bernoulli measure $\rho \doteq (\delta_1 + \delta_{-1})/2$, which defines the case of spins taking values $\{\pm 1\}$. The hypothesis of bounded support allows one to avoid a number of technicalities which arise for unbounded ρ. With extra work, one should be able to extend our results to unbounded ρ, but we do not carry this out. For various reasons, one may require additional properties of J; e.g., J is translation invariant ($J(s,t) \doteq \tilde{J}(s-t)$ for all $s,t \in \mathbb{T}$, some $\tilde{J}$: $\mathbb{T} \to \mathbb{R}$) or J is symmetric ($J(s,t) = J(t,s)$ for all $s,t \in \mathbb{T}$). One can also generalize the circle model by considering analogous systems on an m-dimensional torus, $m \in \{2,3,\ldots\}$. A number of our results (e.g., Theorems 1.3 and 2.1) go over to this case.

The usual set-up in Gibbsian lattice statistical mechanics is to study a system on a subset Λ of a lattice, say $\mathbb{Z}^r$, $r \in \{1,2,\ldots\}$, in the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^r$. The circle model does not fit this prescription.[1] However, given that one wants to define magnetic systems on the circle $\mathbb{T}$, the measures (1.1)–(1.2) are natural objects of study. The factor $1/n$ multiplying the sums in (1.1)–(1.2), which does not arise in the usual Gibbsian set-up, is necessitated by the compactness of the circle. Without this factor, the asymptotics of the circle model could not be studied. For example, the specific free energy would not even exist. Previous work on the circle model was done in [14, 15] under the restrictive hypothesis that J be the covariance function of a Gaussian probability measure on $\mathscr{C}(\mathbb{T})$. This work is discussed in greater detail at the end of this Introduction and in Appendix C.

Our first main result is the evaluation of the specific free energy $\psi(\beta)$ for the circle model. Defined by the formula

$$-\beta\,\psi(\beta) \doteq \lim_{n\to\infty} \frac{1}{n} \log Z_{n,\beta}, \tag{1.3}$$

$-\beta\,\psi(\beta)$ is shown in Theorem 1.3 to be given by the supremum of some functional G_β, on the real Hilbert space $\mathscr{H} \doteq L_2(\mathbb{T})$. We call G_β the free energy functional. The second main result of this paper is the derivation, for certain interaction functions, of a global and a local law of large numbers for the spin random variables $\{X_\alpha^{(n)}\}$. These laws of large numbers depend crucially upon the functions in $\mathscr{H}$ at which the free energy functional G_β achieves its maximum. We describe these laws of large numbers first.

There are various classes of interesting interaction functions J. The choice $J \equiv 1$ defines the Curie-Weiss model, which is the case in which all the spins interact equally. For $J \equiv 1$, (1.1) takes the simple form

$$P_{n,\beta}^{CW}(dx_1,\ldots,dx_n) \doteq \frac{\exp\left[n\frac{\beta}{2}\left(\sum_{\alpha=1}^{n} x_\alpha/n\right)^2\right] \prod_{\alpha=1}^{n} d\rho(x_\alpha)}{Z_{n,\beta}^{CW}}, \tag{1.4}$$

where $Z_{n,\beta}^{CW}$ is the corresponding normalization constant. The probabilistic behavior of the Curie-Weiss model has been studied extensively in [10, 12] and [13]. Facts about the Curie-Weiss model needed in the present paper are worked out in Appendix B. If the interaction function J is positive on $\mathbb{T}\times\mathbb{T}$, then we speak of a ferromagnetic circle model. If we have $J \leqq 0$ on $\mathbb{T}\times\mathbb{T}$, then we speak of an antiferromagnetic circle model. One of the main points of this paper is to contrast, by means of the laws of large numbers for the spin ran-

[1] The circle model is related to a class of models treated by the Lebowitz-Penrose theorem. See the Note before the References.

dom variables, the relatively simple asymptotic behavior of the ferromagnetic circle model with the much more complicated behavior of the antiferromagnetic model.

We do not consider the problems of defining random fields on the whole circle $\mathbb{T}$ and of studying their ergodic components, known as phases. One approach to random fields would be to study weak limits of the measures $\{P_{n,\beta}\}$ in (1.1), but we omit this. (In the Curie-Weiss case, the weak limits are discussed in [12; Theorem 8].) Even so, as a heuristic aid in interpreting the laws of large numbers, we shall use the picturesque term "phases".

Given an interval Δ in $\mathbb{T}$, we define the total spin in Δ, $W_n(\Delta)$, by the formula

$$W_n(\Delta) \doteq \frac{1}{|\Delta|} \sum_{\{\alpha;\, \alpha/n \in \Delta\}} X_\alpha^{(n)}, \tag{1.5}$$

where $|\Delta|$ denotes the Lebesgue measure of Δ. If $\Delta = \mathbb{T}$ we write W_n instead of $W_n(\mathbb{T})$. The global law of large numbers describes the limiting distribution of the average total spin, W_n/n, as $n \to \infty$. The local law of large numbers describes the limiting joint distribution of the vector of average local spins, $(W_n(\Delta_1)/n, \ldots, W_n(\Delta_r)/n)$, where $\{\Delta_j; j=1, \ldots, r\}$ are r intervals in $\mathbb{T}(r \in \{1, 2, \ldots\})$. Although the global law follows from the local law for $r=1$, $\Delta_1 \doteq \mathbb{T}$, we find it useful pedagogically to discuss both. For now, we restrict ourselves to the case $\rho \doteq (\delta_1 + \delta_{-1})/2$. More general ρ will be treated in Sect. II.

Theorem 1.1 discusses the ferromagnetic case[2] $J > 0$ on $\mathbb{T} \times \mathbb{T}$ where J satisfies the extra condition

$$\int J(s,t)\, dt = \int J(s,t)\, ds = 1 \qquad \text{for each } s, t \in \mathbb{T}. \tag{1.6}$$

Here and for the rest of the paper, all integrations with respect to dt and ds are understood to be over the circle $\mathbb{T}$ unless otherwise noted. The laws of large numbers show that for some critical β, $\beta_c > 0$, the ferromagnetic circle model has a unique phase for $0 < \beta \leqq \beta_c$. As β increases through β_c, the $\pm$-symmetry of the measures in (1.1) is broken, and two distinct phases emerge. The interesting feature is that the limit in the laws of large numbers is completely insensitive to any other details of J. In fact, for any such J, these laws of large numbers are identical to the Curie-Weiss case. We have $\beta_c = 1$, which is the Curie-Weiss critical value, and we find that the magnetization per site in each of the two phases is $\pm m^{CW}(\beta)$, where $m^{CW}(\beta)$ is the value of the Curie-Weiss spontaneous magnetization[3]. In the next theorem, the formula (1.7) defining $m^{CW}(\beta)$ is equivalent to the formula in [28; p. 101], and the notation $E_{n,\beta}\{-\}$ denotes expectation with respect to the measure $P_{n,\beta}$ in (1.1). We write 0 and 1 for the constant functions 0 and 1 on $\mathbb{T}$.

Theorem 1.1. *We assume*[2] *that* $J: \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *is continuous,* $J > 0$, J *satisfies* (1.6), *and* $\rho \doteq (\delta_1 + \delta_{-1})/2$. *We define the Curie-Weiss spontaneous magnetization,* $m^{CW}(\beta)$, *to be the unique positive solution* m *of the equation*

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[2] Actually Theorem 1.1 will be proved for a larger class of J ($J \geqq 0$ which satisfy an irreducibility condition); similarly for Theorems 1.4 and 2.2.

[3] The reason for this nomenclature is explained at the end of Appendix B.

$$\beta m = \tanh^{-1} m, \tag{1.7}$$

where $\tanh^{-1}$ *denotes the inverse function of* $\tanh$.

For $\beta > \beta_c \doteq 1$, $m^{CW}(\beta)$ *is well-defined,* $m^{CW}(\beta) > 0$, *and* $m^{CW}(\beta)$ *is monotonically increasing in* β *with* $m^{CW}(\beta) \uparrow 1$ *as* $\beta \uparrow \infty$. *For any continuous function* $h: \mathbb{R} \to \mathbb{R}$, *we have*

$$\lim_{n\to\infty} E_{n,\beta}\left\{h\left(\frac{W_n}{n}\right)\right\} = \begin{cases} h(0) & \text{for } 0<\beta \leqq \beta_c, \\ \frac{1}{2}[h(m^{CW}(\beta)) + h(-m^{CW}(\beta))] & \text{for } \beta > \beta_c. \end{cases} \tag{1.8}$$

More generally, for any $r \in \{1, 2, \ldots\}$, *any* r *intervals* $\{\Delta_j; j = 1, \ldots, r\}$ *in* $\mathbb{T}$, *and any continuous function* $h: \mathbb{R} \to \mathbb{R}$

$$\lim_{n\to\infty} E_{n,\beta}\left\{h\left(\frac{W_n(\Delta_1)}{n}, \ldots, \frac{W_n(\Delta_r)}{n}\right)\right\}$$
$$= \begin{cases} h(\mathbf{0}) & \text{for } 0<\beta \leqq \beta_c, \\ \frac{1}{2}[h(\mathbf{m}^{CW}(\beta)) + h(-\mathbf{m}^{CW}(\beta))] & \text{for } \beta > \beta_c \end{cases} \tag{1.9}$$

where $\mathbf{0} = (0, \ldots, 0)$ *and* $\mathbf{m}^{CW}(\beta) = (m^{CW}(\beta), \ldots, m^{CW}(\beta)) \in \mathbb{R}^r$.

Remarks. 1. Assume that instead of equalling 1, the integrals in (1.6) equal $c \neq 1$ for all $s, t \in \mathbb{T}$. (For example, if J is symmetric about 0 and translation invariant, then either (1.6) or this assumption holds.) Then $J_0(s,t) \doteq c^{-1} J(s,t)$ satisfies (1.6). Replacing J by J_0 is the same as replacing ρ by $d\rho_0(x) \doteq d\rho(x/\sqrt{c})$.

2. The limits on the first and second lines of (1.8) and of (1.9) are continuous since $m^{CW}(\beta) \downarrow 0$ as $\beta \downarrow \beta_c$.

3. Since as $\beta \to \infty$ $m^{CW}(\beta) \uparrow 1 = |X_j^{(n)}|$ for each $j \in \{1, \ldots, n\}$, the limit (1.8) shows that $+1$ and -1 are the two possible realizations of the ferromagnetic ground state.

According to Theorem 1.1, in the plus phase for $\beta > 1$, the number $m^{CW}(\beta)$ gives the limiting value, as $n \to \infty$, of both W_n/n and of $W_n(\Delta)/n$ for any interval Δ in $\mathbb{T}$. In other words, the local structure of the plus phase mimics its global structure. (A similar discussion holds for the minus phase.) This is no longer true in the antiferromagnetic case, as we shall now see.

We now require J to be translation invariant. Then for any $k \in \{1, 2, \ldots, n-1\}$ the random variables $\{X^{(n)}_{\alpha+k}; \alpha = 1, \ldots, n\}$ are also distributed by $P_{n,\beta}$ in (1.1). We refer to this as the $\mathbb{Z}_n$-rotational symmetry of $P_{n,\beta}$ $\left(\text{under the rotations } \frac{\alpha}{n} \to \frac{\alpha}{n} + \frac{k}{n}\right)$ since $\{X^{(n)}_{\alpha+k}\}$ represent the magnetic moments at the sites $\left\{\frac{\alpha+k}{n}\right\} \subseteq \mathbb{T}$.

For the circle model the intuitive picture we have of antiferromagnetism is for all $\beta > 0$ zero magnetization per site as $n \to \infty$ (global average), but for β sufficiently large ($\beta > \beta_a$, some $\beta_a \in (0, \infty)$) the spins clustering into alternating islands of plus spins and minus spins as $n \to \infty$. The number of islands and their size, but not their phase shift, should be determined by the interaction function. In other words, as $n\to\infty$, the spins should be described locally by a wave with fixed shape but random phase shift. Although in contrast to the ferromagnetic case we are as yet unable to treat the antiferromagnetic case in any generality, we verify this intuitive picture for examples of specific J's.

The interaction functions considered in the next theorem are parametrized by three numbers $b\geqq 0$, $v\neq 0$, and $p\in\{1,2,\ldots\}$. While the interaction is antiferromagnetic only for $b\geqq|v|$, there is no point in restricting b and v to values satisfying this inequality since the theorem is true for all $b\geqq 0$, $v\neq 0$. For convenience we shall continue to refer to the next theorem as describing the antiferromagnetic circle model.

Theorem 1.2. *We set* $\rho \doteq (\delta_1+\delta_{-1})/2$ *and take for* J *the interaction function* $J(s,t)=\tilde{J}(s-t)$, *where for some* $b\geqq 0$, $v\neq 0$, *and* $p\in\{1,2,\ldots\}$

$$\tilde{J}(t)=\tilde{J}_{b,v,p}(t)\doteq -b+v\cos(2\pi p t), \qquad t\in\mathbb{T}. \tag{1.10}$$

For any continuous function $h\colon \mathbb{R}\to\mathbb{R}$,

$$\lim_{n\to\infty} E_{n,\beta}\left\{h\left(\frac{W_n}{n}\right)\right\}=h(0) \qquad \text{for all } \beta>0. \tag{1.11}$$

More generally, for any $r\in\{1,2,\ldots\}$, *any* r *intervals* $\{\Delta_j;\, j=1,\ldots,r\}$ *in* $\mathbb{T}$, *and any continuous function* $h\colon \mathbb{R}^r\to\mathbb{R}$,

$$\begin{aligned}\lim_{n\to\infty} E_{n,\beta}&\left\{h\left(\frac{W_n(\Delta_1)}{n},\ldots,\frac{W_n(\Delta_r)}{n}\right)\right\}\\ &=\begin{cases} h(0) & \text{for } 0<\beta\leqq\beta_a\doteq 2/|v|,\\ \int h(\bar{f}(s;\Delta_1),\ldots,\bar{f}(s;\Delta_r))\,ds & \text{for } \beta>\beta_a,\end{cases}\end{aligned} \tag{1.12}$$

where for $\beta>\beta_a$ $\bar{f}(s;\Delta)\doteq|\Delta|^{-1}\int_\Delta \tilde{f}(s+\lambda)\,d\lambda$ *and* $\tilde{f}=\tilde{f}_{\beta,v,p}$ *is a non-constant continuous function on* $\mathbb{T}$ *which is independent of* b, *is an odd-function of* $\cos(2\pi p t)$, *and has the same periodic behavior as* $\tilde{J}$. *Specifically,*

$$\tilde{f}\left(t+\frac{1}{2p}\right)=-\tilde{f}(t),\ t\in\mathbb{T};\quad \int\tilde{f}(t)\,dt=0;\quad \tilde{f} \text{ has } 2p \text{ nodes.} \tag{1.13}$$

See (2.12) *for an explicit formula for* $\tilde{f}$.

Remarks. 1. The limits (1.11) and (1.12) are consistent since by (1.13) $\bar{f}(s;\mathbb{T})=0$ for each $s\in\mathbb{T}$. The limits on the first and second lines of (1.12) are continuous since for each $t\in\mathbb{T}$ $\tilde{f}_{\beta,v,p}(t)\to 0$ as $\beta\downarrow\beta_a$.

2. We also prove that if $v>0$, then

$$\lim_{\beta\to\infty}\tilde{f}_{\beta,v,p}(t)=\begin{cases}+1 & \text{if } \cos(2\pi p t)>0,\\ \ \ 0 & \text{if } \cos(2\pi p t)=0,\\ -1 & \text{if } \cos(2\pi p t)<0.\end{cases} \tag{1.14}$$

If $v<0$, then the $+1$ and -1 in (1.14) are exchanged.

Since $|X_j^{(n)}|=1$ for each $j\in\{1,\ldots,n\}$, (1.13) and (1.14) imply that all possible realizations of the antiferromagnetic ground state are given by $\{\lim_{\beta\to\infty} \tilde{f}_{\beta,v,p}(\cdot+s), s\in\mathbb{T}\}$.

The local law of large numbers (1.12) shows that as $n\to\infty$, for $\beta>2/|v|$ local averages of the spins are described by local averages of the random wave $\tilde{f}(\cdot+s(\omega))$, where the random phase shift $s(\omega)$ is uniformly distributed in $\mathbb{T}$. Heuristically, as $n\to\infty$ one may think of the spins as clustering into $2p$ alternating islands of plus spins and minus spins, where the plus and minus islands correspond respectively to the alternating intervals on which $\tilde{f}>0$ and $\tilde{f}<0$. The second line of (1.12) represents a breaking of the $\mathbb{Z}_n$-rotational symmetry of the measures $\{P_{n,\beta}\}$ in (1.1).

We next turn to the specific free energy $\psi(\beta)$, defined by (1.3). For general J and ρ as in (1.1)–(1.2), Theorem 1.3 below shows that

$$-\beta\,\psi(\beta)=\sup_{f\in\mathscr{H}} G_\beta(f), \tag{1.15}$$

where $\mathscr{H}\doteq L_2(\mathbb{T})$ and G_β is some functional on $\mathscr{H}$. Afterwards, we discuss how Theorems 1.1 and 1.2 follow from the asymptotics of certain probability measures on $\mathscr{H}$. The asymptotics of these measures depend upon the functions in $\mathscr{H}$ at which G_β achieves its maximum.

For $f\in\mathscr{H}$, we define the operator $\mathscr{J}\colon \mathscr{H}\to\mathscr{H}$ and the functional F on $\mathscr{H}$ by the formulae

$$(\mathscr{J}f)(s)\doteq\int J(s,t)\,f(t)\,dt, \tag{1.16}$$

$$F(f)\doteq\tfrac{1}{2}\langle\mathscr{J}f,f\rangle, \tag{1.17}$$

where $\langle -,-\rangle$ denotes the inner product on $\mathscr{H}$. We also define the functional I on $\mathscr{H}$ by the formula

$$I(f)\doteq\int i_\rho(f(t))\,dt, \tag{1.18}$$

where for u real

$$i_\rho(u)\doteq\sup_{t\in\mathbb{R}}\{tu-\log\int\exp(tx)\,\rho(dx)\}. \tag{1.19}$$

In (1.19) and for the rest of this paper, all integrations with respect to $d\rho$ are understood to be over $\mathbb{R}$. The function i_ρ is convex. Formula (1.19) expresses i_ρ as the Legendre transformation of the convex function

$$\gamma_\rho(t)\doteq\log\int\exp(tx)\,\rho(dx); \tag{1.20}$$

we write $i_\rho=\gamma_\rho^*$. One can prove that $i_\rho^*=(\gamma_\rho^*)^*=\gamma_\rho$.

For example, if $\rho=(\delta_1+\delta_{-1})/2$, then

$$i_\rho(u)=\begin{cases}\dfrac{1+u}{2}\log(1+u)+\dfrac{1-u}{2}\log(1-u) & \text{for } |u|\leqq 1,\\ +\infty & \text{for } |u|>1.\end{cases} \tag{1.21}$$

Theorem 1.3. *We assume that* $J\colon \mathbb{T}\times\mathbb{T}\to\mathbb{R}$ *is continuous and that* $\rho\in\mathcal{M}_b$. *Then* $\psi(\beta)$, *defined by* (1.3), *is given by*

$$-\beta\,\psi(\beta)=\sup_{f\in\mathcal{H}}[\beta\,F(f)-I(f)]<\infty. \tag{1.22}$$

We now see what Theorem 1.3 says in the Curie-Weiss case. For $J\equiv 1$, (1.22) becomes

$$-\beta\,\psi(\beta)=\sup_{f\in\mathcal{H}}\left[\frac{\beta}{2}\langle 1,f\rangle^2-I(f)\right], \tag{1.23}$$

In Theorem B.2 (Appendix B) we show that in the variational formula (1.23) we may replace $f\in\mathcal{H}$ by $f=u1$, $u\in\mathbb{R}$. Hence we find

$$-\beta\psi(\beta)=\sup_{u\in\mathbb{R}}\left[\frac{\beta}{2}u^2-i_\rho(u)\right]. \tag{1.24}$$

In Theorem B.1, we verify that (1.24) is the formula for the Curie-Weiss specific free energy

$$\psi^{CW}(\beta)\doteq-\beta^{-1}\lim_{n\to\infty}n^{-1}\log Z^{CW}_{n,\beta}, \tag{1.25}$$

where $Z^{CW}_{n,\beta}$ is the normalization constant in (1.4). Using the equation[4]

$$\sup_{x\in\mathbb{R},\,g(x)<\infty}[f(x)-g(x)]=\sup_{y\in\mathbb{R},\,f^*(y)<+\infty}[g^*(y)-f^*(y)], \tag{1.26}$$

valid for closed convex functions f and g on $\mathbb{R}$, we obtain from (1.24)

$$-\beta\,\psi(\beta)=\sup_{t\in\mathbb{R}}[\gamma_\rho(t)-t^2/(2\beta)]. \tag{1.27}$$

For $\rho\doteq(\delta_1+\delta_{-1})/2$, this formula is well-known [28; p. 100].

There is an important relationship between the Curie-Weiss spontaneous magnetization $m^{CW}(\beta)$ in Theorem 1.1 and the Curie-Weiss specific free energy $\psi^{CW}(\beta)$ in (1.24) (with $\rho\doteq(\delta_1+\delta_{-1})/2$). We define

$$\mathcal{D}^{CW}_\beta\doteq\{m\in\mathbb{R}\colon \beta m^2/2-i_\rho(m)=\sup_{u\in\mathbb{R}}[\beta u^2/2-i_\rho(u)]\} \tag{1.28}$$

We show in Appendix B that

$$\mathcal{D}^{CW}_\beta\doteq\begin{cases}\{0\} & \text{for } 0<\beta\leqq\beta_c\doteq 1,\\ \{m^{CW}(\beta),\,-m^{CW}(\beta)\} & \text{for } \beta>\beta_c.\end{cases} \tag{1.29}$$

We will come back to (1.29) in Theorem 1.4 below.

We prove Theorem 1.3 by finding a doubly indexed stochastic process $\xi_{p,n}(t)$, $n\in\{1,2,\ldots\}$, $p\in\{1,2,\ldots\}$, $t\in\mathbb{T}$, which takes values in $\mathcal{H}$ and which has the property that

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4 See Appendix C for the proof of a more general result.

$$\begin{aligned}-\beta\psi(\beta) &= \lim_{p\to\infty}\lim_{n\to\infty}\frac{1}{n}\log E\{\exp[n\beta F(\xi_{p,n})]\}\\ &= \lim_{p\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\int_{\mathscr{H}}\exp[n\beta F(f)]\,dQ_{p,n}(f).\end{aligned} \tag{1.30}$$

In (1.30), $E\{-\}$ denotes expectation with respect to $\xi_{p,n}$, and $Q_{p,n}$ denotes the distribution of $\xi_{p,n}$: for subsets $\mathscr{A}$ of $\mathscr{H}$

$$Q_{p,n}(\mathscr{A}) \doteq \text{Prob}\{\xi_{p,n}\in\mathscr{A}\}. \tag{1.31}$$

In order to prove that the right-hand side of (1.30) equals the right-hand side of (1.22), we need an extension of the version of Laplace's method for function space integrals given in [29; §3]; namely, a version that applies to doubly indexed integrals, such as the integrals involving $\{Q_{p,n}\}$ in (1.30). Such an extension is proved in Appendix A of the present paper. According to this extension, we will essentially be done once we prove the following large deviation result. For all weakly closed subsets $\mathscr{K}$ in $\mathscr{H}$ and all weakly open subsets $\mathscr{G}$ in $\mathscr{H}$

$$\limsup_{p\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log Q_{p,n}(\mathscr{K}) \leqq -I(\mathscr{K}), \tag{1.32}$$

$$\liminf_{p\to\infty}\liminf_{n\to\infty}\frac{1}{n}\log Q_{p,n}(\mathscr{G}) \geqq -I(\mathscr{G}), \tag{1.33}$$

where for a subset $\mathscr{A}$ in $\mathscr{H}$, $I(\mathscr{A})$ is the infimum of I over $\mathscr{A}$. The reason for our use of the weak topology will be clear in the proof. (See the comment after the proof of Lemma 3.1.)

The process $\xi_{p,n}$ appearing in (1.30) is defined in terms of a sequence of independent, identically distributed (i.i.d.) random variables $\{Y_j^{(n)}; j=1,\ldots,n\}$, each distributed by ρ. In the Curie-Weiss case, $\xi_{p,n}$ is easy to find: $\xi_{p,n}(t) = \sum_{j=1}^{n} Y_j^{(n)}/n$ for all $p\in\{1,2,\ldots\}$ and all $t\in\mathbb{T}$; also the second equality in (1.30) holds without any limits (see (1.4)). However, for general J one cannot get away this cheaply. Intuitively, (1.32) and (1.33) hold for general J because the function governing large deviations of sums of the $\{Y_j^{(n)}\}$ is the function i_ρ in (1.19); see Lemma 3.8. In summary, the point of the calculations in the previous paragraph is that in the limit $n\to\infty$, the circle model can be expressed in terms of a stochastic process $\xi_{p,n}$ which by (1.32)-(1.33) has I as its entropy functional. The formula (1.22) for $\psi(\beta)$ is an instance of the Gibbs variational formula [25; §14], [26; §19].

The local laws of large numbers in Theorems 1.1 and 1.2 are a consequence of Theorem 1.3 and some additional facts. We find probability measures $\{R_n; n=1,2,\ldots\}$ on $\mathscr{H}$, closely related to the measures $\{Q_{p,n}\}$ above, with the following property. For any $r\in\{1,2,\ldots\}$, any r intervals $\{\Delta_j; j=1,\ldots,r\}$ in $\mathbb{T}$, and any continuous function h: $\mathbb{R}^r\to\mathbb{R}$,

$$\lim_{n\to\infty}\left|E_{n,\beta}\left\{h\left(\frac{W_n(\Delta_1)}{n},\ldots,\frac{W_n(\Delta_r)}{n}\right)\right\}-\int_{\mathscr{H}} h(\langle g_1,f\rangle,\ldots,\langle g_r,f\rangle)\,d\Phi_{n,\beta}(f)\right|=0. \tag{1.34}$$

In (1.34) $g_j \doteq |\Delta_j|^{-1} 1_{\Delta_j}$, $j\in\{1,\ldots,r\}$ (1_{Δ_j} is the indicator function of Δ_j). The $\{\Phi_{n,\beta}\}$ are probability measures on $\mathscr{H}$ defined by the formula[5]

$$\Phi_{n,\beta}(\mathscr{A}) \doteq \frac{\int_{\mathscr{A}} \exp[n\beta F_n(f)]\,dR_n(f)}{\int_{\mathscr{H}} \exp[n\beta F_n(f)]\,dR_n(f)} \tag{1.35}$$

for subsets $\mathscr{A}$ of $\mathscr{H}$. In (1.35), the $\{F_n\}$ are a certain sequence of functionals which approximate F. We need Theorem 1.3 to prove the crucial fact that if it exists, the weak limit of the $\{\Phi_{n,\beta}\}$ is concentrated on the set

$$\mathscr{D}_\beta \doteq \{\tilde{f}\in\mathscr{H}: (\beta F - I)(\tilde{f}) = \sup_{\mathscr{H}}(\beta F - I)\}. \tag{1.36}$$

We show that the limit exists and describe it under the hypotheses first of Theorem 1.1, then of Theorem 1.2.

Theorem 1.4. *We assume the hypotheses and notation of Theorem 1.1 (ferromagnetic circle model). For all $\beta>0$ we have $\psi(\beta)=\psi^{CW}(\beta)$ where the latter denotes the Curie-Weiss specific free energy given in* (1.24) *with $\rho \doteq (\delta_1 + \delta_{-1})/2$. Specifically, we have that $\mathscr{D}_\beta \subseteq \mathscr{H}$ and $\mathscr{D}_\beta^{CW} \subseteq \mathbb{R}$ in* (1.29) *are essentially equal; i.e.*

$$\mathscr{D}_\beta = \begin{cases} \{0\} & \text{for } 0<\beta\leqq\beta_c \doteq 1, \\ \{m^{CW}(\beta)1,\ -m^{CW}(\beta)1\} & \text{for } \beta>\beta_c. \end{cases} \tag{1.37}$$

Also for any weakly continuous functional φ on $\mathscr{H}$

$$\lim_{n\to\infty}\int_{\mathscr{H}} \varphi(f)\,d\Phi_{n,\beta}(f) = \begin{cases} \varphi(0) & \text{for } 0<\beta\leqq\beta_c \\ \tfrac{1}{2}[\varphi(m^{CW}(\beta)1)+\varphi(-m^{CW}(\beta)1)] & \text{for } \beta>\beta_c, \end{cases} \tag{1.38}$$

Remarks. 1. The limit (1.9) in Theorem 1.1 follows from (1.34) and (1.38) with

$$\varphi(f) \doteq h(\langle g_1,f\rangle,\ldots,\langle g_r,f\rangle),\ g_j \doteq |\Delta_j|^{-1} 1_{\Delta_j}. \tag{1.39}$$

2. The proof of statement (1.37) about $\mathscr{D}_\beta$ depends in part upon Theorem 5.1(iii) which characterizes $f\in\mathscr{D}_\beta$ only up to a set of measure zero. Since we work in $\mathscr{H}$, this information suffices to prove the limit theorems. These same comments apply to statement (1.40) about $\mathscr{D}_\beta$ in Theorem 1.5.

Theorem 1.5. *We assume the hypotheses and notation of Theorem 1.2 (antiferromagnetic circle model). We have*

[5] We could have worked with measures defined as in (1.35) but with R_n replaced by $Q_{p,n}$. However, these measures would have been more cumbersome than those in (1.35). See the remark after the statement of Theorem 4.1 for more details.

$$\mathscr{D}_\beta = \begin{cases} \{0\} & \text{for } 0<\beta \leqq \beta_a \doteq 2/|v|, \\ \{f(\cdot + s), s \in \mathbb{T}\} & \text{for } \beta > \beta_a, \end{cases} \tag{1.40}$$

and for any weakly continuous functional φ *on* $\mathscr{H}$

$$\lim_{n\to\infty} \int_{\mathscr{H}} \varphi(f)\, d\Phi_{n,\beta}(f) = \begin{cases} \varphi(0) & \text{for } 0<\beta\leqq\beta_a, \\ \int \varphi(f(\cdot + s))\, ds & \text{for } \beta>\beta_a. \end{cases} \tag{1.41}$$

Remark. The limit (1.12) in Theorem 1.2 follows from (1.34) and (1.41) with φ defined in (1.39).

The circle model was first studied in [14, 15], but under the restrictive hypothesis that J be the covariance function of a Gaussian probability measure on $\mathscr{C}(\mathbb{T})$. Because a translation invariant covariance function cannot be everywhere nonpositive, the interesting antiferromagnetic behavior of the circle model was not treated in these earlier papers. Also, these earlier papers could not handle the case of an arbitrary d-body interaction. Formula (1.22) for the circle model specific free energy reduces to the formula for this quantity derived in [15; Theorem 1.2], when the hypotheses on J assumed in the latter paper hold. See Appendix C below for details.

In Sect. II, the results in Sect. I are stated for more general single-spin measures ρ. In addition, Theorem 1.3 is generalized to d-body interactions. In Sect. III, we prove Theorem 1.3 and its generalization (Theorem 2.1) to more general ρ and to d-body interactions. In Sect. IV we prove the laws of large numbers in Theorems 1.1, 1.2, 1.4, 1.5 and their generalizations in Sect. II, assuming the validity of the statements about $\mathscr{D}_\beta$ in (1.37) and (1.40) and the generalizations of these statements in Sect. II. In order to prove these statements about $\mathscr{D}_\beta$, we need the crucial fact that if $\tilde{f} \in \mathscr{D}_\beta$, the $\tilde{f}$ is equivalent to a solution of a certain nonlinear integral equation. The latter fact together with additional useful information is proved in Sect. V. In Sect. VI, under the hypotheses of Theorems 1.1, 1.2, 1.4, 1.5 and their generalization in Sect. II, we find all relevant solutions of the nonlinear integral equation of Sect. V. This leads to a proof of the statements about $\mathscr{D}_\beta$ in (1.37) and (1.40) and of their Sect. II generalizations. In Appendix A, we prove a version of Laplace's method for doubly indexed function space integrals which is needed for the proof of Theorems 1.3 and 2.1. The proof of the theorem in Appendix A is based upon unpublished notes of S.R.S. Varadhan. Appendix B contains all of the facts about Curie-Weiss models needed in the main body of the paper. In Appendix C, we prove a theorem about Legendre transformations which generalizes (1.26), and then we use it to derive alternate formulae for the Curie-Weiss and the circle model specific free energies.

Acknowledgements. Part of the research on this paper was carried out during a visit by R.S. Ellis to the Institut für Angewandte Mathematik, Universität Heidelberg (July, 1981). Support by a grant from the Sonderforschungsbereich 123 (Stochastische mathematische Modelle) is gratefully acknowledged. We thank S.R.S. Varadhan for a useful suggestion concerning the proof of Theorem C.1 (Appendix C) and for permission to use his unpublished notes which form the basis of the proof of Theorem A.1 (Appendix A).

II. General Statements of Theorems

We generalize the results in Sect. I to more general single-spin measures ρ. In addition, we generalize Theorem 1.3 to d-body interactions. An extension of Theorems 1.1 and 1.4 which treats the relationship between d-body ferromagnetic circle models and the d-body Curie-Weiss model will be presented in another paper.

Given $d\in\{2,3,\ldots\}$, a d-body interaction is defined by d functions $\{J_j; j=1,\ldots,d\}$, where for each j $J_j\colon \mathbb{T}^j\to\mathbb{R}$ is continuous $\left(\mathbb{T}^j \doteq \mathop{\times}\limits_{k=1}^{j} \mathbb{T}\right)$. For $j\geqq 2$, the function J_j defines the pure j-body part of the general d-body interaction; J_1 defines the external magnetic field. As a generalization of (1.1), we define the joint distribution of the spin random variables $\{X_\alpha^{(n)}; \alpha=1,\ldots,n\}$ to be

$$P_{n,\beta}(dx_1,\ldots,dx_n) \doteq \frac{\exp\left[\beta \sum_{j=1}^{d} \frac{1}{j!}\frac{1}{n^{j-1}} \sum_{\alpha_1,\ldots,\alpha_j=1}^{n} J_j\left(\frac{\alpha_1}{n},\ldots,\frac{\alpha_j}{n}\right) x_{\alpha_1}\ldots x_{\alpha_j}\right] \prod_{\alpha=1}^{n} d\rho(x_\alpha)}{Z_{n,\beta}}, \tag{2.1}$$

where $\beta>0$ and $Z_{n,\beta}$ is the corresponding normalization constant. As in (1.1), we assume that $\rho\in\mathcal{M}_b$. We say that (2.1) defines a ferromagnetic model if each $J_j\geqq 0$ on $\mathbb{T}^j$. For various reasons, we may require certain symmetry properties of the $\{J_j\}$; e.g., for j odd $J_j\equiv 0$ or for all j $J_j(s_1,\ldots,s_j)$, $(s_1,\ldots,s_j)\in\mathbb{T}^j$, is invariant under all permutations of $s_1,\ldots,s_j$.

Our first result is a formula for the specific free energy $\psi(\beta)$ corresponding to the measures $\{P_{n,\beta}\}$. For each $j\in\{1,\ldots,d\}$, we define the multilinear operator $F^{(j)}$ on $\mathcal{H}\doteq L_2(\mathbb{T})$ by the formula

$$F^{(j)}(f) \doteq \frac{1}{j!} \int_{\mathbb{T}^j} J_j(s_1,\ldots,s_j)\, f(s_1)\ldots f(s_j)\, ds_1\ldots ds_j \tag{2.2}$$

and set

$$F(f) \doteq \sum_{j=1}^{d} F^{(j)}(f). \tag{2.3}$$

The following theorem generalizes Theorem 1.3.

Theorem 2.1. *Given $d\in\{1,2,\ldots\}$, d continuous functions $\{J_j; j=1,\ldots,d\}$ with $J_j\colon \mathbb{T}^j\to\mathbb{R}$, and $\rho\in\mathcal{M}_b$, we define by* (1.3) *the specific free energy $\psi(\beta)$ corresponding to the measures $\{P_{n,\beta}\}$ in* (2.1). *Then*

$$-\beta\,\psi(\beta) = \sup_{f\in\mathcal{H}} [\beta F(f) - I(f)] < \infty. \tag{2.4}$$

The functionals F and I are defined in (2.3) *and in* (1.18), *respectively.*

In Theorems 1.1, 1.2, 1.4, and 1.5, we assumed that $\rho\doteq(\delta_1+\delta_{-1})/2$. The natural generalization is to measures ρ which satisfy the one-site GHS inequality. We define $\mathscr{G}$ to be the set of even Borel probability measures ρ on $\mathbb{R}$, $\rho \neq \delta_0$, for which

$$\int_{\mathbb{R}} \exp(kx^2)\, d\rho(x) < \infty \qquad \text{for all } k \geqq 0 \tag{2.5}$$

and

$$\frac{d^3 \gamma_\rho(t)}{dt^3} \leqq 0 \qquad \text{for all } t > 0, \tag{2.6}$$

where γ_ρ is defined in (1.20). Inequality (2.6) implies that γ'_ρ is concave on $(0, \infty)$. It is not hard to prove that in fact γ'_ρ is strictly concave on $(0, \infty)$[6]. We define $\mathscr{G}_b$ to be the subset of measures in $\mathscr{G}$ which have bounded support. The following measures[7] are known to belong to $\mathscr{G}_b$:

$$\rho \doteq \sum_{j=-(k-1)/2}^{(k-1)/2} \delta_{ja}/k \qquad \text{for any } k \in \{1, 2, \ldots\} \quad \text{and} \quad a > 0; \tag{2.7}$$

ρ absolutely continuous with respect to Lebesgue measure and for some $L \in (0, \infty)$

$$\frac{d\rho}{dx}(x) \doteq \begin{cases} \text{const} \exp\left(-\int_0^{|x|} H(y)\, dy\right) & \text{for } |x| < L, \\ 0 & \text{for } |x| \geqq L, \end{cases} \tag{2.8}$$

where $H(0) = 0$ and H is convex on $[0, L)$ [9; Theorem 1.2(d)]. For $L = \infty$, (2.8) yields measures in $\mathscr{G}$. Not every even bounded measure ρ belongs to $\mathscr{G}_b$; e.g., [27; p. 153], [9; Theorem 1.2(b)]

$$\rho \doteq a\delta_0 + (1-a)(\delta_1 + \delta_{-1})/2 \begin{cases} \in \mathscr{G}_b & \text{for } a \in [0, 2/3], \\ \notin \mathscr{G}_b & \text{for } a \in (2/3, 1). \end{cases}$$

We next generalize Theorems 1.1 and 1.4 to arbitrary $\rho \in \mathscr{G}_b$. The content of Theorem 1.1 and 1.4 was that with $\rho \doteq (\delta_1 + \delta_{-1})/2$, the asymptotic behavior of the circle model for $d = 2$ and a suitably normalized ferromagnetic interaction function is identical to the asymptotic behavior of the Curie-Weiss model.

Theorem 2.2. *We assume the hypotheses and notations of Theorems 1.1 and 1.4 (ferromagnetic circle model) except that $\rho \doteq (\delta_1 + \delta_{-1})/2$ is replaced by another $\rho \in \mathscr{G}_b$. Then all of the conclusions of Theorems 1.1 and 1.4 remain valid after the following changes. The critical $\beta, \beta_c = 1$, is replaced by*

$$\beta_c(\rho) \doteq 1 / \int x^2\, d\rho(x).$$

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[6] If $\gamma'''_\rho = 0$ on some subinterval of $(0, \infty)$, then by real analyticity, $\gamma'''_\rho(u) = 0$ for all $u > 0$. This implies that $\gamma_\rho(u) = cu^2$ for some real C and thus that ρ is Gaussian. But a Gaussian measure cannot satisfy (2.5).

[7] In a general even ferromagnet with single-spin measure (2.7) or (2.8), the multisite GHS inequality is valid [18, 19], [9; Theorem 1.2(d)]. This implies that the average magnetization is a concave function of internal field. See [9] and [11] for this and other implications of the GHS inequality.

The Curie-Weiss spontaneous magnetization, $m^{CW}(\beta)$, is replaced by $m^{CW}(\rho;\beta)$, which is defined to be the unique positive root m of the equation

$$\beta m = i'_\rho(m). \tag{2.9}$$

For $\beta > \beta_c(\rho)$, $m^{CW}(\rho;\beta)$ is well-defined, $m^{CW}(\rho;\beta) > 0$, and $m^{CW}(\rho;\beta)$ is monotonically increasing in β with

$$m^{CW}(\rho;\beta) \uparrow L \doteq \sup\{x\colon x \text{ in support of } \rho\} \quad \text{as} \quad \beta \to \infty. \tag{2.10}$$

Remarks. 1. Assume that J not only satisfies the hypotheses of Theorem 1.1 but also is translation invariant and positive definite. Then one can prove that for any even $\rho \in \mathcal{M}_b$ (not necessarily in $\mathcal{G}_b$) the circle model specific free energy reduces to $\psi^{CW}(\beta)$ in (1.24). However, in general it is only for $\rho \in \mathcal{G}_b$ that one has a simple description of $\mathcal{D}_\beta$ (cf., Theorem 1.4) and thus obtains explicit limit theorems.

2. Equation (3.65) below implies that i'_ρ is the inverse function of γ'_ρ; i.e. $i'_\rho = (\gamma'_\rho)^{-1}$. Hence, (2.10) is equivalent to the equation $\gamma'_\rho(\beta m) = m$.

The next theorem, Theorem 2.3, generalizes Theorems 1.2 and 1.5 to arbitrary $\rho \in \mathcal{G}_b$. The latter theorems investigated the asymptotic behavior of the antiferromagnetic circle model for $\rho \doteq (\delta_1 + \delta_{-1})/2$, $d=2$ (with $J_1 \equiv 0$), and $J_2(s,t) \doteq \tilde{J}(s-t)$, $s,t \in \mathbb{T}$, where for some $b \geqq 0$, $v \neq 0$, and $p \in \{1,2,\ldots\}$

$$\tilde{J}(t) = -b + v\cos(2\pi p t), \qquad t \in \mathbb{T}. \tag{2.11}$$

Theorem 2.3. *We assume the hypotheses and notation of Theorems 1.2 and 1.5 (antiferromagnetic circle model) except that $\rho \doteq (\delta_1 + \delta_{-1})/2$ is replaced by another $\rho \in \mathcal{G}_b$. Then all of the conclusions of Theorems 1.2 and 1.5 remain valid after the following changes. The number $\beta_a \doteq 2/|v|$ is replaced by $\beta_a(\rho) \doteq 2/(|v| \int x^2\, d\rho(x))$. The function $\tilde{f} = \tilde{f}_{\beta,v,p}$ is replaced by a function $\tilde{f} = \tilde{f}_{\beta,v,p,\rho}$. The latter is defined by the formula*

$$\tilde{f}(t) \doteq \gamma'_\rho(\beta v \mu \cos(2\pi p t)), \tag{2.12}$$

where $\mu = \mu(\beta, v, p, \rho)$ is defined as the unique positive root μ of the equation

$$\mu = \int \gamma'_\rho(\beta v \mu \cos(2\pi p t)) \cdot \cos(2\pi p t)\, dt. \tag{2.13}$$

For $\beta > \beta_a(\rho)$, μ is well-defined.

Remark. We define $L = L(\rho)$ by (2.10). Then in the limit (1.14), the values -1, $+1$ are replaced by $-L$, $+L$, respectively. In this case, $L = \max|X_j^{(n)}|$ for $j \in \{1,\ldots,n\}$.

III. Proof of Theorem 2.1

In order to ease the notation, we set $\beta = 1$ and write ψ and Z_n instead of $\psi(1)$ and $Z_{n,1}$. We write $\mathcal{H}$ for $L_2(\mathbb{T})$ and denote the norm of $\mathcal{H}$ by $\|-\|$. The proof of Theorem 2.1 follows the same steps that were sketched in the third paragraph after the statement of Theorem 1.3, which is Theorem 2.1 in the case $d=2$. There are three main steps.

(1) Define the stochastic process $\xi_{p,n}$ taking values in $\mathscr{H}$.

(2) Assuming that the limits exists, prove that

$$-\psi \doteq \lim_{n\to\infty} \frac{1}{n} \log Z_n = \lim_{p\to\infty} \lim_{n\to\infty} \frac{1}{n} \log \int_{\mathscr{H}} \exp[nF(f)]\, dQ_{p,n}(f), \tag{3.1}$$

where $Q_{p,n}$ denotes the distribution of $\xi_{p,n}$.

(3) Prove that

$$\lim_{p\to\infty} \lim_{n\to\infty} \frac{1}{n} \log \int_{\mathscr{H}} \exp[nF(f)]\, dQ_{p,n}(f) = \sup_{f\in\mathscr{H}} [F(f)-I(f)] < \infty. \tag{3.2}$$

We first give motivation, then carry out the three steps.

III.1 Motivation

We first motivate our definition of $\xi_{n,p}$ since $\xi_{n,p}$ is not the simplest process for which the limit (3.1) is valid. However, as we shall see, it is in a sense the simplest process for which both limits (3.1) and (3.2) hold for general $\{J_j\}$. (See the comment in the paragraph after (1.33) concerning a simple choice of $\xi_{n,p}$ for the Curie-Weiss model.)

We first seek to write Z_n for each $n\in\{1,2,\ldots\}$ in terms of a stochastic process taking values in $\mathscr{H}$. In other words, we try to prove an equality like (3.1) but without limits on the right-hand side. Given $f\in\mathscr{H}$, we define

$$(\pi^{(n)}f)(t) \doteq n \int_{(k-1)/n}^{k/n} f(s)\,ds \quad \text{if} \quad t\in\left(\frac{k-1}{n}, \frac{k}{n}\right], \quad k\in\{1,\ldots,n\}. \tag{3.3}$$

The operation $f\to\pi^{(n)}f$ is the conditional expectation of f with respect to the σ-algebra generated by the intervals $\{((k-1)/n, k/n], k\in\{1,\ldots,n\}\}$. We define the functional F_n on $\mathscr{H}$ by the formula

$$F_n(f) \doteq \sum_{j=1}^{d} \frac{1}{j!} \frac{1}{n^j} \sum_{\alpha_1,\ldots,\alpha_j=1}^{n} J_j\left(\frac{\alpha_1}{n},\ldots,\frac{\alpha_j}{n}\right) (\pi^{(n)}f)\left(\frac{\alpha_1}{n}\right) \ldots (\pi^{(n)}f)\left(\frac{\alpha_j}{n}\right). \tag{3.4}$$

For each n, we define $\{Y_k^{(n)}; k=1,2,\ldots,n\}$ to be a sequence of i.i.d. random variables each distributed by ρ and ζ_n to be the stochastic process

$$\zeta_n(t) \doteq \sum_{k=1}^{n} Y_k^{(n)} 1_{((k-1)/n, k/n]}(t). \tag{3.5}$$

Since ρ has bounded support, the $\{Y_k\}$ are uniformly bounded a.s. and so ζ_n takes values in $\mathscr{H}$. We express Z_n in terms of ζ_n by the formula

$$Z_n = E\{\exp[nF_n(\zeta_n)]\} = \int_{\mathscr{H}} \exp[nF_n(f)]\, dR_n(f), \tag{3.6}$$

where $E\{-\}$ denotes expectation with respect to ζ_n and R_n denotes the distribution of ζ_n. Thus, one may think of each $Y_k^{(n)}$ as representing the spin at the site $k/n \in \mathbb{T}$. Since each J_j is continuous, F_n is a good approximation to F, and so it is plausible that we should have (provided the limits exist)

$$-\psi \doteq \lim_{n\to\infty} \frac{1}{n} \log Z_n = \lim_{n\to\infty} \frac{1}{n} \log \int_{\mathscr{H}} \exp[nF(f)]\, dR_n(f). \tag{3.7}$$

In order to complete the proof of Theorem 2.1, we would try to prove that the limit on the right-hand side of (3.7) exists and equals $\sup(F-I)$ on $\mathscr{H}$. Theorem A.1 (Appendix A) gives sufficient conditions which imply this. A main step (cf., (1.32)-(1.33)) is to find a functional I on $\mathscr{H}$ such that for all weakly closed subsets $\mathscr{K}$ in $\mathscr{H}$ and all weakly open subsets $\mathscr{G}$ in $\mathscr{H}$

$$\limsup_{n\to\infty} \frac{1}{n} \log R_n(\mathscr{K}) \leqq -I(\mathscr{K}), \tag{3.8}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log R_n(\mathscr{G}) \geqq -I(\mathscr{G}). \tag{3.9}$$

One can prove (3.8) with I given by (1.18); in fact, we use this in Sect. 5 (see the proof of Lemma 5.2). However, (3.9) fails in general[8]. The basic reason is that the process ζ_n is not random enough.

The process ζ_n is replaced by another process $\xi_{p,n}$. For simplicity we now define $\xi_{p,n}$ only for n of the form $n=2^p q$, $p,q\in\{1,2,\dots\}$, where eventually $q\to\infty$, $p\to\infty$. We define π_p to be the conditional expectation operator on $\mathscr{H}$ with respect to the σ-algebra generated by the intervals

$$\{\Lambda_{p,k};\, k=1,\dots,2^p\}, \quad \text{where} \quad \Lambda_{p,k} \doteq \left(\frac{k-1}{2^p}, \frac{k}{2^p}\right].$$

Thus for $f\in\mathscr{H}$

$$(\pi_p f)(t) = 2^p \int_{\Lambda_{p,k}} f(s)\,ds \quad \text{if} \quad t\in\Lambda_{p,k}, \quad k\in\{1,\dots,2^p\}. \tag{3.10}$$

Clearly, $\pi_p = \pi^{(2^p)}$ in (3.3). We now define

$$\begin{aligned} \xi_{p,n} &= (\pi_p \zeta_n)(t) \\ &= \sum_{k=1}^{2^p} \left(\frac{1}{q} \sum_{j=(k-1)q+1}^{kq} Y_j^{(n)}\right) 1_{\Lambda_{p,k}}(t). \end{aligned} \tag{3.11}$$

The conditional expectation π_p acts like a block spin transformation since it groups together blocks of q spins. Again, $\xi_{p,n}$ takes values in $\mathscr{H}$.

The proof of (3.1) uses (3.6) together with the fact that the functional F is nearly invariant with respect to the block spin transformation π_p; i.e., as $p\to\infty$,

[8] If $\rho \doteq (\delta_1+\delta_{-1})/2$, then $\|\zeta_n\|=1$. Hence if $\mathscr{G} \doteq \{f\in\mathscr{H}: \|f\|<\frac{1}{2}\}$, then $\liminf n^{-1}\log R_n(\mathscr{G}) = -\infty$ while $\inf_{\mathscr{G}} I = 0$.

$F \circ \pi_p \to F$ uniformly on weakly compact subsets of $\mathscr{H}$ (see Lemma 3.2). The latter depends upon the fact that as $p \to \infty$ $\|f - \pi_p f\| \to 0$ for each $f \in \mathscr{H}$ (see Lemma 3.4). This limit is a consequence of the martingale convergence theorem.

The main step in the proof of (3.2) is to show the estimates (1.32)–(1.33). Intuitively, the latter reduce to the facts that for different k the sums

$$S_{q,k}^{(n)} \doteq \sum_{j=(k-1)q+1}^{kq} Y_j^{(n)} \tag{3.12}$$

are i.i.d. and that for nice subsets $\mathscr{A}$ in $\mathbb{R}$

$$\lim_{q\to\infty} \frac{1}{q} \log \operatorname{Prob}\left\{\frac{S_{q,k}^{(n)}}{q} \in \mathscr{A}\right\} = -i_\rho(\mathscr{A}). \tag{3.13}$$

where i_ρ is the entropy function of ρ, defined in (1.19) (see Lemma 3.8). Because the proofs of (1.32)–(1.33) are rather involved, we motivate these estimates by a heuristic calculation that shows for continuous f

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Prob}\{\xi_{p,n} \sim f\} \text{“}=\text{”} -I(f), \tag{3.14}$$

where the symbol $\sim$ means near. Since f is continuous, we should have for large p

$$\{\xi_{p,n} \sim f\} \simeq \bigcap_{k=1}^{2^p} \{q^{-1} S_{q,k}^{(n)} \sim f(k/2^p)\}, \tag{3.15}$$

and so by independence and (3.13).

$$\begin{aligned} &\lim_{p\to\infty} \lim_{q\to\infty} \frac{1}{2^p q} \log \operatorname{Prob}\{\xi_{p,n} \sim f\} \\ &\quad \text{“}=\text{”} \lim_{p\to\infty} \lim_{q\to\infty} \frac{1}{2^p} \sum_{k=1}^{2^p} \frac{1}{q} \log \operatorname{Prob}\left\{q^{-1} S_{q,k}^{(n)} \sim f\left(\frac{k}{2^p}\right)\right\} \\ &\quad \text{“}=\text{”} -\lim_{p\to\infty} \sum_{k=1}^{2^p} \frac{1}{2^p} i_\rho(f(k/2^p)) = -I(f). \end{aligned} \tag{3.16}$$

This gives (3.14) since $n = 2^p q$.

III.2 Definition of $\xi_{p,n}$ for General n

In this paragraph we denote by $[-]$ the greatest integer function. Given $n \in \{1, 2, \ldots\}$, $p \in \{1, \ldots, [\log_2 n]\}$, we define numbers $0 \doteq n_0 < n_1 < \ldots < n_{2^p - 1} < n_{2^p} \doteq n$ by the formulae

$$n_k \doteq \begin{cases} k[n/2^p] & \text{for } k \in \{0, 1, \ldots, 2^p - 1\}, \\ n & \text{for } k = 2^p. \end{cases}$$

Since $0 \leqq p \leqq [\log_2 n]$, $[n/2^p] \geqq 1$. We define $\pi_{p,n}$ to be the conditional expectation operator on $\mathscr{H}$ with respect to the σ-algebra generated by the intervals

$$\{\Lambda_{p,n,k};\ k=1,\dots,2^p\}, \qquad \text{where } \Lambda_{p,n,k} \doteq \left(\frac{n_{k-1}}{n}, \frac{n_k}{n}\right].$$

Thus for $f \in \mathscr{H}$

$$(\pi_{p,n} f)(t) = \frac{n}{n_k - n_{k-1}} \int_{\Lambda_{p,n,k}} f(s)\, ds \quad \text{if } t \in \Lambda_{p,n,k}, \quad k \in \{1,\dots,2^p\}. \tag{3.17}$$

We define the process $\xi_{p,n}$ by the formula

$$\begin{aligned} \xi_{p,n}(t) &\doteq (\pi_{p,n} \zeta_n)(t) \\ &= \sum_{k=1}^{2^p} \left(\frac{1}{n_k - n_{k-1}} \sum_{j=n_{k-1}+1}^{n_k} Y_j^{(n)} \right) 1_{\Lambda_{p,n,k}}(t). \end{aligned} \tag{3.18}$$

This definition is close to the definition (3.11) since $n_k/n \to k/2^p$ as $n \to \infty$. It will simplify matters somewhat if we define $\xi_{p,n}$ for $p > [\log_2 n]$. We arbitrarily set

$$\xi_{p,n} \doteq Y_1^{(n)} \quad \text{for } p > [\log_2 n]. \tag{3.19}$$

Since the measure ρ has bounded support and is not a point mass, we have

$$-\infty < l \doteq \inf\{x\colon x \text{ in support of } \rho\} < L \doteq \sup\{x\colon x \text{ in support of } \rho\} < \infty. \tag{3.20}$$

We define

$$\mathscr{B} \doteq \{h \in \mathscr{H}\colon l \leqq \operatorname{ess\,inf} h \leqq \operatorname{ess\,sup} h \leqq L\}. \tag{3.21}$$

Clearly, ζ_n and $\xi_{p,n}$ take values in $\mathscr{B}$. A useful property of $\mathscr{B}$ is given in the next lemma.

Lemma 3.1. *$\mathscr{B}$ is a weakly compact subset of $\mathscr{H}$ and*

$$\sup_{f \in \mathscr{B}} \|f\|_\infty \leqq \max(|l|, |L|). \tag{3.22}$$

Proof. $\mathscr{B}$ is a subset of $\{f \in \mathscr{H}\colon \|f\| \leqq \max(|l|, |L|)\}$ which is weakly compact [6; Thm V.4.7]. Hence it suffices to prove that $\mathscr{B}$ is weakly closed. Given $\{f_m\}$ a sequence in $\mathscr{B}$ such that $f_m \rightharpoonup f \in \mathscr{H}$, we prove that $\operatorname{ess\,sup} f \leqq L$. For if the set $\mathscr{A} \doteq \{t\colon f(t) > L\}$ has non-zero Lebesgue measure, then the limit $\langle f_m, 1_{\mathscr{A}}\rangle \to \langle f, 1_{\mathscr{A}}\rangle$ contradicts the fact that $\langle f_m, 1_{\mathscr{A}}\rangle \leqq L|\mathscr{A}| < \langle f, 1_{\mathscr{A}}\rangle$. A similar proof shows that if $f_m \rightharpoonup f \in \mathscr{H}$, $\{f_m\} \subseteq \mathscr{B}$, then $\operatorname{ess\,inf} f \geqq l$. The bound (3.22) follows from the definition of $\mathscr{B}$. □

The weak compactness of $\mathscr{B}$ greatly simplifies the proofs of (3.1) and (3.2) and explains why we assume ρ to have bounded support. It also explains why we use the weak topology in proving the large deviation estimates (1.32)–(1.33) for $\xi_{p,n}$.

III.3 Proof of (3.1)

We prove the following lemma after we show how it yields (3.1).

Lemma 3.2. *We define functionals $\{F_n\}$ by* (3.4) *and operators $\{\pi_{p,n}\}$ by* (3.17). *Given $\varepsilon>0$ there exists $p_0=p_0(\varepsilon)$ and for all $p\geqq p_0$ there exists $n_0=n_0(p)$ such that whenever $n\geqq n_0$,*

$$\sup_{f\in\mathscr{B}}|F_n(f)-F(\pi_{p,n}f)|<\varepsilon. \tag{3.23}$$

To prove (3.1), we use the fact that the measures $\{R_n\}$, which are the distribution of the processes $\{\zeta_n\}$ in (3.5), are concentrated on the set $\mathscr{B}$. We have by (3.6)

$$\begin{aligned}\frac{1}{n}&|\log Z_n-\log\int\exp[nF(f)]\,dQ_{p,n}|\\ &=\frac{1}{n}|\log\int\exp[nF_n(f)]\,dR_n-\log\int\exp[nF(f)]\,dQ_{p,n}|\\ &\leqq\frac{1}{n}\left|\log\left[\frac{\int\exp[nF_n(f)]\,dR_n}{\int\exp[nF(\pi_{p,n}f)]\,dR_n}\right]\right|+\frac{1}{n}\left|\log\left[\frac{\int\exp[nF(\pi_{p,n}f)]\,dR_n}{\int\exp[nF(f)]\,dQ_{p,n}}\right]\right|.\end{aligned} \tag{3.24}$$

By (3.18), we have for all n and $p\leqq[\log_2 n]$ $R_n\circ\pi_{p,n}^{-1}=Q_{p,n}$. Hence the last term in (3.24) is zero for all sufficiently large n. Since each measure R_n is concentrated on $\mathscr{B}$, we see by Lemma 3.2 that given $\varepsilon>0$ the first term on the right-hand side of (3.24) is less than ε for all $p\geqq p_0(\varepsilon)$ and all $n\geqq n_0(p)$. This proves (3.1). □

In order to prove Lemma 3.2, we need three facts.

Lemma 3.3. *The functional F is weakly continuous and uniformly bounded on $\mathscr{B}$.*

Proof. For any $f\in\mathscr{B}$, we have

$$|F(f)|\leqq\sum_{j=1}^{d}\frac{1}{j!}[\max(|l|,|L|)]^j\int_{\mathbb{T}^j}|J_j|,$$

which proves the uniform boundedness of F *on* $\mathscr{B}$. It suffices to prove that each operator $F^{(j)}$ defined in (2.2) is weakly continuous. By the Stone-Weierstrass Theorem [6; Thm IV.6.16], since each function J_j: $\mathbb{T}^j\to\mathbb{R}$ is continuous, given $\varepsilon>0$ there exist $N=N(\varepsilon)$ and functions $\{b_{\alpha k};\ \alpha=1,\dots,N;\ k=1,\dots,j\}$ such that

$$\sup_{(s_1,\dots,s_j)\in\mathbb{T}^j}\left|J_j(s_1,\dots,s_j)-\sum_{\alpha=1}^{N}\prod_{k=1}^{j}b_{\alpha k}(s_k)\right|<\varepsilon. \tag{3.25}$$

The weak continuity of F follows from (3.25), the uniform bound (3.22), and the weak continuity of the operators

$$f\to\prod_{k=1}^{j}\langle b_{\alpha k},f\rangle.\quad\square$$

Lemma 3.4. *We define* $\pi^{(n)}$ *by* (3.3) *and* π_p *by* (3.10). *For any* $h \in \mathscr{H}$, *as* $p \to \infty$ *we have*

$$\pi_p h \to h \quad a.e., \quad \|\pi_p h - h\| \to 0, \quad \sup_{f \in \mathscr{B}} |\langle h, \pi_p f - f\rangle| \to 0. \tag{3.26}$$

The last two limits in (3.26) *also hold for* $\pi^{(n)}$.

Proof. Since each interval $\Lambda_{p+1,j}$ is a subset of some $\Lambda_{p,k}$, the sequence $\{\pi_p h\}$ forms a martingale and $\pi_p h \to h$ a.e. on $\mathbb{T}$ [5; Example 1, p. 344]. That $\|\pi_p f - f\| \to 0$ follows from the bound $\sup_p \|\pi_p h\| \leqq \|h\|$.

To prove the third limit in (3.26), we have

$$\sup_{f \in \mathscr{B}} |\langle h, \pi_p f - f\rangle| = \sup_{f \in \mathscr{B}} |\langle \pi_p h - h, f\rangle| \leqq \|\pi_p h - h\| \cdot \sup_{f \in \mathscr{B}} \|f\|_\infty.$$

We now use the second limit in (3.26) and the bound in (3.22). For $\pi^{(n)}$ the last two limits in (3.26) follow from [23; Prop. 5.1.2] and (3.22). □

Lemma 3.5. $F_n \to F$ *uniformly on* $\mathscr{B}$.

Proof. We define the functional $F_n^{(j)}$ by $1/(n^j j!)$ times the inner summation in (3.4). It suffices to prove that for each j $F_n^{(j)} \to F^{(j)}$ uniformly on $\mathscr{B}$. For $f \in \mathscr{B}$, since $(\pi^{(n)} f)(t) = (\pi^{(n)} f)(\alpha/n)$ if $t \in ((\alpha-1)/n, \alpha/n]$, we have

$$\begin{aligned} & j! \, |F^{(j)}(f) - F_n^{(j)}(f)| \\ & \leqq \sum_{\alpha_1, \ldots, \alpha_j = 1}^{n} \left| \int_{(\alpha_1-1)/n}^{\alpha_1/n} \cdots \int_{(\alpha_j-1)/n}^{\alpha_j/n} J_j(s_1, \ldots, s_j) \, [f(s_1) \ldots f(s_j) \right. \\ & \quad \left. - (\pi^{(n)} f)(s_1) \ldots (\pi^{(n)} f)(s_j)] \right| ds_1 \ldots ds_j \\ & \quad + \sum_{\alpha_1, \ldots, \alpha_j = 1}^{n} \int_{(\alpha_1-1)/n}^{\alpha_1/n} \cdots \int_{(\alpha_j-1)/n}^{\alpha_j/n} \left| J_j(s_1, \ldots, s_j) - J_j\left(\frac{\alpha_1}{n}, \ldots, \frac{\alpha_j}{n}\right) \right| \\ & \quad \cdot |(\pi^{(n)} f)(s_1) \ldots (\pi^{(n)} f)(s_j)| \, ds_1 \ldots ds_j. \end{aligned} \tag{3.27}$$

For any $f \in \mathscr{B}$, $\|\pi^{(n)} f\|_\infty \leqq \|f\|_\infty$, and so by (3.22)

$$\sup_{\substack{n \in \{1, 2, \ldots\} \\ f \in \mathscr{B}}} \|\pi^{(n)} f\|_\infty < \infty. \tag{3.28}$$

The uniform continuity of J_j on $\mathbb{T}^j$ and (3.28) imply that the second term on the right-hand side of (3.27) is arbitrarily small, uniformly for $f \in \mathscr{B}$, for all sufficiently large n. Concerning the first term on the right-hand side of (3.27) we approximate J_j as in (3.25). By (3.22) and (3.28), we can prove that for all sufficiently large n this term is arbitrarily small, uniformly for $f \in \mathscr{B}$, by proving that for any $h \in \mathscr{H}$,

$$\sup_{f \in \mathscr{B}} |\langle h, f - \pi^{(n)} f\rangle| \to \infty \qquad \text{as } n \to \infty. \tag{3.29}$$

The latter holds by Lemma 3.4. □

Proof of Lemma 3.2. By Lemma 3.5, it suffices to prove (3.23) with F_n replaced by F. Let $\mathscr{B}^w$ denote the set of weakly continuous functionals on $\mathscr{B}$. By Lemma 3.3, we have $F\in\mathscr{B}^w$. We consider the point-separating algebra $\mathscr{B}_0^w$ of functionals $\tilde{F}\in\mathscr{B}^w$ of the form

$$\tilde{F}(f)=h(\langle g_1,f\rangle,\dots,\langle g_r,f\rangle), \tag{3.30}$$

where $r\in\{1,2,\dots\}$, $h\colon \mathbb{R}^r\to\mathbb{R}$ is continuous, and $g_1,\dots,g_r\in\mathscr{C}(\mathbb{T})$. Since $\mathscr{B}$ is weakly compact, the Stone-Weierstrass Theorem [6; Thm. IV.6.16] implies that $\mathscr{B}_0^w$ is dense in $\mathscr{B}^w$ with respect to the topology of uniform convergence. Hence, since $\pi_{p,n}$ maps $\mathscr{B}$ to $\mathscr{B}$ it suffices to prove (3.23) for functionals $\tilde{F}$ of the form (3.30). Finally, since h is uniformly continuous on compact subsets of $\mathbb{R}^r$, it suffices to prove that for each $\varepsilon>0$ and $g\in\mathscr{C}(\mathbb{T})$ there exists p_0 and for all $p\geqq p_0$ there exists $n_0=n_0(p)$ such that whenever $n\geqq n_0$

$$\sup_{f\in\mathscr{B}}|\langle g,\pi_{p,n}f\rangle-\langle g,f\rangle|<\varepsilon. \tag{3.31}$$

By (3.26), $\langle g,f\rangle$ and $\langle g,\pi_p f\rangle$ are arbitrarily close, uniformly for $f\in\mathscr{B}$, for all sufficiently large p. Hence, it is enough to prove (3.31) with $\langle g,f\rangle$ replaced by $\langle g,\pi_p f\rangle$. For any $f\in\mathscr{B}$,

$$\begin{aligned}|\langle g,\pi_{p,n}f\rangle-\langle g,\pi_p f\rangle|&=|\langle\pi_{p,n}g-\pi_p g,f\rangle|\\&\leqq\|\pi_{p,n}g-\pi_p g\|\cdot\|f\|_\infty.\end{aligned} \tag{3.32}$$

Below we prove

$$\lim_{n\to\infty}(\pi_{p,n}g)(t)=(\pi_p g)(t). \tag{3.33}$$

By the uniform bound (3.22) and the dominated convergence theorem ($\|\pi_{p,n}g\|_\infty\leqq\|g\|_\infty$, $\|\pi_p g\|\leqq\|g\|_\infty$), we will then be finished.

The following inequalities are valid for $n>2^{2p}$, $k\in\{1,\dots,2^p\}$:

$$\frac{k}{2^p}\geqq\frac{n_k}{n}\geqq\frac{k}{2^p}-\frac{k}{n}>\frac{k-1}{2^p}\geqq\frac{n_{k-1}}{n}\geqq\frac{k-1}{2^p}-\frac{k-1}{n}. \tag{3.34}$$

Hence, if $t\in((k-1)/2^p,k/2^p)$, then $t\in(n_{k-1}/n,n_k/n)$ for all n sufficiently large, and so for such n

$$\begin{aligned}|(\pi_p g)(t)-(\pi_{p,n}g)(t)|&=\left|2^p\int_{A_{p,k}}g-\frac{n_k-n_{k-1}}{n}\int_{A_{p,n,k}}g\right|\\&\leqq 2^p\left|\int_{A_{p,k}}g-\int_{A_{p,n,k}}g\right|+\left|2^p-\frac{n}{n_k-n_{k-1}}\right|\int_{A_{p,n,k}}|g|\\&\leqq\left[2^p\left\{\frac{k}{2^p}-\frac{n_k}{n}+\frac{k-1}{2^p}-\frac{n_{k-1}}{n}\right\}+\left|2^p\left(\frac{n_k-n_{k-1}}{n}\right)-1\right|\right]\|g\|_\infty\\&=\|g\|_\infty O\left(\frac{2^{2p}}{n}\right).\end{aligned} \tag{3.35}$$

The last equality uses (3.34). The bound (3.35) implies (3.33). □

III.4 Proof of (3.2)

We apply Theorem A.1 in Appendix A. In the latter, we set $\Omega \doteq \mathscr{B}$, where $\mathscr{B}$ is defined in (3.21), and define Ψ to be the σ-algebra of subsets of $\mathscr{B}$ generated by all weakly open and weakly closed subsets. We set $\{a_n\} \doteq \{n\}$ and $P_{m,n} \doteq Q_{p,n}$ for $n \in \{1, 2, \ldots\}$ and $m = p \in \{1, 2, \ldots\}$, where $Q_{p,n}$ denotes the distribution of the process $\xi_{p,n}$ defined in (3.18)-(3.19). This process takes values in $\mathscr{B}$. We must check that each $Q_{p,n}$ defines a measure on Ψ. Given $\mathbf{y} \doteq (y_1, \ldots, y_n) \in \mathbb{R}^n$, we define $T\mathbf{y}$ by the formula (3.18) with each $Y_j^{(n)}$ replaced by y_j $(j \in \{1, \ldots, n\})$. It is easily checked that for any basic weakly open subset $\mathscr{G}$ in $\mathscr{B}$, $T^{-1}\mathscr{G}$ is open in $\mathbb{R}^n$. Thus, $T^{-1}\mathscr{A}$ is a Borel subset of $\mathbb{R}^n$ for any $\mathscr{A} \in \Psi$ and

$$Q_{p,n}(\mathscr{A}) = \int_{T^{-1}\mathscr{A}} \prod_{j=1}^{n} d\rho(y_j).$$

We define the functionals F and I by (2.3) and (1.18) respectively. By Lemma 3.2, F is weakly continuous and uniformly bounded on $\mathscr{B}$, and so satisfies condition (A.3). Hence (3.2) follows from Theorem A.1 once we prove that for all weakly closed subsets $\mathscr{K}$ in $\mathscr{B}$ and all weakly open subsets $\mathscr{G}$ in $\mathscr{B}$

$$\limsup_{p\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log Q_{p,n}(\mathscr{K}) \leqq -I(\mathscr{K}), \tag{3.36}$$

$$\liminf_{p\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log Q_{p,n}(\mathscr{G}) \geqq -I(\mathscr{G}). \tag{3.37}$$

Proof of (3.36). We need a lemma, the proof of which we save for the end of Sect. 3.3.

Lemma 3.6. *For $f \in \mathscr{H}$ and $\rho \in \mathscr{M}_b$ we define the functional $\Gamma(f)$ by the formula*

$$\Gamma(f) \doteq \int \gamma_\rho(f(t))\, dt, \tag{3.38}$$

where γ_ρ is defined in (1.20). *Then for any $h \in \mathscr{H}$*

$$I(h) = \sup_{f \in \mathscr{H}} \{\langle f, h\rangle - \Gamma(f)\}. \tag{3.39}$$

The functional I is weakly lower semicontinuous on $\mathscr{H}$.

Remark. Formula (3.39) exhibits I as the Legendre transformation of the convex functional Γ (see Appendix C); i.e., $I = \Gamma^*$. This formula is reasonable since $i_\rho \doteq \gamma_\rho^*$ (see (1.19)). One can also prove $I^* = \Gamma$.

We now prove (3.36). We denote by $E\{-\}$ expectation with respect to $\xi_{p,n}$. For any $g \in \mathscr{H}$, we have

$$\begin{aligned} E\{\exp[n\langle g, \xi_{p,n}\rangle]\} &\geqq E\{1_{\{\xi_{p,n} \in \mathscr{K}\}} \exp[n\langle g, \xi_{p,n}\rangle]\} \\ &\geqq \operatorname{Prob}\{\xi_{n,p} \in \mathscr{K}\} \cdot \inf_{h \in \mathscr{K}} (\exp[n\langle g, h\rangle]). \end{aligned} \tag{3.40}$$

Thus,

$$Q_{n,p}(\mathscr{K}) = \text{Prob}\{\xi_{p,n} \in \mathscr{K}\}$$
$$\leqq \sup_{h \in \mathscr{K}} (\exp[-n\langle g, h\rangle]) E\{\exp[n\langle g, \xi_{p,n}\rangle]\}$$
$$= \sup_{h \in \mathscr{K}} (\exp[-n\langle g, h\rangle]) \prod_{k=1}^{2^p} \left[\int \exp\left[(\pi_{p,n} g)\left(\frac{n_k}{n}\right) \cdot x\right] d\rho(x)\right]^{n_k - n_{k-1}}. \tag{3.41}$$

Hence we have

$$\frac{1}{n} \log Q_{n,p}(\mathscr{K}) \leqq \sup_{h \in \mathscr{K}} (-\langle g, h\rangle) + \sum_{k=1}^{2^p} \frac{n_k - n_{k-1}}{n} \gamma_\rho \left((\pi_{p,n} g)\left(\frac{n_k}{n}\right)\right)$$
$$\leqq \sup_{h \in \mathscr{K}} (-\langle g, h\rangle) + \sum_{k=1}^{2^p} \int_{A_{p,n,k}} \gamma_\rho(g(t))\, dt$$
$$= \sup_{h \in \mathscr{K}} (-\langle g, h\rangle) + \Gamma(g). \tag{3.42}$$

The second inequality in (3.42) uses Jensen's inequality.

Here is the idea of the rest of the proof. Since $g \in \mathscr{H}$ is arbitrary, we have

$$\frac{1}{n} \log Q_{p,n}(\mathscr{K}) \leqq -\sup_{g \in \mathscr{H}} \inf_{h \in \mathscr{K}} (\langle g, h\rangle - \Gamma(g)). \tag{3.43}$$

$\mathscr{K}$, being a weakly closed subset of the weakly compact set $\mathscr{B}$, is also weakly compact. We use this to reverse the infimum and supremum in (3.43). After this reveral, (3.39) implies (3.36).

The proof to follow would be much simpler if $\sup_{\mathscr{B}} I$ were finite. However, this holds if and only if ρ has atoms at l and at L (see (3.64)).

We complete the proof of (3.36) under the assumption that $I(\mathscr{K}) < \infty$. If $I(\mathscr{K}) = \infty$, then (3.36) follows by a straightforward modification of this proof. Given $\varepsilon > 0$ and $h \in \mathscr{K}$ such that $I(h) < \infty$, we find by Lemma 3.6 $g_h \in \mathscr{H}$ and a weak neighborhood

$$\mathscr{A}(h) \text{ of } h \ (\mathscr{A}(h) \doteq \{f \in \mathscr{B}: |\langle g_h, h\rangle - \langle g_h, f\rangle| < \varepsilon/2\})$$

such that

$$\langle g_h, f\rangle - \Gamma(g_h) > I(h) - \varepsilon \qquad \text{for all } f \in \mathscr{K}(h) \doteq \overline{\mathscr{A}(h)} \cap \mathscr{K}. \tag{3.44}$$

In (3.44), $\overline{\mathscr{A}(h)}$ denotes the closure of $\mathscr{A}(h)$. Thus,

$$\inf_{f \in \mathscr{K}(h)} \langle g_h, f\rangle - \Gamma(g_h) \geqq I(\mathscr{K}(h)) - \varepsilon. \tag{3.45}$$

On the other hand, if $I(h) = \infty$, then by Lemma 3.6 there exists $g_h \in \mathscr{H}$ and a weak neighborhood $\mathscr{A}(h)$ of h such that

$$\langle g_h, f\rangle - \Gamma(g_h) > I(\mathscr{K}) + 1 \doteq R \qquad \text{for all } f \in \overline{\mathscr{A}(h)} \cap \mathscr{K}. \tag{3.46}$$

By the weak lower semicontinuity of I (Lemma 3.6), there exists another weak neighborhood $\mathscr{A}_1(h)$ of h such that $I(g) \geqq R$ for all $g \in \mathscr{A}_1(h)$. In particular,

$$\inf_{f\in\mathscr{K}(h)} \langle g_h, f\rangle - \Gamma(g_h) > I(\mathscr{K}) + 1,$$

$$I(f) \geqq R \qquad \text{for all } f\in\mathscr{K}(h) \doteq \overline{\mathscr{A}(h)} \cap \overline{\mathscr{A}_1(h)} \cap \mathscr{K}. \tag{3.47}$$

By the weak compactness of $\mathscr{K}$, we find $r\in\{1,2,\ldots\}$ and elements $\{h_1,\ldots,h_r\}$ and sets $\{\mathscr{K}(h_1),\ldots,\mathscr{K}(h_r)\}$ which cover $\mathscr{K}$, where if $I(h_j)<\infty$ then $\mathscr{K}(h_j)$ is as in (3.44)-(3.45) and if $I(h_j)=\infty$ then $\mathscr{K}(h_j)$ is as in (3.47). For each h_j we pick $g_j \doteq g_{h_j}\in\mathscr{H}$ satisfying (3.45) or (3.47). Clearly

$$\infty > I(\mathscr{K}) = \min_{j\in\{1,\ldots,r\}} I(\mathscr{K}(h_j)) = \min\{I(\mathscr{K}(h_j)) \colon I(\mathscr{K}(h_j)) < R\}. \tag{3.48}$$

By (3.42), (3.45), and (3.47), we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\log Q_{p,n}(\mathscr{K}) &\leqq \max_{j\in\{1,\ldots,r\}} \left[\limsup_{n\to\infty}\frac{1}{n}\log Q_{p,n}(\mathscr{K}(h_j))\right]\\ &\leqq -\min_{j\in\{1,\ldots,r\}} \inf_{f\in\mathscr{K}(h_j)} (\langle g_j, f\rangle - \Gamma(g_j))\\ &\leqq -\min_{j\in\{1,\ldots,r\}} \delta_j,\end{aligned} \tag{3.49}$$

where

$$\delta_j \doteq \begin{cases} I(\mathscr{K}(h_j)) - \varepsilon & \text{if } I(h_j) < R,\\ I(\mathscr{K}) + 1 & \text{if } I(h_j) \geqq R.\end{cases} \tag{3.50}$$

By (3.48), $\min_{j\in\{1,\ldots,r\}} \delta_j = I(\mathscr{K}) - \varepsilon$, and so (3.49) yields (3.36) by taking $\varepsilon\to 0$. □

Proof of (3.37). We need two lemmas, Lemma 3.7 and 3.8. We save the proof of Lemma 3.7 for the end of this section. Lemma 3.8 is proved in [4; Thm. 5.3] (in much greater generality).

Lemma 3.7. *We define $\mathscr{B}_{p,n}$ to be the image of $\mathscr{B}$ under $\pi_{p,n}$. Let $\mathscr{G}$ be any weakly open subset in $\mathscr{B}$ and pick $h\in\mathscr{G}$. Then there exists $p_0 = p_0(h)$ and for all $p \geqq p_0$ there exists $\delta = \delta(p)$ and $n_0 = n_0(p)$ such that whenever $n \geqq n_0$*

$$\left\{f\in\mathscr{B}_{p,n} \colon \left| f\left(\frac{n_k}{n}\right) - (\pi_p h)\left(\frac{k}{2^p}\right)\right| < \delta, \text{ all } k = 1,\ldots,2^p\right\} \subseteq \mathscr{G}. \tag{3.51}$$

Lemma 3.8. *Let $\{Y_j,\ j=1,2,\ldots\}$ be a sequence of i.i.d. random variables each distributed by a probability measure ρ on $\mathbb{R}$ which satisfies $\int \exp(k|x|)\,d\rho(x) < +\infty$ for all $k>0$.*

Set $S_r = \sum_{j=1}^{r} Y_j$, $r\in\{1,2,\ldots\}$. For any closed subset K in $\mathbb{R}$ and any open subset G in $\mathbb{R}$

$$\limsup_{r\to\infty} \frac{1}{r}\log \operatorname{Prob}\left\{\frac{S_r}{r}\in K\right\} \leqq -i_\rho(K), \tag{3.52}$$

$$\liminf_{r\to\infty} \frac{1}{r}\log \operatorname{Prob}\left\{\frac{S_r}{r}\in G\right\} \geqq -i_\rho(G). \tag{3.53}$$

We now prove (3.37). For any $h\in\mathscr{G}$, we have by Lemma 3.7 for $p\geqq p_0(h)$, $n\geqq\max(n_0, 2^p)$

$$\begin{aligned} Q_{p,n}(\mathscr{G}) &= \operatorname{Prob}\{\xi_{p,n}\in\mathscr{G}\}\geqq \operatorname{Prob}\left\{\left|\xi_{p,n}\left(\frac{n_k}{n}\right)-(\pi_p h)\left(\frac{k}{2^p}\right)\right|<\delta\right\} \\ &= \prod_{k=1}^{2^p}\operatorname{Prob}\left\{\left|\frac{1}{r_k}\sum_{j=n_{k-1}+1}^{n_k} Y_j^{(n)}-(\pi_p h)\left(\frac{k}{2^p}\right)\right|<\delta\right\}\equiv\prod_{k=1}^{2^p}\operatorname{Prob}\{\mathscr{A}_{p,k}\}. \end{aligned} \tag{3.54}$$

In (3.54), $r_k\doteq n_k-n_{k-1}$ and the last equality defines the sets $\mathscr{A}_{p,k}$. By (3.53), the fact that $\lim\limits_{n\to\infty} r_k/n=1/2^p$, and the choice of h, we have for all $p\geqq p_0(h)$

$$\begin{aligned} \liminf_{n\to\infty}\frac{1}{n}\log Q_{n,p}(\mathscr{G}) &\geqq \frac{1}{2^p}\sum_{k=1}^{2^p}\liminf_{n\to\infty}\frac{1}{r_k}\log\operatorname{Prob}\{\mathscr{A}_{k,p}\} \\ &= \frac{1}{2^p}\sum_{k=1}^{2^p}(-\inf\{i_p(u):\ |u-(\pi_p h)(k/2^p)|<\delta\}) \\ &\geqq -\frac{1}{2^p}\sum_{k=1}^{2^p} i_p((\pi_p h)(k/2^p))\geqq -\sum_{k=1}^{2^p}\int_{\Delta_{p,k}} i_p(h(t))\,dt=-I(h). \end{aligned} \tag{3.55}$$

The last inequality in (3.55) follows from Jensen's inequality. Inequality (3.55) implies that

$$\liminf_{p\to\infty}\liminf_{n\to\infty}\frac{1}{n}\log Q_{n,p}(\mathscr{G})\geqq -I(h).$$

Taking the supremum over $h\in\mathscr{G}$ of $-I(h)$, we conclude (3.27).

We will have completed the proof of (3.2) once we prove Lemmas 3.6 and 3.7.

Proof of Lemma 3.6. We write i and γ instead of i_p and γ_p. The key step is the following fact. Given a real number h and $N\in\{1,2,\ldots\}$, we define

$$g\doteq\begin{cases} +\infty & \text{for } h\geqq L,\\ (\gamma')^{-1}(h) & \text{for } l<h<L,\\ -\infty & \text{for } h\leqq l,\end{cases} \tag{3.56}$$

and

$$g_N\doteq\max(\min(g,N),-N). \tag{3.57}$$

Then we have

$$0\leqq g_N h-\gamma(g_N)\uparrow i(h)\qquad\text{as } N\to\infty. \tag{3.58}$$

In (3.56), $[l,L]$ is the smallest closed interval containing the support of ρ. In (3.56) $(\gamma')^{-1}$ denotes the inverse of the function γ'. The quantity $(\gamma')^{-1}(h)$ is well defined for $h\in(l,L)$. Indeed, for all real t,

$$\gamma''(t)=\int\left(x-\int x\,d\alpha_t(x)\right)^2 d\alpha_t(x)>0, \tag{3.59}$$

where $d\alpha_t(x)\doteq\exp(tx)\,d\rho(x)/\int\exp(ty)\,d\rho(y)$. Thus, $\gamma'(t)=\int x\,d\alpha_t(x)$ is a strictly monotonically increasing function of t; $\gamma'(t)\in(l,L)$; $\lim\limits_{t\to\infty}\gamma'(t)=L$, $\lim\limits_{t\to-\infty}\gamma'(t)=l$.

We show how (3.58) yields (3.39) in the lemma and then prove (3.58). Given $h \in \mathscr{H}$, we define $g(t)$ and $g_N(t)$ by (3.56) and (3.57) with $h(t)$ written for h; since $\|g_N\|_\infty \leqq N$, $g_N \in \mathscr{H}$. For any $f \in \mathscr{H}$, by (1.19) we have

$$I(h) = \int i(h(t))\,dt = \int \sup_{u \in \mathbb{R}} \{u\,h(t) - \gamma(u)\}\,dt$$
$$\geqq \int (f(t)\,h(t) - \gamma(f(t)))\,dt. \tag{3.60}$$

Thus, by the arbitrariness of $f \in \mathscr{H}$, (3.58), and the monotone convergence theorem

$$I(h) \geqq \sup_{f \in \mathscr{H}} (\langle f, h\rangle - \Gamma(f)) \geqq \sup_N (\langle g_N, h\rangle - \Gamma(g_N))$$
$$= \int \lim_{N \to \infty} (g_N(t)\,h(t) - \gamma(g_N(t)))\,dt = \int i(h(t))\,dt = I(h). \tag{3.61}$$

This is (3.39) in Lemma 3.6.

We now prove (3.58). We first consider $h = L$. Then $g_N = N$ and

$$NL - \gamma(N) = -\log \int \exp(N(x-L))\,d\rho(x). \tag{3.62}$$

We write A for the support of ρ. Since

$$\exp(N(x-L))\,1_A(x) \downarrow 1_{\{L\}}(x) \qquad \text{as } N \uparrow \infty, \tag{3.63}$$

we see that as $N \uparrow \infty$ $NL - \gamma(N)$ is strictly increasing. In other words, for $u = L$ the supremum in the definition of $i(u)$ is not attained, and

$$0 \leqq i(L) = \lim_{N \uparrow \infty} (NL - \gamma(N)) = \begin{cases} -\log \rho(\{L\}) & \text{if } \rho \text{ has an atom at } L, \\ +\infty & \text{if } \rho \text{ has no atom at } L. \end{cases} \tag{3.64}$$

Thus for $h = L$ (3.58) is valid. By similar reasoning, if $h > L$, then $i(h) = +\infty$, and (3.58) holds. The case $h \leqq l$ is handled similarly. We now consider $h \in (l, L)$. By calculus and the fact that the range of γ' is (l, L), we have for any $h \in (l, L)$

$$i(h) = h\,t(h) - \gamma(t(h)) < \infty, \qquad t(h) \doteq (\gamma')^{-1}(h). \tag{3.65}$$

For $h \in (l, L)$, we have by the definition of g_N

$$g_N \geqq 0, \quad g_N \uparrow (\gamma')^{-1}(h) \quad \text{if } (\gamma')^{-1}(h) \geqq 0,$$
$$g_N \leqq 0, \quad g_N \downarrow (\gamma')^{-1}(h) \quad \text{if } (\gamma')^{-1}(h) \leqq 0. \tag{3.66}$$

The fact that for $h \in (l, L)$ $g_N h - \gamma(g_N) \uparrow i(h)$ follows from (3.65), (3.66), and the fact that

$$\frac{d}{dt}(t\,h - \gamma(t)) \begin{cases} > 0 & \text{for } t < (\gamma')^{-1}(h), \\ < 0 & \text{for } t > (\gamma')^{-1}(h). \end{cases} \tag{3.67}$$

To prove $g_N h - \gamma(g_N) \geqq 0$ for $h \in (l, L)$, we use (3.66), (3.67), and the fact that at $t = 0$ $(t\,h - \gamma(t)) = 0$. This completes the proof of (3.58).

We now prove that I is weakly lower semi-continuous. Let $\{h_j\}$ be a sequence in $\mathscr{H}$ tending weakly to some $h\in\mathscr{H}$. We prove

$$\liminf_{j\to\infty} I(h_j)\geqq I(h). \tag{3.68}$$

We first assume $I(h)<\infty$. Let $\varepsilon>0$ be given. There exists $j_0\in\{1,2,\ldots\}$ and by (3.39) there exists $f\in\mathscr{H}$ such that for all $j\geqq j_0$

$$I(h_j)\geqq\langle f,h_j\rangle-\Gamma(f)\geqq\langle f,h\rangle-\Gamma(f)-\varepsilon\geqq I(h)-2\varepsilon. \tag{3.69}$$

Taking $\varepsilon\to 0$, we obtain (3.68). If $I(h)=\infty$, then given $R\in(0,\infty)$, there exists $j_0\in\{1,2,\ldots\}$ and by (3.39) there exists $f\in\mathscr{H}$ such that for all $j\geqq j_0$

$$I(h_j)\geqq\langle g,h_j\rangle-\Gamma(f)\geqq\langle f,h\rangle-\Gamma(f)-1\geqq R. \tag{3.70}$$

Taking $R\to\infty$, we obtain (3.68). $\square$

Proof of Lemma 3.7. Since $\mathscr{G}$ is weakly open in $\mathscr{B}$, we can find $\delta_0>0$, $r\in\{1,2,\ldots\}$, and r functions $\{g_j; j=1,\ldots,r\}$ in $\mathscr{H}$ such that

$$\{f\in\mathscr{B}\colon |\langle f,g_j\rangle-\langle h,g_j\rangle|<\delta_0,\ j=1,\ldots,r\}\subseteq\mathscr{G}. \tag{3.71}$$

By Lemma 3.4, there exists p_0 such that for all $p\geqq p_0$

$$\{f\in\mathscr{B}\colon |\langle f,g_j\rangle-\langle \pi_p h,g_j\rangle|<\delta_0/2,\ j=1,\ldots,r\}\subseteq\mathscr{G}. \tag{3.72}$$

We are done once we have proved that for each $p\geqq p_0$ there exists $n_0=n_0(p)$ and $A=A(p)$ such that whenever $n\geqq n_0$

$$\left\{f\in\mathscr{B}_{p,n}\colon \left|f\left(\frac{n_k}{n}\right)-(\pi_p h)\left(\frac{k}{2^p}\right)\right|<\delta_0/A,\ k=1,\ldots,2^p\right\}$$
$$\subseteq\{f\in\mathscr{B}\colon |\langle f,g_j\rangle-\langle \pi_p h,g_j\rangle|<\delta_0/2,\ j=1,\ldots,r\}. \tag{3.73}$$

In (3.51), set $\delta\doteq\delta_0/A$.

Fix $f\in\mathscr{B}_{p,n}$ and $g\doteq g_j$, $j\in\{1,\ldots,r\}$. Since f is constant on the intervals $\{\Lambda_{p,n,k}\}$ and $\pi_p h$ is constant on the intervals $\{\Lambda_{p,k}\}$, we have

$$|\langle f,g\rangle-\langle\pi_p h,g\rangle|\leqq\sum_{k=1}^{2^p}\left|f\left(\frac{n_k}{n}\right)\int_{\Lambda_{p,n,k}} g-(\pi_p h)\left(\frac{k}{2^p}\right)\int_{\Lambda_{p,k}} g\right|$$
$$\leqq\sum_{k=1}^{2^p}\left|f\left(\frac{n_k}{n}\right)-(\pi_p h)\left(\frac{k}{2^p}\right)\right|\int_{\Lambda_{p,n,k}}|g|$$
$$+\sum_{k=1}^{2^p}\left|(\pi_p h)\left(\frac{k}{2^p}\right)\right|\left|\int_{\Lambda_{p,n,k}} g-\int_{\Lambda_{p,k}} g\right|. \tag{3.74}$$

By (3.34), for $n\geqq 2^{2p}$ and $k\in\{1,\ldots,2^p\}$ there exist two intervals $\Lambda^{(1)}_{p,n,k}$ and $\Lambda^{(2)}_{p,n,k}$ with $|\Lambda^{(j)}_{p,n,k}|\leqq k/n$ $(j=1,2)$ such that

$$\left|\int_{\Lambda_{p,n,k}} g-\int_{\Lambda_{p,k}} g\right|=\left|\int_{\Lambda^{(1)}_{p,n,k}} g\right|+\left|\int_{\Lambda^{(2)}_{p,n,k}} g\right|\leqq 2\frac{2^{p/2}}{\sqrt{n}}\|g\|.$$

From (3.74), we conclude

$$|\langle f, g\rangle - \langle \pi_p h, g\rangle| \leqq \sup_{k\in\{1,\ldots,\ 2^p\}} \left| f\left(\frac{n_k}{n}\right) - (\pi_p h)\left(\frac{k}{2^p}\right)\right| \|g\| + \frac{2^p \|h\| \cdot 2 \cdot 2^{p/2} \|g\|}{\sqrt{n}}. \quad (3.75)$$

Clearly we can find $n_0 = n_0(p)$ and $A = A(p)$ sufficiently large such that if the inequalities on the left-hand side of (3.73) are satisfied for all $n \geqq n_0$, then f belongs to the set on the right-hand side of (3.73). □

IV. Laws of Large Numbers

We prove the laws of large numbers in Theorems 1.1, 1.2, 1.4, and 1.5 and their generalizations in Sect. II, assuming the validity of the statements about $\mathscr{D}_\beta$ in (1.37) and (1.40) and the generalizations of these statements in Sect. II. The latter will be proved in Sect. VI by means of ideas to be developed in Sect. V.

We formulate and prove the laws of large numbers for the general d-body case defined in Sect. II. Given $\rho \in \mathscr{M}_b$, we consider spin random variables $\{X_\alpha^{(n)}; \alpha = 1, \ldots, n\}$ with joint distribution given by (2.1). For Δ an interval in $\mathbb{T}$, we define

$$W_n(\Delta) \doteq |\Delta|^{-1} \sum_{\{\alpha:\alpha/n\in\Delta\}} X_\alpha^{(n)}. \quad (4.1)$$

We define the functional F on $\mathscr{H}$ by (2.3) and the functional F_n on $\mathscr{H}$ by (3.4). For the two-body case considered in Sect. I, F is given by (1.17) and F_n by

$$F_n(f) = \sum_{\alpha_1, \alpha_2 = 1}^{n} \frac{1}{n^2} J\left(\frac{\alpha_1}{n}, \frac{\alpha_2}{n}\right) (\pi^{(n)} f)\left(\frac{\alpha_1}{n}\right) \cdot (\pi^{(n)} f)\left(\frac{\alpha_2}{n}\right), \quad (4.2)$$

where $\pi^{(n)} f$ is defined in (3.3). We define the stochastic process ζ_n by (3.5) and denote by R_n the distribution of ζ_n. R_n defines a measure on the σ-algebra Ψ generated by all weakly open and weakly closed subsets of $\mathscr{B}$. We recall the formula

$$Z_{n,\beta} = \int_{\mathscr{B}} \exp[n\beta F_n(f)]\, dR_n(f), \quad (4.3)$$

first noted in (3.6). Because of (4.3), a probability measure $\Phi_{n,\beta}$ is defined on $(\mathscr{B}, \Psi)$ by the formula

$$\Phi_{n,\beta}(\mathscr{A}) \doteq \frac{\int_{\mathscr{A}} \exp[n\beta F_n(f)]\, dR_n(f)}{Z_{n,\beta}}, \qquad \mathscr{A} \in \Psi. \quad (4.4)$$

This definition coincides with the definition (1.35) since each measure R_n is concentrated on $\mathscr{B}$. Finally we define the set

$$\mathscr{D}_\beta \doteq \{f \in \mathscr{B}\colon\ (\beta F - I)(f) = \sup_{\mathscr{B}} (\beta F - I)\}. \quad (4.5)$$

This coincides with the definition (1.36) since $I(f) = +\infty$ for $f \in \mathscr{H} \setminus \mathscr{B}$.

Theorem 4.1. (i) *For any* $r \in \{1, 2, \ldots\}$, *any* r *intervals* $\{\Delta_j; j = 1, \ldots, r\}$ *in* $\mathbb{T}$, *and any continuous function* $h\colon \mathbb{R}^r \to \mathbb{R}$

$$\lim_{n\to\infty}\left|E_{n,\beta}\left\{h\left(\frac{W_n(\Delta_1)}{n},\ldots,\frac{W_n(\Delta_r)}{n}\right)\right\}-\int_{\mathscr{B}} h(\langle g_1,f\rangle,\ldots,\langle g_r,f\rangle)\,d\Phi_{n,\beta}(f)\right|=0, \tag{4.6}$$

where $g_j \doteq |\Delta_j|^{-1}1_{\Delta_j}$.

(ii) *The set of measures* $\{\Phi_{n,\beta};\ n=1,2,\ldots\}$ *is relatively compact with respect to weak convergence and the support of any accumulation point of* $\{\Phi_{n,\beta}\}$ *is contained in* $\mathscr{D}_\beta$.

(iii) *Let* φ *be any weakly continuous functional on* $\mathscr{H}$.

(a) *If* $\mathscr{D}_\beta=\{0\}$, *then*

$$\lim_{n\to\infty}\int_{\mathscr{B}}\varphi(f)\,d\Phi_{n,\beta}(f)=\varphi(0). \tag{4.7}$$

(b) *Assume that* ρ *is even and that in formulae* (2.3) *for* F *and* (3.4) *for* F_n, $J_j\equiv 0$ *for odd* j. *If there exists a constant* $m\neq 0$ *such that* $\mathscr{D}_\beta=\{m1,\,-m1\}$, *then*

$$\lim_{n\to\infty}\int_{\mathscr{B}}\varphi(f)\,d\Phi_{n,\beta}(f)=\tfrac{1}{2}[\varphi(m1)+\varphi(-m1)]. \tag{4.8}$$

(c) *Assume that* $d=2$ *and that* F *and* F_n *are defined by* (1.17) *and* (4.2), *respectively. If the interaction function* J *is translation invariant and there exists a function* $\tilde f\in\mathscr{H}$, $\tilde f\neq 0$, *such that* $\mathscr{D}_\beta=\{\tilde f(\cdot+s),\ s\in\mathbb{T}\}$, *then*

$$\lim_{n\to\infty}\int_{\mathscr{B}}\varphi(f)\,d\Phi_{n,\beta}(f)=\int\varphi(\tilde f(\cdot+s))\,ds. \tag{4.9}$$

Remark. Let us define measures $\{\Xi_{n,p,\beta}\}$ on $(\mathscr{B},\Psi)$ by the formula

$$\Xi_{p,n,\beta}(\mathscr{A})\doteq\frac{\int_{\mathscr{A}}\exp[n\beta F_n(f)]\,dQ_{n,p}(f)}{\int_{\mathscr{B}}\exp[n\beta F_n(f)]\,dQ_{n,p}(f)},\qquad \mathscr{A}\in\Psi,$$

where $Q_{p,n}$ is the distribution of the process $\xi_{p,n}$ in (3.18). Then the limits (4.6)–(4.9) all hold with $\{\Phi_{n,\beta}\}$ replaced by $\{\Xi_{p,n,\beta}\}$ (with $\lim_{p\to\infty}\lim_{n\to\infty}$ instead of $\lim_{n\to\infty}$). We have chosen to work with $\{\Phi_{n,\beta}\}$ because they are less cumbersome than $\{\Xi_{p,n,\beta}\}$.

In order to prove part (ii), we need a lemma.

Lemma 4.2. *For any weakly closed subset* $\mathscr{K}$ *in* $\mathscr{B}$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\int_{\mathscr{K}}\exp[n\beta F_n(f)\,dR_n(f)]\leq\sup_{\mathscr{K}}[\beta F-I]. \tag{4.10}$$

Proof. By Lemma 3.5, $F_n\to F$ uniformly on $\mathscr{B}$. Hence it suffices to prove the lemma with $\{F_n\}$ replaced by F. By Remark 1 following the statement of Theorem A.1 (with $P_{m,n}\doteq R_n$ for all $m\in\{1,2,\ldots\}$) and by (A.5) it suffices to prove that for all weakly closed subsets $\mathscr{K}$ in $\mathscr{B}$

$$\limsup_{n\to\infty}\frac{1}{n}\log R_n(\mathscr{K})\leqq -I(\mathscr{K}). \tag{4.11}$$

This is proved like (3.36). We have for any $g\in\mathscr{H}$ (cf. (3.40)–(3.42))

$$\begin{aligned}\frac{1}{n}\log R_n(\mathscr{K}) &\leqq \sup_{h\in\mathscr{K}}(-\langle g,h\rangle)+\frac{1}{n}\log E\{\exp[n\langle g,\zeta_n\rangle]\}\\ &=\sup_{h\in\mathscr{K}}(-\langle g,h\rangle)+\sum_{k=1}^{n}\frac{1}{n}\gamma_\rho\left((\pi^{(n)}g)\left(\frac{k}{n}\right)\right)\\ &\leqq \sup_{h\in\mathscr{K}}(-\langle g,h\rangle)+\Gamma(g),\end{aligned} \tag{4.12}$$

where the last step uses Jensen's inequality. Now (4.12) is shown to imply (4.11) in exactly the same way that we proved (3.42) implies (3.36). □

Proof of Theorem 4.1. We set $\beta=1$, writing E_n, Φ_n, and $\mathscr{D}$ instead of $E_{n,\beta}$, $\Phi_{n,\beta}$ and $\mathscr{D}_\beta$.

(i) We consider only $r=1$ and $\Delta_1\doteq(a,b]$, where $a,b\in\mathbb{T}$, $a<b$. The case of general $r\in\{1,2,\dots\}$ and general intervals $\{\Delta_j;\ j=1,\dots,r\}$ has handled similarly. The random variables $W_n(\Delta_1)/n$ and $\langle|\Delta|^{-1}1_\Delta,\zeta_n\rangle$ take values in the compact subset $A\doteq[l,L]$ in $\mathbb{R}$ (see (3.20)). It suffices to prove (4.6) for $h\in\mathscr{C}^1(A)$ since $\mathscr{C}^1(A)$ is dense in $\mathscr{C}(A)$. We pick integers $j_0=j_0(n)$, $k_0=k_0(n)\in\{0,\dots,n-1\}$ such that

$$\frac{j_0}{n}\leqq a<\frac{j_0+1}{n},\qquad \frac{k_0}{n}<b\leqq\frac{k_0+1}{n}.$$

Then

$$\begin{aligned}\left\langle\frac{1_{\Delta_1}}{|\Delta_1|},\zeta_n\right\rangle &=\frac{1}{|\Delta_1|}\sum_{k=1}^{n}Y_k^{(n)}\left|\Delta_1\cap\left(\frac{k-1}{n},\frac{k}{n}\right]\right|\\ &=\frac{1}{|\Delta_1|}\frac{1}{n}\sum_{j_0+1\leqq k\leqq k_0}Y_k^{(n)}+\left(\frac{j_0+1}{n}-a\right)Y_{j_0}^{(n)}+\left(b-\frac{k_0}{n}\right)Y_{k_0+1}^{(n)}.\end{aligned} \tag{4.13}$$

Hence

$$\begin{aligned}&\left|E_n\left\{h\left(\frac{W_n(\Delta_1)}{n}\right)\right\}-\int h\left(\left\langle\frac{1_{\Delta_1}}{|\Delta_1|},f\right\rangle\right)d\Phi_n(f)\right|\\ &=\left|E_n\left\{h\left(\frac{W_n(\Delta_1)}{n}\right)-h\left(\frac{W_n(\Delta_1)}{n}+\left(\frac{j_0+1}{n}-a\right)X_{j_0}^{(n)}+\left(b-\frac{k_0}{n}\right)X_{k_0+1}^{(n)}\right)\right\}\right|\\ &\leqq[\sup_{t\in A}|h'(t)|]\cdot\left\{\left(\frac{j_0+1}{n}-a\right)E_n|X_{j_0}^{(n)}|+\left(b-\frac{k_0}{n}\right)E_n|X_{k_0+1}^{(n)}|\right\}\\ &=O(1/n).\end{aligned} \tag{4.14}$$

The last step uses the fact that since ρ has bounded support $\sup\{E_n|X_j^{(n)}|;\ n=1,2,\ldots;j=1,\ldots,n\}<\infty$.

(ii) Let $\mathscr{S}$ be any weakly open set in $\mathscr{B}$ containing $\mathscr{D}$. We prove below that there exist $C\in(0,\infty)$, $\delta\in(0,\infty)$ such that for all $n\in\{1,2,\ldots\}$

$$\Phi_n(\mathscr{B}\setminus\bar{\mathscr{S}})\leqq C\,e^{-n\delta}. \tag{4.15}$$

Here $\bar{\mathscr{S}}$ denotes the weak closure of $\mathscr{S}$. Since $\bar{\mathscr{S}}$ is also weakly compact, we see that the set of measures $\{\Phi_n\}$ is tight, and thus by Prohorov's theorem [2; Thm. 6.1] $\{\Phi_n\}$ is relatively compact. Let Φ be any accumulation point of $\{\Phi_n\}$; say $\Phi_{n'}\Rightarrow\Phi$ for some subsequence $\{n'\}$. By (4.15)

$$\Phi(\mathscr{B}\setminus\bar{\mathscr{S}})\leqq\liminf_{n'\to\infty}\Phi_{n'}(\mathscr{B}\setminus\bar{\mathscr{S}})=0, \tag{4.16}$$

and so Φ is supported on $\bar{\mathscr{S}}$. By the arbitrariness of $\mathscr{S}$ and the fact that $\mathscr{D}$ is weakly closed (see Theorem 5.1(ii) below), we conclude that Φ is supported on

$$\bigcap_{\{\mathscr{S}\supseteq\mathscr{D},\,\mathscr{S}\text{ weakly open}\}}\bar{\mathscr{S}}=\mathscr{D}. \tag{4.17}$$

We now prove (4.15). We have by Lemma 4.2 and Theorem 2.1

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\Phi_n(\mathscr{B}\setminus\bar{\mathscr{S}})&\leqq\limsup_{n\to\infty}\frac{1}{n}\log\Phi_n(\mathscr{B}\setminus\mathscr{S})\\&\leqq\sup_{\mathscr{B}\setminus\mathscr{S}}[F-I]-\sup_{\mathscr{B}}[F-I].\end{aligned} \tag{4.18}$$

We are done once we prove

$$\sup_{\mathscr{B}\setminus\mathscr{S}}[F-I]<\sup_{\mathscr{B}}[F-I]. \tag{4.19}$$

If we had equality in (4.19), then by the weak compactness of $\mathscr{B}\setminus\mathscr{S}$ (Lemma 3.1), there would exist a sequence $\{f_j\}$ and an element f in $\mathscr{B}\setminus\mathscr{S}$ such that

$$f_j\rightharpoonup f,\quad (F-I)(f_j)\to\sup_{\mathscr{B}}[F-I]. \tag{4.20}$$

The weak upper semicontinuity of $F-I$ on $\mathscr{B}$ (Lemmas 3.3 and 3.6) would then imply $(F-I)(f)=\sup_{\mathscr{B}}[F-I]$ or $f\in\mathscr{D}$. But the latter is impossible since $(\mathscr{B}\setminus\mathscr{S})\cap\mathscr{D}=\emptyset$.

(iii) The right-hand sides of (4.7), (4.8), and (4.9) define probability measures Φ_A, Φ_B, and Φ_C, respectively; i.e., $\Phi_A\doteq\delta_0$, $\Phi_B\doteq(\delta_{m1}+\delta_{-m1})/2$, and Φ_C denotes the circular average in (4.9). We first consider case (a). Since Φ_A is the unique probability measure supported on $\mathscr{D}\doteq\{0\}$, by part (ii) any accumulation point of $\{\Phi_n\}$ must be Φ_A.

Thus, $\Phi_n\Rightarrow\Phi_A$, which is (4.7). In case (b), by the evenness of ρ and the fact that $J_j\equiv0$ for odd j, each Φ_n is signature invariant (invariant with respect to the action $f \to -f$ on $\mathscr{B}$). Hence any accumulation point of $\{\Phi_n\}$ must be signature invariant. Since Φ_B is the unique signature invariant probability measure supported on $\mathscr{D} \doteq \{m1, -m1\}$, we conclude as in the proof of case (a) that $\Phi_n \Rightarrow \Phi_B$. This is (4.8). In case (c), by the uniqueness of Haar measure [21; Theorem C, § 60], Φ_C is the unique translation invariant probability measure supported on $\mathscr{D} \doteq \{\tilde{f}(\cdot + s), s \in \mathbb{T}\}$. (Translation invariant means invariant with respect to the action $f(\cdot) \to f(\cdot + s)$, $s \in \mathbb{T}$, on $\mathscr{B}$.) We will be able to conclude that $\Phi_n \Rightarrow \Phi_C$, which is (4.9), once we prove that any accumulation point Φ of $\{\Phi_n\}$ is translation invariant. We denote by $\mathscr{B}^w$ the set of weakly continuous functionals on $\mathscr{B}$. Assuming that $\Phi_{n'} \Rightarrow \Phi$ for some subsequence $\{n'\}$, we prove that Φ is translation invariant by showing that for any $\varphi \in \mathscr{B}^w$ and $s \in \mathbb{T}$

$$\int_{\mathscr{B}} \varphi(f(\cdot + s))\, d\Phi(f) = \int_{\mathscr{B}} \varphi(f)\, d\Phi(f). \tag{4.21}$$

The idea of the proof is to show that at the expense of a small error we can replace Φ by some Φ_n and s by the nearest real number of the form j/n for some $j \in \{1, \dots, n\}$, $n \in \{n'\}$. With these replacements (4.21) is easily shown to hold. The latter is closely relately to the $\mathbb{Z}_n$-rotational symmetry of the measures $\{P_{n,\beta}\}$ noted in the third paragraph after the statement of Theorem 1.1.

We consider the set $\mathscr{B}_0^w$ of weakly continuous functionals $\tilde{\varphi}$ on $\mathscr{B}$ of the form

$$\tilde{\varphi}(f) = h(\langle g_1, f\rangle, \dots, \langle g, f\rangle), \tag{4.22}$$

where $r \in \{1, 2, \dots\}$, $h: \mathbb{R}^r \to \mathbb{R}$ is continuous, and $g_1, \dots, g_r \in \mathscr{C}(\mathbb{T})$. As we noted in the proof of Lemma 3.2, $\mathscr{B}_0^w$ is dense in $\mathscr{B}^w$ with respect to the topology of uniform convergence. Hence it suffices to prove (4.21) with $\varphi \doteq \tilde{\varphi}$ of the form (4.22). For any $n \in \{n'\}$ and any $j \in \{1, \dots, n\}$, we have

$$\begin{aligned} &|\int \tilde{\varphi}(f(\cdot + s))\, d\Phi - \int \tilde{\varphi}(f)\, d\Phi| \\ &\quad \leqq |\int \tilde{\varphi}(f(\cdot + s))\, d\Phi - \int \tilde{\varphi}(f(\cdot + s))\, d\Phi_n| + \int |\tilde{\varphi}(f(\cdot + s)) - \tilde{\varphi}(f(\cdot + j/n))|\, d\Phi_n \\ &\qquad + |\int \tilde{\varphi}(f(\cdot + j/n))\, d\Phi_n - \int \tilde{\varphi}(f)\, d\Phi_n| + |\int \tilde{\varphi}(f)\, d\Phi_n - \int \tilde{\varphi}(f)\, d\Phi| \\ &\quad \equiv L_1(n) + L_2(j, n) + L_3(j, n) + L_4(n). \end{aligned} \tag{4.23}$$

Since $\Phi_{n'} \Rightarrow \Phi$, given $\varepsilon > 0$ we can pick $n_0 > 0$ such that $L_1(n) \leqq \varepsilon$, $L_4(n) \leqq \varepsilon$ whenever $n \in \{n'\}$, $n \geqq n_0$. Given $\delta > 0$, we may pick $n \in \{n'\}$, $n \geqq n_0$, $j \in \{1, \dots, n\}$ such that $|s - j/n| < \delta$. We show that there exists $\delta > 0$ such that $L_2(j, n) \leqq \varepsilon$. Since h in (4.22) is uniformly continuous on compact subsets of $\mathbb{R}^r$, it suffices to prove that for each $\eta > 0$ and $g \in \mathscr{C}(\mathbb{T})$ there exists $\delta > 0$ such that whenever $|s - j/n| < \delta$

$$\sup_{f \in \mathscr{B}} |\langle g, f(\cdot + s)\rangle - \langle g, f(\cdot + j/n)\rangle| < \eta. \tag{4.24}$$

The left-hand side of (4.24) equals

$$\sup_{f \in \mathscr{B}} |\langle g(\cdot - s) - g(\cdot - j/n), f\rangle| \leqq \sup_{t \in \mathbb{T}} |g(t - s) - g(t - j/n)| \cdot \sup_{f \in \mathscr{B}} \|f\|_\infty. \tag{4.25}$$

Hence by the uniform continuity of g on $\mathbb{T}$ and the uniform bound (3.22), there exists $\delta>0$ such that (4.24) holds if $|s-j/n|<\delta$. Thus, we have proved $L_2(j,n)\leqq\varepsilon$ for suitable j and n. We claim that for all $n\in\{1,2,\ldots\}$ and all $j\in\{1,\ldots,n\}$ $L_3(j,n)=0$. Once this is proved, we will be done. By (4.4), it suffices to prove that for all $f\in\mathscr{B}$

$$dR_n(f(\cdot-j/n))=dR_n(f) \tag{4.26}$$

and

$$F_n(f(\cdot-j/n))=F_n(f). \tag{4.27}$$

We have (4.26) since R_n is the distribution of the process ζ_n in (3.5) and the distributions of $\zeta_n(\cdot+j/n)$ and of ζ_n are equal. Since J is translation invariant, there exists $\tilde{J}$: $\mathbb{T}\to\mathbb{R}$ such that $J(s,t)=\tilde{J}(s-t)$ for all $s,t\in\mathbb{T}$. Hence for any $f\in\mathscr{B}$

$$\begin{aligned} n^2F_n(f(\cdot-j/n)) &= \sum_{\alpha_1,\alpha_2=1}^{n}\tilde{J}\left(\frac{\alpha_1-\alpha_2}{n}\right)(\pi^{(n)}f(\cdot-j/n))\left(\frac{\alpha_1}{n}\right)\cdot(\pi^{(n)}f(\cdot-j/n))\left(\frac{\alpha_2}{n}\right)\\ &= \sum_{\alpha_1,\alpha_2=1}^{n}\tilde{J}\left(\frac{\alpha_1-\alpha_2}{n}\right)(\pi^{(n)}f)\left(\frac{\alpha_1+j}{n}\right)\cdot(\pi^{(n)}f)\left(\frac{\alpha_2+j}{n}\right)\\ &= n^2F_n(f). \end{aligned}$$

This proves (4.27). □

V. Solving the Variational Problem $\sup[\beta F-I]$ on $\mathscr{B}$

In order to complete the proofs of Theorems 1.1, 1.2, 1.4, and 1.5 and their generalizations in Sect. II we must prove the statements about $\mathscr{D}_\beta$ in (1.37) and (1.40) and the generalizations of these statements in Sect. II. The key to doing this is proved by Theorem 5.1 below. We prove that the maximum of $\beta F-I$ on $\mathscr{B}$ is always achieved, so that $\mathscr{D}_\beta\neq\emptyset$; that any maximizing $\tilde{f}$ lies in the set

$$\mathscr{B}^0\doteq\{f\in\mathscr{B}\colon\ l<f<L \text{ a.e.}\}, \tag{5.1}$$

which is the interior of $\mathscr{B}$ with respect to the weak topology (the numbers l and L are defined in (3.20)); that such an $\tilde{f}$ equals a.e. a solution f_0 of the nonlinear integral equation

$$i'_\rho(f_0(t))=\beta(F'(f_0))(t), \qquad t\in\mathbb{T}, \tag{5.2}$$

and that f_0 is continuous on $\mathbb{T}$. In (5.2), F' is an operator mapping $\mathscr{H}\to\mathscr{H}$ which generates the Frechet derivative of F at f_0. F' is defined in (5.6) below.

The proof of (5.2) is subtle. If it were true (say) that I were Gateaux differentiable on $\mathscr{B}$, then (5.2) would follow from general theory [1; Theorem 6.1.1]. But this differentiability is not valid. Given H a Frechet differentiable function on $\mathscr{H}$, we define the operator H': $\mathscr{H}\to\mathscr{H}$ by the formula

$$DH(f)h=\langle H'(f),h\rangle, \qquad f,h\in\mathscr{H}, \tag{5.3}$$

where $DH(f)$ denotes the Frechet derivative of H at f.

Given $j\in\{2,3,\dots\}$ and numbers $s_1,\dots,s_{j-1}$, $t\in\mathbb{T}$, we define quantities $\mathbf{s}_k^{(j)}$, $k\in\{1,\dots,j\}$, as follows: for $k=j$, $\mathbf{s}_k^{(j)}=(s_1,\dots,s_{k-1},t)$ while for $k\in\{1,\dots,j-1\}$ $\mathbf{s}_k^{(j)}$ is obtained from $(s_1,\dots,s_{j-1},t)$ by interchanging the k'th coordinate, s_k, and the j'th coordinate, t (e.g., $\mathbf{s}_2^{(4)}\doteq(s_1,t,s_3,s_2)$).

Theorem 5.1. *We define the functional F by* (2.2)–(2.3), *where $d\in\{2,3,\dots\}$ and each function $J_j:\mathbb{T}^j\to\mathbb{R}$ is continuous, $j\in\{1,\dots,d\}$. For $s_1,\dots,s_{d-1}$, $t\in\mathbb{T}$, $j\in\{2,\dots,d\}$, we define*

$$\bar{J}_1(t)=J_1(t),\ \bar{J}_j(s_1,\dots,s_{j-1},t)=\sum_{k=1}^{j} J_j(\mathbf{s}_k^{(j)}). \tag{5.4}$$

We define the set $\mathcal{D}_\beta$ by (4.5). *We have the following facts.*

(i) *F is Frechet differentiable on $\mathcal{H}$ and*

$$DF(f)h=\langle F'(f),h\rangle,\qquad f,h\in\mathcal{H}, \tag{5.5}$$

where

$$(F'(f))(t)=J_1(t)+\sum_{j=2}^{d}\frac{1}{j!}\int_{\mathbb{T}^{j-1}}\bar{J}_j(s_1,\dots,s_{j-1},t)\,f(s_1)\dots f(s_{j-1})\,ds_1\dots ds_{j-1}. \tag{5.6}$$

(ii) *$\mathcal{D}_\beta\neq\emptyset$, $\mathcal{D}_\beta$ is weakly closed, and $\mathcal{D}_\beta\subseteq\mathcal{B}^0$, where $\mathcal{B}^0$ is defined in* (5.1).

(iii) *If $\tilde{f}\in\mathcal{D}_\beta$, then there exists a solution f_0 of* (5.2) *which is continuous and which equals $\tilde{f}$ a.e.*

We need a lemma.

Lemma 5.2. *We have*

$$\lim_{u\uparrow L} i'(u)=+\infty,\ \lim_{u\downarrow l} i'(u)=-\infty. \tag{5.7}$$

Proof. For $u\in(l,L)$, (3.65) implies that $i'(u)=(\gamma')^{-1}(u)$. The lemma follows by properties of γ' stated after (3.59). □

Proof of Theorem 5.1. To ease the notation, we set $\beta\doteq 1$ and write $\mathcal{D}$ instead of $\mathcal{D}_\beta$.

(i) This is a direct calculation.

(ii) The proof that $\mathcal{D}$ is non-empty and weakly closed is standard. By Theorem 2.1, $\alpha\doteq\sup_{\mathcal{B}}(F-I)<\infty$. We pick a sequence $\{f_j\}$ in $\mathcal{B}$ such that $(F-I)(f_j)\to\alpha$. By the weak compactness of $\mathcal{B}$ (Lemma 3.1), there exists a weakly convergent subsequence $\{f_{j'}\}$ with weak limit $f\in\mathcal{B}$. Since $F-I$ is weakly upper semicontinuous on $\mathcal{B}$ (Lemmas 3.3 and 3.6),

$$\alpha=\lim((F-I)(f_{j'}))\leqq(F-I)(f)\leqq\alpha. \tag{5.8}$$

Thus, $f\in\mathcal{D}$. A similar proof shows that $\mathcal{D}$ is weakly closed.

We now prove that $\mathcal{D}\subseteq\mathcal{B}^0$. Given $\tilde{f}\in\mathcal{D}$, we prove that $|\{t:\tilde{f}(t)=L\}|=0$; that $|\{t:\tilde{f}(t)=l\}|=0$ is proved similarly. We argue by contradiction. If $\mathcal{A}\doteq\{t:\tilde{f}(t)=L\}$ were not a null set, then by convexity of i and Lemma 5.2, we would have as $\varepsilon\downarrow 0$

$$\begin{aligned}\frac{1}{\varepsilon}(I(\tilde{f})-I(\tilde{f}-\varepsilon 1_{\mathscr{A}}))&=\frac{1}{\varepsilon}\int[i(\tilde{f})-i(\tilde{f}-\varepsilon 1_{\mathscr{A}})]\,dt\\&=\int_{\mathscr{A}}\frac{1}{\varepsilon}(i(L)-i(L-\varepsilon))\,dt\geqq|\mathscr{A}|\,i'(L-\varepsilon)\to\infty.\end{aligned}\tag{5.9}$$

By part (i) $\varepsilon^{-1}(F(\tilde{f})-F(\tilde{f}-\varepsilon 1_{\mathscr{A}}))$ stays bounded as $\varepsilon\to 0$. Hence we can find a small $\varepsilon_0>0$ such that $\tilde{f}-\varepsilon_0 1_{\mathscr{A}}\in\mathscr{B}$ and

$$\frac{1}{\varepsilon_0}[I(\tilde{f})-I(\tilde{f}-\varepsilon_0 1_{\mathscr{A}})]>\frac{1}{\varepsilon_0}[F(\tilde{f})-F(\tilde{f}-\varepsilon_0 1_{\mathscr{A}})].\tag{5.10}$$

The latter is the same as

$$F(\tilde{f}-\varepsilon_0 1_{\mathscr{A}})-I(\tilde{f}-\varepsilon_0 1_{\mathscr{A}})>F(\tilde{f})-I(\tilde{f}),\tag{5.11}$$

but (5.11) contradicts $\tilde{f}\in\mathscr{D}$.

(iii) By part (ii), we may find an a.e. strictly positive function $h\in\mathscr{H}$ such that for all $|\varepsilon|\leqq 1$, $l<\tilde{f}+\varepsilon h<L$ a.e. Let A be any measurable subset of $\mathbb{T}$. Given $N\in\{1,2,\ldots\}$, we define

$$A_N\doteq A\cap\{t:\ -N<i'(\tilde{f}(t)-h(t))\leqq i'(\tilde{f}(t)+h(t))<N\}.\tag{5.12}$$

Since $i'(u)<\infty$ for $u\in(l,L)$, we have $A_N\uparrow A$ as $N\to\infty$. For any $\varepsilon\in(0,1]$, we have by part (i) and Taylor's theorem with remainder for $n=0$ [1; Theorem 2.1.33],

$$\begin{aligned}0&\geqq F(\tilde{f}+\varepsilon h 1_{A_N})-F(\tilde{f})-I(\tilde{f}+\varepsilon h 1_{A_N})+I(\tilde{f})\\&=\varepsilon\langle F'(\tilde{f}),h1_{A_N}\rangle+o(\varepsilon)-\varepsilon\int i'(\tilde{f})\,h1_{A_N}\,dt\\&\quad-\varepsilon\int\left(\int[i'(\tilde{f}+s\varepsilon h1_{A_N})-i'(\tilde{f})]\,ds\right)h1_{A_N}\,dt.\end{aligned}\tag{5.13}$$

Since on A_N

$$i'(\tilde{f}+s\varepsilon h\cdot 1_{A_N})\leqq i'(\tilde{f}+h)<N,\qquad s\in[0,1],\ \varepsilon\in(0,1],\tag{5.14}$$

we have by dominated convergence that the second integral in (5.13) tends to zero as $\varepsilon\downarrow 0$. Hence (5.13) implies

$$\langle i'(\tilde{f}),h1_{A_N}\rangle\geqq\langle F'(\tilde{f}),h1_{A_N}\rangle.\tag{5.15}$$

Repeating the argument with $\varepsilon\in[-1,0)$, we conclude (5.15) but with the sense of the inequality reversed. Thus,

$$\langle i'(\tilde{f}),h1_{A_N}\rangle=\langle F'(\tilde{f}),h1_{A_N}\rangle.\tag{5.16}$$

Since $A_N\uparrow A$, A is an arbitrary measurable subset of $\mathbb{T}$, and $h>0$ a.e., (5.16) implies that

$$i'(\tilde{f})=F'(\tilde{f})\qquad\text{a.e.}\tag{5.17}$$

Let us assume that (5.17) fails on a null set N. We define

$$f_0(t) \doteq \begin{cases} \tilde{f}(t) & \text{for } t \in \mathbb{T} \setminus \mathsf{N}, \\ (i')^{-1}((F'(\tilde{f}))(t)) & \text{for } t \in \mathsf{N}. \end{cases} \tag{5.18}$$

Since $f_0 = \tilde{f}$ a.e. and F' is an integral operator with a smooth kernel, we have $F'(f_0) = F'(\tilde{f})$ on all of $\mathbb{T}$. Thus, $i'(f_0) = F'(f_0)$ on all of $\mathbb{T}$. The function f_0 is continuous since $f_0 = (i')^{-1}(F'(f_0))$ and each $\bar{J}_j$ in (5.4) is continuous. We remark that

$$-\infty < \operatorname{ess\,inf} i'(\tilde{f}) \leqq \operatorname{ess\,sup} i'(\tilde{f}) < \infty \tag{5.19}$$

and thus that $i'(\tilde{f}) \in \mathscr{H}$. □

VI. Proofs of Theorems 1.1, 1.2, 1.4, 1.5, 2.2–2.3

Under the hypotheses of these theorems, we find all relevant solutions of the equation (5.2). By our work in Sect. IV and V, we will then be able to complete the proofs of the theorems.

VI.1 Proof of Theorems 1.1, 1.4, *and* 2.2

We prove these theorems for a larger class of interactions than the class of continuous $J > 0$. Given $J \geqq 0$ on $\mathbb{T} \times \mathbb{T}$, we say that J is *irreducible* if for all $s, t \in \mathbb{T}$ there exists a sequence $u_0 = s, u_1, u_2, \ldots, u_n = t$ such that $J(u_i, u_{i+1}) > 0$ for each $i \in \{0, 1, \ldots, n-1\}$. Clearly J is irreducible if $J > 0$ on $\mathbb{T} \times \mathbb{T}$.

We prove Theorems 1.1, 1.4, and 2.2 assuming that $J: \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is continuous, non-negative, and irreducible and satisfies (1.6).

It suffices to prove Theorem 2.2, which generalizes Theorems 1.1 and 1.4 from $\rho \doteq (\delta_1 + \delta_{-1})/2$ to arbitrary $\rho \in \mathscr{G}_b$. The statements about the Curie-Weiss spontaneous magnetization $m^{CW}(\rho; \beta)$ are checked in Appendix B. Below we prove for $\rho \in \mathscr{G}_b$ that

$$\mathscr{D}_\beta = \begin{cases} \{0\} & \text{for } 0 < \beta \leqq \beta_c(\rho) \doteq 1/\int x^2\, d\rho(x), \\ \{m^{CW}(\rho; \beta)\, 1, -m^{CW}(\rho; \beta)\, 1\} & \text{for } \beta > \beta_c(\rho). \end{cases} \tag{6.1}$$

Then the ρ-versions of the limits (1.8), (1.9), and (1.38) follow from parts (i), (iii)(a), and (iii)(b) of Theorem 4.1.

We prove the first line of (6.1). By Theorem 5.1(ii)–(iii), it suffices to prove that if $0 < \beta \leqq \beta_c(\rho)$, then $f_0 \equiv 0$ is the unique continuous solution in $\mathscr{B}^0$ of the equation

$$i'_\rho(f_0(t)) = \frac{\beta}{2} \int \bar{J}(s, t)\, f_0(s)\, ds, \qquad t \in \mathbb{T}, \tag{6.2}$$

where $\bar{J}(s, t) \doteq J(s, t) + J(t, s)$. Let $f_0 \in \mathscr{C}(\mathbb{T}) \cap \mathscr{B}^0$ solve (6.2) and pick $t_0 \in \mathbb{T}$ such that $|f_0(t_0)| = \max\{|f_0(t)|: t \in \mathbb{T}\}$. Since $f_0 \in \mathscr{B}^0$, we have $|f_0(t_0)| < L$, where L is defined in (2.10). Since i'_ρ is odd and J satisfies (1.5), we have

$$i'_\rho(|f_0(t_0)|) = |i'_\rho(f_0(t_0))| = \beta\, |\int \bar{J}(s, t_0)\, f_0(s)\, ds| \leqq \beta\, |f_0(t_0)| \leqq \beta_c(\rho)\, |f_0(t_0)|. \tag{6.3}$$

By (B.14) (in Appendix B), we conclude that $|f_0(t_0)| = 0$ and so $f_0 \equiv 0$.

We now prove the second line of (6.1). For $\beta>\beta_c(\rho)$, the function $f_0\equiv 0$ continues to solve (6.2) but we claim that $0\notin\mathscr{D}_\beta$. In fact, if $f=u1$, some $u\in\mathbb{R}$, then

$$(\beta F-I)(f)=\frac{\beta u^2}{2}-i_\rho(u). \tag{6.4}$$

Since

$$(\beta u^2/2-i_\rho(u))'|_{u=0}=0,\ (\beta u^2/2-i_\rho(u))''|_{u=0}=\beta-1/\int x^2\,d\rho(x)=\beta-\beta_c(\rho)>0,$$

$\beta u^2/2-i_\rho(u)>(\beta F-I)(0)$ for all u sufficiently small, and thus $0\notin\mathscr{D}_\beta$. We are done once we prove that if $\tilde{f}\not\equiv 0$ belongs to $\mathscr{D}_\beta$ and is continuous, then

$$|\tilde{f}(t)|=m^{CW}(\rho;\beta) \qquad \text{for all } t\in\mathbb{T}. \tag{6.5}$$

We first prove that $|\tilde{f}|\in\mathscr{D}_\beta$. Since i_ρ is even and $J\geqq 0$, we have

$$\sup_{f\in\mathscr{B}}(\beta F-I)(f)=(\beta F-I)(\tilde{f})\leqq(\beta F-I)(|\tilde{f}|)\leqq\sup_{f\in\mathscr{B}}(\beta F-I)(f). \tag{6.6}$$

This implies that $|\tilde{f}|\in\mathscr{D}_\beta$ and thus by Theorem 5.1(iii)

$$i'_\rho(|\tilde{f}(t)|)=\beta\int\bar{J}(s,t)\,|\tilde{f}(s)|\,ds, \qquad t\in\mathbb{T}. \tag{6.7}$$

Since $|\tilde{f}|$ is continuous and by Theorem 5.1(ii) $|\tilde{f}|\in\mathscr{B}^0$, we can pick $t_0,t_1\in\mathbb{T}$ such that

$$L>|\tilde{f}(t_0)|\doteq\max_{t\in\mathbb{T}}|\tilde{f}(t)|\geqq\min_{t\in\mathbb{T}}|\tilde{f}(t)|\doteq|\tilde{f}(t_1)|\geqq 0. \tag{6.8}$$

By (6.7) and (1.6),

$$i'_\rho(|\tilde{f}(t_0)|)=\frac{\beta}{2}\int\bar{J}(s,t_0)\,|\tilde{f}(s)|\,ds\leqq\beta\,|\tilde{f}(t_0)|, \tag{6.9}$$

$$i'_\rho(|\tilde{f}(t_1)|)=\frac{\beta}{2}\int\bar{J}(s,t_1)\,|\tilde{f}(s)|\,ds\geqq\beta\,|\tilde{f}(t_1)|. \tag{6.10}$$

By (B.16) (in Appendix B), we have

$$0\leqq|\tilde{f}(t_0)|\leqq m^{CW}(\rho;\beta),\ |\tilde{f}(t_1)|=0 \quad\text{or}\quad m^{CW}(\rho;\beta)\leqq|\tilde{f}(t_1)|<L. \tag{6.11}$$

We consider two possibilities. If $|\tilde{f}(t_1)|>0$, then (6.11) implies $|\tilde{f}(t_0)|=|\tilde{f}(t_1)|=m^{CW}(\rho;\beta)$. By (6.8), this yields (6.5). If $|\tilde{f}(t_1)|=0$, then we would have equality in (6.10); hence $\int|\tilde{f}(s)|\,\bar{J}(s,t_1)\,ds=0$. We prove that this leads to a contradiction. Given $u,v\in\mathbb{T}$ and an integer $k\geqq 1$, we define

$$\bar{J}^{*k}\doteq\int\ldots\int ds_1\ldots ds_k\,\bar{J}(u,s_1)\prod_{i=1}^{k-1}\bar{J}(s_i,s_{i+1})\,\bar{J}(s_k,v),$$

where for $k=1$ the empty product is replaced by 1. Since $|\tilde{f}|\not\equiv 0$, we have $L>|\tilde{f}(t_0)|>0$. Since J and thus $\bar{J}$ are continuous and irreducible, there exists an integer $n \geqq 1$ such that $|\tilde{f}(t_0)| \cdot \bar{J}^{*n}(t_0, t_1) > 0$. We define $\alpha \doteq |\tilde{f}|(t_0)/i'_\rho(|\tilde{f}|(t_0)) > 0$. By the convexity[9] of i'_ρ on $[0, L)$ and (6.7),

$$0 = \int |\tilde{f}|(s_n)\, \bar{J}(s_n, t_1)\, ds_n \geqq \alpha \int i'_\rho(|\tilde{f}|(s_n))\, \bar{J}(s_n, t_1)\, ds_n$$
$$= \beta\alpha \int |\tilde{f}|(s_{n-1})\, \bar{J}^{*1}(s_{n-1}, t_1)\, ds_{n-1}$$
$$\geqq \ldots \geqq (\beta\alpha)^n \int |\tilde{f}|(s_0)\, \bar{J}^{*n}(s_0, t_1)\, ds_0 > 0.$$

This contradicts shows that $|\tilde{f}|(t_1) = 0$ cannot hold, and we are done. □

VI.2 Proof of Theorems 1.2, 1.5, *and* 2.3

It suffices to prove Theorem 2.3, which generalizes Theorems 1.2 and 1.5, from $\rho \doteq (\delta_1 + \delta_{-1})/2$ to arbitrary $\rho \in \mathscr{G}_b$. We are given $\rho \in \mathscr{G}_b$ and $J(s,t) \doteq \tilde{J}(s-t)$, where $\tilde{J}$ is defined by (2.11). We prove that

$$\mathscr{D}_\beta = \begin{cases} \{0\} & \text{for } 0 < \beta \leqq \beta_a(\rho) \doteq 2/|v|, \\ \{\tilde{f}(\cdot + s), s \in \mathbb{T}\} & \text{for } \beta > \beta_a(\rho), \end{cases} \tag{6.12}$$

where $\tilde{f}$ is given by (2.12) in terms of a quantity μ that satisfies $\mu = \langle \tilde{f}, \cos(2\pi p t) \rangle$ (this is (2.13)). Given (6.12), the ρ-versions of the limits (1.11), (1.12), and (1.41) follow from parts (i), (iii)(a), and (iii)(c) of Theorem 4.1.

By Theorem 5.1(iii) one can characterize $\tilde{f} \in \mathscr{D}_\beta$ up to a null set by studying solutions of (6.2). We first prove that $\tilde{f}$ satisfies (6.2) if and only if $\tilde{f}$ has the form

$$\tilde{f}(t) = \gamma'_\rho(\beta v \mu \cos(2\pi p(t+s))), \qquad t \in \mathbb{T}, \tag{6.13}$$

where s is some number in $\mathbb{T}$ and μ satisfies

$$\mu = \int \gamma'_\rho(\beta v \mu \cos(2\pi p t)) \cos(2\pi p t)\, dt. \tag{6.14}$$

We then study solutions of (6.14) for different values of β. Given $f \in \mathscr{H}$ and $k \in \{0, 1, \ldots\}$, we define the Fourier coefficients

$$f^\wedge(k) \doteq \int \exp(2\pi i k t)\, f(t)\, dt,\; f^\#(k) \doteq \mathrm{Re}[f^\wedge(k)] = \langle \cos 2\pi k t, f \rangle.$$

For J in (2.11), equation (6.2) for $\tilde{f}$ becomes

$$i'_\rho(\tilde{f}(t)) = -\beta b \tilde{f}^\wedge(0) + \beta v\, \mathrm{Re}[\tilde{f}^\wedge(p) \exp(-2\pi i p t)]. \tag{6.15}$$

We show that $\tilde{f}$ of the form (6.13) with $s \in \mathbb{T}$ and μ satisfying (6.14) is a solution of (6.15). We first consider $s = 0$. Since γ'_ρ is odd, we have

$$\tilde{f}(t + 1/(2p)) = -\tilde{f}(t), \qquad \text{all } t \in \mathbb{T},$$

so that $\tilde{f}^\wedge(0) = 0$. If μ satisfies (6.14), then since $\tilde{f}$ is symmetric about 0, $\mu = \tilde{f}^\wedge(p) = \tilde{f}^\#(p)$. Since $i'_\rho = (\gamma'_\rho)^{-1}$, (6.13) with $s = 0$ is exactly (6.15). We now consider (6.13) with general $s \in \mathbb{T}$. It suffices to prove that if $\tilde{f}$ is any solution of

[9] $i'_\rho(x)/x \leqq i'_\rho(y)/y$ for $0 < x \leqq y < L$.

(6.15), then $h \doteq \tilde{f}(\cdot+s)$ also solves (6.15). But $h^\wedge(0)=\tilde{f}^\wedge(0)$ and

$$h^\wedge(p)=\tilde{f}^\wedge(p)\exp(-2\pi i p s).$$

Hence h satisfies (6.15) if and only if for all $t\in\mathbb{T}$

$$i'_p(\tilde{f}(t+s))=-\beta\tilde{f}^\wedge(0)+\beta v\,\mathrm{Re}[\tilde{f}^\wedge(p)\exp(-2\pi i p(t+s))]. \tag{6.16}$$

But (6.16) follows from (6.15) by substituting $(t+s)$ for t in the latter equation.

We now prove the converse, namely that any solution of (6.15) has the form (6.13). We need a lemma.

Lemma 6.1. *Let $\tilde{f}$ be any solution of* (6.15). *Then there exists $s\in\mathbb{T}$ such that*

$$\tilde{f}(s+t)=\tilde{f}(s-t) \quad \textit{for all } t\in\mathbb{T}. \tag{6.17}$$

Proof. We consider two cases, $\tilde{f}^\wedge(p)=0$ and $\tilde{f}^\wedge(p)\neq 0$. If $\tilde{f}^\wedge(p)=0$, then all solutions of (6.15) are of the form $\tilde{f}=u1$, u constant, where u must satisfy $i'_p(u)=-\beta u$. The only solution of this equation is $u=0$, so that $\tilde{f}\equiv 0$ and (6.17) is valid for any $s\in\mathbb{T}$. We now consider the case $\tilde{f}^\wedge(p)\neq 0$. Equation (6.17) is equivalent to

$$\tilde{f}(2s-t)=\tilde{f}(t) \quad \text{for all } t\in\mathbb{T}. \tag{6.18}$$

We prove that (6.18) holds with

$$s\doteq\frac{1}{2\pi p}\arg(\tilde{f}^\wedge(p)). \tag{6.19}$$

By (6.15), it suffices to check that for all $t\in\mathbb{T}$

$$\mathrm{Re}[x\exp(-2\pi i p(2s-t))]=\mathrm{Re}[x\exp(-2\pi i p t)], \quad \text{where } x\doteq f^\wedge(p)\neq 0. \tag{6.20}$$

Writing $x=\exp(\log|x|+i\arg x)$, we check (6.20). □

We now prove that if $\tilde{f}$ satisfies (6.15), then $\tilde{f}$ has the form (6.13). Again we consider two cases. If $\tilde{f}^\wedge(p)=0$, then as in the proof of Lemma 6.1, $\tilde{f}\equiv 0$. This has the form (6.13) with $\mu=0$ ($\mu=0$ satisfies (6.14)). We now consider the case $\tilde{f}^\wedge(p)\neq 0$. We pick $s\in\mathbb{T}$ so that (6.17) holds and define $h(t)\doteq\tilde{f}(t+s)$, $t\in\mathbb{T}$. It suffices to prove that h has the form

$$h(t)=\gamma'_p(\beta v\mu\cos(2\pi p t)), \quad t\in\mathbb{T} \tag{6.21}$$

for some μ satisfying (6.14). By the calculation leading up to (6.16), h solves (6.15), and since $\tilde{f}$ satisfies (6.17), h is symmetric about 0. Thus $h^\wedge(p)=h^\#(p)$, which is non-zero since $\tilde{f}^\wedge(p)\neq 0$, and

$$h(t)=\gamma'_p(-\beta b h^\#(0)+\beta v h^\#(p)\cos(2\pi p t)), \quad t\in\mathbb{T}. \tag{6.22}$$

We claim that once we have proved that $h^\#(0)=0$, we will be done. In fact, setting $\mu\doteq h^\#(p)$, we see that (6.22) reduces to (6.21) and the equation $\mu=h^\#(p)$ is exactly (6.14). We now prove that $h^\#(0)=0$. If $b=0$, then from (6.22)

$h(t+1/(2p))=-h(t)$, so that $h^{\#}(0)=0$. If $b>0$, then we assume $h^{\#}(0)\neq 0$ and obtain a contradiction. If $h^{\#}(0)>0$, then since γ_ρ' is strictly increasing and odd and $h^{\#}(p)\neq 0$, we have from (6.22)

$$h(t)<\gamma_\rho'(\beta b h^{\#}(0)+\beta v h^{\#}(p)\cos(2\pi p t))=-h\left(t+\frac{1}{2p}\right) \tag{6.23}$$

for all $t\in\mathbb{T}$. This leads to the contradiction

$$h^{\#}(0)=\frac{1}{2}\int\left(h(t)+h\left(t+\frac{1}{2p}\right)\right)dt<0. \tag{6.24}$$

Similarly, assuming $h^{\#}(0)<0$ leads to $h^{\#}(0)>0$. Thus, $h^{\#}(0)=0$. This completes the proof that any solution of (6.15) has the form (6.13).

We now study solutions $\mu=\mu(\beta)$ of (6.14). We assume that $v>0$; $v<0$ can be handled similarly. We need only consider $\mu(\beta)\geqq 0$. If μ solves (6.14), then $-\mu$ solves (6.14), and $h_{-\mu}=-h_\mu$, where h_μ is defined by (6.21). Since $h(t+1/(2p))=-h(t)$, h and $-h$ generate the same orbit $\{h(\cdot+s), s\in\mathbb{T}\}$ in $\mathscr{B}$. Hence, if we knew that for $\beta>\beta_a(\rho)$, (6.14) has a unique solution $\mu(\beta)>0$ (this is proved below), then whether we use $\mu(\beta)$ or $-\mu(\beta)$, the second line of (6.12) remains unchanged.

We write $H(\mu)=H_{\beta,v,p,\rho}(\mu)$ for the right-hand side of (6.14) and note the following properties:

(a) $H(0)=0$, $H(\mu)>0$ for $\mu>0$;

(b) H is strictly increasing on $\mathbb{R}$; (6.25)

(c) H is strictly concave on $[0,\infty)$;

(d) for each $\mu>0$ $\lim_{\beta\to\infty} H_{\beta,v,p,\rho}(\mu)=2L/\pi=\lim_{\mu\to\infty} H(\mu)$,

where $L\doteq\sup\{x\colon x \text{ in support of } \rho\}$. Since γ_ρ' is odd and $\gamma_\rho'(t)>0$ for $t>0$, (a) follows from the equation

$$H(\mu)=\int\gamma_\rho'(\beta v\mu|\cos 2\pi p t|)|\cos 2\pi p t|\,dt. \tag{6.26}$$

Property (b) follows from the strict convexity of γ_ρ and (c) from the oddness of γ_ρ''' (as in (6.26)) and the strict concavity of γ_ρ' on $(0,\infty)$. The latter holds since $\rho\in\mathscr{G}_b$ (see (2.6) and the discussion that follows (2.6)). Property (d) is a consequence of the limits

$$\lim_{t\to\infty}\gamma_\rho'(t)=L,\quad \lim_{t\to-\infty}\gamma_\rho'(t)=-L. \tag{6.27}$$

Properties (a)–(c) imply that (6.14) has the unique solution $\mu=\mu(\beta)=0$ whenever β satisfies

$$\left.\frac{dH_{\beta,v,p,\rho}(\mu)}{d\mu}\right|_{\mu=0}\leqq\frac{d\mu}{d\mu}=1; \tag{6.28}$$

i.e., whenever

$$0<\beta\leqq\frac{1}{v\gamma_\rho''(0)/2}=\frac{2}{v\int x^2\,d\rho(x)}\doteq\beta_a(\rho). \tag{6.29}$$

For $\beta>\beta_a(\rho)$, (6.14) has the solution $\mu=0$ as well as a unique positive root $\mu=\mu(\beta)$. By (6.25) (b) and (d), we see that

$$\mu(\beta)\text{ is monotonically increasing and }\lim_{\beta\to\infty}\mu(\beta)=2L/\pi. \tag{6.30}$$

See Fig. 1

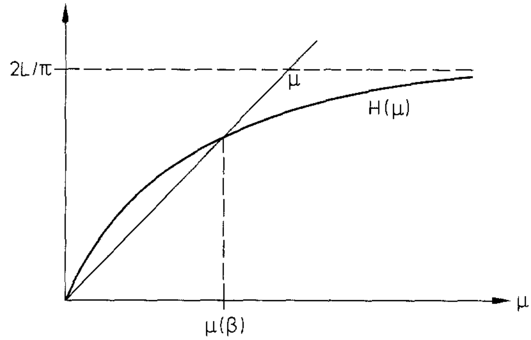

Fig. 1. Solving (6.14) for $\beta>\beta_a(\rho)$

When substituted into (6.21), $\mu=0$ gives rise to $h\equiv 0$ while $\mu(\beta)$ gives rise to $h\not\equiv 0$. Hence the second line of (6.12) will follow once we prove that $0\notin\mathscr{D}_\beta$ for $\beta>\beta_a(\rho)$. We prove this by showing that for all sufficiently small $u\in\mathbb{R}\setminus\{0\}$

$$(\beta F-I)(u\cos 2\pi pt)>(\beta F-I)(0). \tag{6.31}$$

Since $d\{(\beta F-I)(u\cos 2\pi pt)\}/du=0$ at $u=0$, it suffices to prove that

$$\frac{du^2}{du^2}\{(\beta F-I)(u\cos 2\pi pt)\}|_{u=0}>0. \tag{6.32}$$

A short calculation shows that the left-hand side of (6.32) equals

$$\frac{\beta v}{4}-\frac{1}{2\int x^2\,d\rho(x)}=\frac{v}{4}(\beta-\beta_a(\rho)). \tag{6.33}$$

Since $\beta>\beta_a(\rho)$ and we have assumed that $v>0$, (6.32) holds. The properties of $\tilde{f}_{\beta,v,p,\rho}$ listed in Theorems 1.2 and 2.3 follow from (6.21). The limit (1.14) and its generalization to $\rho\in\mathscr{G}_b$ (see the Remark after the statement of Theorem 2.3) follow from (6.25), (6.27), and (6.30). □

Appendix A. Laplace's Method for Doubly Indexed Function Space Integrals

The proof of the following theorem is based upon unpublished notes of S.R.S. Varadhan.

Theorem A.1. *Let Ω be a topological space and Ψ a σ-algebra of subsets of Ω which includes all open and closed subsets of Ω. Let $\{P_{m,n};\ m=1,2,\ldots;\ n=1,2,\ldots\}$ be probability measures on (Ω,Ψ). Let $\{a_n;\ n=1,2,\ldots\}$ be a sequence of positive numbers such that $a_n\to\infty$ and for all closed subsets $\mathscr{K}$ in Ω and open subsets $\mathscr{G}$ in Ω*

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{a_n}\log P_{m,n}(\mathscr{K})\leqq -I(\mathscr{K}), \tag{A.1}$$

$$\liminf_{m\to\infty}\liminf_{n\to\infty}\frac{1}{a_n}\log P_{m,n}(\mathscr{G})\geqq -I(\mathscr{G}). \tag{A.2}$$

In (A.1)-(A.2), I is some extended real-valued functional on Ω, and for $\mathscr{A}\subseteq\Omega$, $I(\mathscr{A})=\inf\{I(f): f\in\mathscr{A}\}$. Let F be a continuous functional on Ω such that

$$\lim_{L\to\infty}\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{a_n}\log\int_{\{f:\,F(f)\geqq L\}}\exp[a_n F(f)]\,dP_{m,n}(f)=-\infty. \tag{A.3}$$

Then $\sup_{\Omega}[F-I]<\infty$ and

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{a_n}\log\int_{\Omega}\exp[a_n F(f)]\,dP_{m,n}(f)=\sup_{f\in\Omega}[F(f)-I(f)], \tag{A.4}$$

and for any closed subset $\mathscr{K}$ in Ω

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{a_n}\log\int_{\mathscr{K}}\exp a_n F(f)\,dP_{m,n}(f)\leqq\sup_{f\in\mathscr{K}}[F(f)-I(f)]. \tag{A.5}$$

Remarks. 1. The proof of the bound (A.5) requires only (A.1), not (A.2).

2. Setting $\mathscr{K}=\mathscr{G}=\Omega$ in (A.1) and (A.2), we see that $\inf_{\Omega} I=0$. Thus, $I:\Omega\to[0,\infty]$. If F is bounded above, then (A.3) is trivially satisfied and $\sup_{\Omega}[F-I]\leqq\sup_{\Omega}[F]<\infty$.

3. [29; §3] considers the case of sequences of probability measures $\{P_n;\ n=1,2,\ldots\}$ and of functionals $\{F_n;\ n=1,2,\ldots\}$. If the analogues of (A.1)-(A.2) hold with $\{P_{m,n}\}$ replaced by $\{P_n\}$ and F by F_n, if F_n tends to F suitably as $n\to\infty$, if I is lower semi-continuous on Ω, and if $\{f\in\Omega:\ I(f)\leqq M\}$ is a compact of Ω for every finite M, then

$$\lim_{n\to\infty}\frac{1}{a_n}\int_{\Omega}\exp[a_n F_n(f)]\,dP_n(f)=\sup_{f\in\Omega}[F(f)-I(f)].$$

For the case where F is continuous and $F_n=F$ for all n, Theorem A.1 (applied to $P_{m,n}\doteq P_n$ for all m,n) is stronger since it does not require that I be lower semicontinuous or that $\{f\in\Omega:\ I(f)\leqq M\}$ be compact. The real use of [29; §3] is its ability to handle sequences of functionals $\{F_n\}$ under a fairly weak hypothesis on their convergence to F.

Proof of Theorem A.1. For $\mathscr{A}\in\Psi$ we define

$$U_{m,n}(\mathscr{A})\doteq\int_{\mathscr{A}}\exp[a_n F(f)]\,dP_{m,n}(f),$$
$$U_{m,n}\doteq U_{m,n}(\Omega)\doteq\int_{\Omega}\exp[a_n F(f)]\,dP_{m,n}(f). \tag{A.6}$$

The first step is to prove $\sup_{\Omega}[F-I]<\infty$. By (A.3) there exists a real number C such that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{a_n}\log U_{m,n}\leqq C. \tag{A.7}$$

Given any $g\in\Omega$ and $\varepsilon>0$, we define the open set

$$\mathscr{G}\doteq\{f\in\Omega\colon F(f)>F(g)-\varepsilon\}.$$

By (A.2) and (A.7)

$$C\geqq\liminf_{m\to\infty}\liminf_{n\to\infty}\frac{1}{a_n}\log U_{m,n}\geqq\liminf_{m\to\infty}\liminf_{n\to\infty}\frac{1}{a_n}\log U_{m,n}(\mathscr{G})$$
$$\geqq F(g)-\varepsilon-I(\mathscr{G})\geqq F(g)-I(g)-\varepsilon. \tag{A.8}$$

Since g and $\varepsilon>0$ are arbitrary, we conclude from (A.8)

$$\sup_{\Omega}[F-I]\leqq C<+\infty$$

and

$$\liminf_{m\to\infty}\liminf_{n\to\infty}\frac{1}{a_n}\log U_{m,n}\geqq\sup_{\Omega}[F-I]. \tag{A.9}$$

The second step is to prove

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{a_n}\log U_{m,n}\leqq\sup_{\Omega}[F-I], \tag{A.10}$$

which with (A.9) gives (A.4). We prove (A.10) first under the assumption that $S\doteq\sup_{\Omega}F<\infty$. (For such an F, condition (A.3) is trivially satisfied.) We define $D\doteq\max\{-\sup_{\Omega}[F-I],S\}$, $\Gamma_1\doteq\{f\in\Omega\colon -D\leqq F(f)\leqq D\}$, and $\Gamma_2\doteq\{f\in\Omega\colon F(f)<-D\}$. Since $D\geqq S$, we have $\Gamma_1\cup\Gamma_2=\Omega$. Let us suppose that we can prove that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{a_n}\log U_{m,n}(\Gamma_1)\leqq\sup_{f\in\Gamma_1}[F(f)-I(f)]. \tag{A.11}$$

Then

$$\limsup_{m,n}\frac{1}{a_n}\log U_{m,n}=\max\left(\limsup_{m,n}\frac{1}{a_n}\log U_{m,n}(\Gamma_1),\ \limsup_{m,n}\frac{1}{a_n}\log U_{m,n}(\Gamma_2)\right)$$
$$\leqq\max(\sup_{\Gamma_1}[F-I],-D)$$
$$\leqq\sup_{\Omega}[F-I],$$

which is (A.10). Hence it suffices to prove (A.11). This is equivalent to proving (A.10) under the assumption that $M \doteq \sup_{\Omega} |F| < \infty$.

We pick $N \in \{1, 2, \ldots\}$ and for $j \in \{-N, -N+1, \ldots, N-1\}$ define the closed subsets

$$\mathscr{K}_{N,j} \doteq \left\{ f \in \Omega : \frac{jM}{N} \leqq F(f) \leqq \frac{(j+1)M}{N} \right\}. \tag{A.12}$$

We have $\bigcup_{j=-N}^{N-1} \mathscr{K}_{N,j} = \Omega$. By property (A.1), for each $j \in \{-N, \ldots, N-1\}$

$$\limsup_{m,n \to \infty} \frac{1}{a_n} \log P_{m,n}(\mathscr{K}_{N,j}) \leqq -I(\mathscr{K}_{N,j}). \tag{A.13}$$

We have

$$U_{n,m} \leqq \sum_{j=-N}^{N-1} \exp\left[a_n \frac{(j+1)M}{N}\right] P_{m,n}(\mathscr{K}_{N,j})$$

and so by (A.1),

$$\begin{aligned}
\limsup_{m,n \to \infty} \frac{1}{a_n} \log U_{m,n} &\leqq \max_{j \in \{-N, \ldots, N-1\}} \left[\frac{(j+1)M}{N} - I(\mathscr{K}_{N,j})\right] \\
&\leqq \max_{j \in \{-N, \ldots, N-1\}} \sup_{f \in \mathscr{K}_{N,j}} [F(f) - I(f)] + \frac{M}{N} \\
&= \sup_{f \in \Omega} [F(f) - I(f)] + \frac{M}{N}.
\end{aligned}$$

Taking $N \to \infty$, conclude (A.10).

We now prove (A.10) for a continuous F unbounded above but satisfying condition (A.3). We pick L sufficiently large so that

$$\limsup_{m,n} \frac{1}{a_n} \log U_{m,n}(\{F \geqq L\}) \leqq \sup_{\Omega} [F - I]. \tag{A.14}$$

This is possible by condition (A.3). We define $\bar{F} \doteq \min(F, L)$ and $\bar{U}_{m,n}$ as in (A.6) with $\bar{F}$ written for F. Since $\bar{F}$ is continuous and $\bar{F} \leqq L$, we have by the earlier case, by (A.14), and by the inequality $\bar{F} \leqq F$

$$\begin{aligned}
\limsup_{m,n} \frac{1}{a_n} \log U_{m,n} &= \max\left(\limsup_{m,n} \frac{1}{a_n} \log \bar{U}_{m,n}, \limsup_{m,n} \frac{1}{a_n} \log U_{m,n}(\{F \geqq L\})\right) \\
&\leqq \max(\sup_{\Omega} [\bar{F} - I], \sup_{\Omega} [F - I]) = \sup_{\Omega} [F - I].
\end{aligned}$$

This is (A.10).

To prove (A.5), we repeat the proof of (A.10) with the sets $\{\mathscr{K}_{N,j}\}$ in (A.12) replaced by $\{\mathscr{K}_{N,j} \cap \mathscr{K}\}$. □

Appendix B. Verification of Statements about the Curie-Weiss Model

Let ρ be a Borel probability measure on $\mathbb{R}$ which is not a one-point mass and which satisfies

$$\int \exp(kx^2)\, d\rho(x) < \infty \qquad \text{for all } k > 0.$$

We denote the set of such ρ by $\mathscr{M}$. Later on, we shall impose additional conditions on ρ. The hypothesis in the main body of this paper that ρ have bounded support is needed only to treat the general case of the circle model. For the Curie-Weiss case, unbounded $\rho \in \mathscr{M}$ can be treated without much additional work.

For $\rho \in \mathscr{M}$, we define the Curie-Weiss model by (1.4). Since ρ satisfies (B.1), $Z_{n,\beta}^{CW}$ is finite. Our first result evaluates the specific free energy $\psi^{CW}(\rho;\beta)$ associated with (1.4).

Theorem B.1. *For* $\rho \in \mathscr{M}$

$$\begin{aligned} -\beta\psi^{CW}(\rho;\beta) &\doteq \lim_{n\to\infty} \frac{1}{n} \log Z_{n,\beta}^{CW} \\ &\doteq \sup_{u\in\mathbb{R}} [\beta u^2/2 - i_\rho(u)] = \sup_{t\in\mathbb{R}} [\gamma_\rho(t) - t^2/(2\beta)], \end{aligned} \tag{B.2}$$

where i_ρ *is defined in* (1.19) *and* γ_ρ *is defined in* (1.20).

Proof. Let $\{Y_j; j=1,2,\ldots\}$ be a sequence of i.i.d. random variables each distributed by ρ and set $S_n \doteq \sum_{j=1}^{n} Y_j$. Let $g(x) \doteq x^2/2$. Then

$$Z_{n,\beta}^{CW} = \int_{\mathbb{R}} \exp[n\beta g(x)]\, \mathrm{Prob}\left\{\frac{S_n}{n} \in dx\right\}.$$

We apply Theorem A.1 in Appendix A with $\Omega \doteq \mathbb{R}$, Ψ the Borel subsets of $\mathbb{R}$, $F \doteq g$, $P_{m,n} \doteq \mathrm{Prob}\{S_n/n \in \cdot\}$ for each $m, n \in \{1,2,\ldots\}$. By Lemma 3.8, the bounds (A.1) and (A.2) are valid with $I \doteq i_\rho$. Hence we will have proved the first equality in (B.7) once we prove (A.3). We have by the convexity of g and Čebyšev's inequality

$$\begin{aligned} &\int_{\{x:\, \beta g(x) \geq L\}} \exp[n\beta g(x)]\, \mathrm{Prob}\left\{\frac{S_n}{n} \in dx\right\} \\ &\quad \leq \sum_{\alpha=1}^{\infty} \exp[n(\alpha+1)L]\, \mathrm{Prob}\left\{\beta g\left(\frac{S_n}{n}\right) \geq \alpha L\right\} \\ &\quad \leq \sum_{\alpha=1}^{\infty} \exp[n(\alpha+1)L]\, \mathrm{Prob}\left\{\frac{\beta}{n} \sum_{j=1}^{n} g(Y_j) \geq \alpha L\right\} \\ &\quad \leq \sum_{\alpha=1}^{\infty} \exp[n(\alpha+1)L] \exp[-3n\alpha L]\, K(\beta) \\ &\quad = \sum_{\alpha=1}^{\infty} \exp[-n(2\alpha-1)L]\, K(\beta), \end{aligned}$$

where $K(\beta) \doteq \int \exp[3\beta g]\, d\rho < \infty$ by (B.1). We conclude that

$$\lim_{L\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \int_{\{x:\, \beta g(x) \geqq L\}} \exp[n\beta g(x)] \operatorname{Prob}\left\{\frac{S_n}{n} \in dx\right\} = -\infty.$$

This is (A.3).

In order to prove the second equality in (B.2), we apply Theorem C.1 in Appendix C to the closed convex functions g and i_ρ. □

As a consistency check, we show that for the Curie-Weiss model, (B.2) in Theorem B.1 agrees with (2.4) in Theorem 2.1. This verifies a claim made after the statement of Theorem 1.3, of which Theorem 2.1 is a generalization. Theorem 2.1 calculates the specific free energy $\psi(\beta)$ corresponding to the joint distributions $\{P_{n,\beta}\}$ in (2.1). For the Curie-Weiss model, F in Theorem 2.1 is given by $F(f) \doteq \frac{1}{2}(\int f(t)\, dt)^2$. In this case, Theorem 2.1 states that

$$-\beta\psi(\beta) = \sup_{f\in\mathscr{H}} \left[\frac{\beta}{2}(\int f(t)\, dt)^2 - I(f)\right]. \tag{B.3}$$

For Theorem 2.1, we need $\rho \in \mathscr{M}_b$.

Theorem B.2. *For* $\rho \in \mathscr{M}_b$ *the supremum in* (B.3) *equals the first supremum in* (B.2).

Proof. Let $g(x) \doteq x^2/2$. If f equals a constant u a.e., then

$$g(\int f) - I(f) = g(u) - i_\rho(u),$$

so that $-\beta\psi^{CW}(\rho;\beta) \leqq -\beta\psi(\beta)$. On the other hand, for any $f \in \mathscr{H}$, $t \in \mathbb{T}$, by (B.2)

$$\beta g(f(t)) - i_\rho(f(t)) \leqq -\beta\psi^{CW}(\rho;\beta) \tag{B.4}$$

and so

$$\int_{\mathbb{T}} \beta g\circ f - I(f) \leqq -\beta\psi^{CW}(\rho;\beta). \tag{B.5}$$

By Jensen's inequality $g(\int f) \leqq \int g\circ f$, so from (B.5) we conclude $-\beta\psi(\beta) \leqq -\beta\psi^{CW}(\rho;\beta)$. □

Finally, we verify the statements in Theorems 1.1, 1.4, and 2.2 concerning the Curie-Weiss model. It suffices to consider Theorem 2.2, which generalizes Theorems 1.1 and 1.4 from the measure $\rho \doteq (\delta_1 + \delta_{-1})/2$ to ρ satisfying the one-site GHS inequality. Again, we do not restrict ourselves to ρ with bounded support. In other words, we now assume that $\rho \in \mathscr{G}$, where the latter class is defined by (2.5) and (2.6). (B.2) states that

$$-\beta\psi^{CW}(\rho;\beta)=\sup_{u\in\mathbb{R}}\left\{\frac{\beta u^2}{2}-i_\rho(u)\right\}. \tag{B.6}$$

We define

$$\mathscr{D}_\beta^{CW}\doteq\left\{m\in\mathbb{R}:\frac{\beta m^2}{2}-i_\rho(m)=\sup_{u\in\mathbb{R}}\left[\frac{\beta u^2}{2}-i_\rho(u)\right]\right\}. \tag{B.7}$$

We prove that

$$\mathscr{D}_\beta^{CW}=\begin{cases}\{0\} & \text{for } 0<\beta<\beta_c(\rho)\doteq 1/\int x^2\,d\rho(x),\\ \{\pm m^{CW}(\rho;\beta)\} & \text{for } \beta>\beta_c(\rho),\end{cases} \tag{B.8}$$

where for $\beta>\beta_c(\rho)$, $m^{CW}(\rho;\beta)$ is the unique positive root of the equation

$$\beta m=i'_\rho(m). \tag{B.9}$$

Defining $L=\sup\{x: x \text{ in the support of } \rho\}\in(0,\infty]$, we also check that $m^{CW}(\rho;\beta)\uparrow L$ as $\beta\to\infty$.

Since $\rho\in\mathscr{G}$, by (2.6) and the discussion that follows (2.6), the function γ'_ρ is strictly concave on $[0,\infty)$. By the strict convexity of i_ρ on $[0,L)$, i'_ρ is strictly increasing on $[0,L)$, and since it is the inverse of γ'_ρ, i'_ρ is strictly convex on $[0,L)$.

We prove that the supremum in (B.6) is achieved for $u\in(-L,L)$. If $L<\infty$, then this follows from Theorems 5.1 (ii) and B.2. If $L=\infty$, then it suffices to prove

$$\frac{\beta u^2}{2}-i_\rho(u)\to-\infty \qquad \text{as } |u|\to\infty. \tag{B.10}$$

By (B.1), for any $\varepsilon>0$

$$\gamma_\rho(t)\doteq\log\int\exp(tx)\,d\rho(x)\leqq\frac{t^2}{2\varepsilon}+K_1(\varepsilon), \tag{B.11}$$

where $K_1(\varepsilon)\doteq\log\int\exp(\varepsilon x^2/2)\,d\rho(x)<\infty$. Thus for any real u,

$$i_\rho(u)\doteq\sup_{t\in\mathbb{R}}[ut-\gamma_\rho(t)]\geqq\sup_{t\in\mathbb{R}}\left[ut-\frac{t^2}{2\varepsilon}-K_1(\varepsilon)\right]=\frac{\varepsilon u^2}{2}-K_1(\varepsilon). \tag{B.12}$$

Taking $\varepsilon>\beta$, we obtain (B.10).

The supremum in (B.6) is achieved at points u which satisfy

$$\beta u=i'_\rho(u). \tag{B.13}$$

For all $\beta>0$, $u\doteq 0$ satisfies this equation. We first consider $0<\beta\leqq\beta_c(\rho)$. Since i'_ρ is strictly convex on $(0,L)$, for $u\in(0,L)$

$$i''_\rho(u)>i''_\rho(0)=1/\int x^2\,d\rho(x)\doteq\beta_c(\rho),$$

and so

$$i'_\rho(u)>\beta_c(\rho)\,u \qquad \text{for } u\in(0,L). \tag{B.14}$$

Thus, if $0<\beta\leqq\beta_c(\rho)$, $u\doteq 0$ is the only solution of (B.14). This gives the first line of (B.8). If $\beta>\beta_c(\rho)$, then by Lemma 5.2 (also valid if $L=\infty$) and the strict convexity of i'_ρ on $[0,L)$, there exists a unique strictly positive solution $m^{CW}(\rho;\beta)$ of (B.13) with $m^{CW}(\rho;\beta)\in(0,L)$. See Fig. 2.

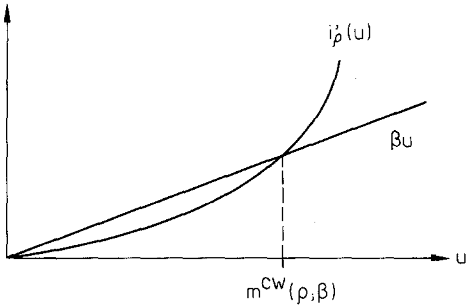

Fig. 2. Solving (B.13) for $\beta>\beta_c(\rho)$

By symmetry, $-m^{CW}(\rho;\beta)$ is also a solution. Since for $\beta>\beta_c(\rho)$

$$\frac{d}{du}\left(\frac{\beta u^2}{2}-i_\rho(u)\right)\Bigg|_{u=0}=0,\ \frac{d^2}{du^2}\left(\frac{\beta u^2}{2}-i_\rho(u)\right)\Bigg|_{u=0}=\beta-\beta_c(\rho)>0, \tag{B.15}$$

the supremum in (B.6) cannot be achieved at $u\doteq 0$. Thus, we obtain the second line of (B.8). From Fig. 2 and the fact that $i'_\rho(u)\uparrow\infty$ as $u\uparrow L$, it is clear that $m^{CW}(\rho;\beta)$ is strictly increasing in β for $\beta>\beta_c(\rho)$ and that $m^{CW}(\rho;\beta)\uparrow L$ as $\beta\uparrow\infty$.

Figure 2 also shows that for $\beta>\beta_c(\rho)$

$$\beta u-i'_\rho(u)\begin{cases}\geqq 0 & \text{for } 0\leqq u\leqq m^{CW}(\rho;\beta),\\ \leqq 0 & \text{for } m^{CW}(\rho;\beta)\leqq u<L \quad\text{and}\quad u=0.\end{cases} \tag{B.16}$$

This is used in Sect. VI.1 in the proof of Theorem 2.2.

We end this section by explaining why $m^{CW}(\rho;\beta)$ is called the Curie-Weiss spontaneous magnetization. Let $h\geqq 0$ be a given real number (the external magnetic field). We define measures $P^{CW}_{m,\beta,h}$ by formulae (1.4) with an additional summand $+\beta h\sum_{\alpha=1}^{n}x_\alpha$ inserted in the exponents of (1.4). We denote by $E^{CW}_{n,\beta,h}$ expectation with respect to $P^{CW}_{n,\beta,h}$. In [8; Thm. 7.2.2(c)(i)], it is proved that for $\rho\in\mathscr{G}$

$$\lim_{h\downarrow 0}\lim_{n\to\infty}\frac{1}{n}\sum_{\alpha=1}^{n}E^{CW}_{n,\beta,h}\{X^{(n)}_\alpha\}=\begin{cases}0 & \text{for } 0\leqq\beta\leqq\beta_c(\rho),\\ m^{CW}(\rho;\beta)>0 & \text{for } \beta>\beta_c(\rho).\end{cases} \tag{B.17}$$

This is the definition of spontaneous magnetization [28; p. 95].

Appendix C. A Theorem Concerning Legendre Transformations

Theorem C.1 gives a result about Legendre transformations which generalizes (1.26). This theorem is then applied to derive (1.27) from (1.24) (two formulae for the Curie-Weiss specific free energy) and to compare formula (1.22) for the circle model free energy with the formula for this quantity derived in [15; Theorem 1.2].

Let $\mathscr{X}$ be a real Banach space and $F_1 : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ a convex functional on $\mathscr{X}$. We assume that $S_{F_1} \doteq \{x: F_1(x) < \infty\} \neq \emptyset$. We say that F_1 is closed if the subset (epigraph of F_1)

$$\mathscr{E}(F_1) \doteq \{(x,u) \in \mathscr{S}_{F_1} \times \mathbb{R}: u \geqq F_1(x)\} \tag{C.1}$$

is closed in $\mathscr{X} \times \mathbb{R}$. We denote by $\mathscr{X}^*$ the dual space of $\mathscr{X}$. The Legendre transformation of F_1 is the function F_1^* with domain

$$\mathscr{S}_{F_1^*} \doteq \{\alpha \in \mathscr{X}^*: \sup_{x \in \mathscr{X}} [\alpha(x) - F_1(x)] < \infty\}. \tag{C.2}$$

For $\alpha \in \mathscr{X}^*$, we define

$$F_1^*(\alpha) \doteq \sup_{x \in \mathscr{X}} [\alpha(x) - F_1(x)]. \tag{C.3}$$

Since $F_1 = +\infty$ on $\mathscr{X} \setminus \mathscr{S}_{F_1}$, we can replace $\mathscr{X}$ in (C.3) by $\mathscr{S}_{F_1}$.

Theorem C.1. *We suppose that F_1 and F_2 are closed convex functionals on $\mathscr{X}$. Then $\mathscr{S}_{F_1^*} \neq \emptyset$ and*

$$\sup_{x \in \mathscr{S}_{F_2}} [F_1(x) - F_2(x)] = \sup_{\alpha \in \mathscr{S}_{F_1^*}} [F_2^*(\alpha) - F_1^*(\alpha)]. \tag{C.4}$$

Proof. We define $M \doteq \sup_{\mathscr{S}_{F_2}} [F_1 - F_2]$ and claim

$$M \geqq F_2^*(\alpha) - F_1^*(\alpha) \qquad \text{for all } \alpha \in \mathscr{S}_{F_1^*}. \tag{C.5}$$

(C.5) is trivial if $M = +\infty$. But if $M < +\infty$, then $\mathscr{S}_{F_2} \subseteq \mathscr{S}_{F_1}$, and for all $x \in \mathscr{S}_{F_2} \subseteq \mathscr{S}_{F_1}$

$$F_1^*(\alpha) + M \geqq \alpha(x) - F_1(x) + M \geqq \alpha(x) - F_2(x). \tag{C.6}$$

For $\alpha \in \mathscr{S}_{F_1^*}$ (C.6) implies (C.5). Therefore

$$\sup_{\mathscr{S}_{F_2}} [F_1 - F_2] \geqq \sup_{\mathscr{S}_{F_1^*}} [F_2^* - F_1^*]. \tag{C.7}$$

By [20; Theorem, p. 45] $\mathscr{S}_{F_1^*} \neq \emptyset$, $\mathscr{S}_{F_2^*} \neq \emptyset$, F_1^* and F_2^* are closed convex functionals and for $x \in \mathscr{X}$, $F_i^{**}(x) = F_i(x)\,(i=1,2)$, where $x \in \mathscr{X}$ is naturally imbedded in $\mathscr{X}^{**}$ by

$$x(x^*) \doteq x^*(x) \qquad \text{for all } x^* \in \mathscr{X}^*.$$

In particular $\mathscr{S}_{F_2^{**}} \cap \mathscr{X} = \mathscr{S}_{F_2}$. Reapplying (C.7) with F_1 replaced by F_2^* and F_2 by F_1^*, we find

$$\begin{aligned} \sup_{x \in \mathscr{S}_{F_2}} [F_1(x) - F_2(x)] &\geqq \sup_{\alpha \in \mathscr{S}_{F_1^*}} [F_2^*(\alpha) - F_1^*(\alpha)] \\ &\geqq \sup_{x^{**} \in \mathscr{S}_{F_2^{**}}} [F_1^{**}(x^{**}) - F_2^{**}(x^{**})] \\ &\geqq \sup_{x \in \mathscr{S}_{F_2^{**}} \cap \mathscr{X}} [F_1^{**}(x) - F_2^{**}(x)] \\ &= \sup_{x \in \mathscr{S}_{F_2}} [F_1^*(x) - F_2^*(x)]. \end{aligned}$$

This gives (C.4). □

We apply Theorem C.1 to formula (1.24) with $F_1(u) \doteq \beta u^2/2$ and $F_2(u) \doteq i_\rho(u)$. These functions are closed convex functions on $\mathbb{R}$ with $\mathscr{S}_{F_1} = \mathbb{R}$ and the interior of $\mathscr{S}_{F_2}$ equal to the interval (l, L) (defined in (3.20)). Hence formula (1.27) follows from Theorem C.1.

We now compare formula (1.22) for the circle model specific free energy $\psi(\beta)$ with the formula for this quantity derived in [15; Theorem 1.2]. In fact, the relationship between the former and the latter is exactly that between the Curie-Weiss formulae (1.24) and (1.27). Formula (1.22) gives $-\beta\psi(\beta)$ in the form of the left-hand side of (C.4) with $\mathscr{X} \doteq \mathscr{H}$, $F_1(x) \doteq \beta F(x) \doteq (\beta/2)\langle \mathscr{J} x, x\rangle$, $F_2(x) \doteq I(x)$. We assume that the kernel $J(s,t)$ of F is translation invariant, symmetric (i.e., $J(s,t) = J(t,s)$) and continuous on $\mathbb{T} \times \mathbb{T}$. We claim that the hypotheses of Theorem C.1 are satisfied if and only if the integral operator $\mathscr{J} h(s) \doteq \int J(s,t)\, h(t)\, dt$ is non-negative semidefinite[10].

We now (partially) verify the last statement. Formula (3.39) expresses $I(x)$ as the Legendre transformation of the convex functional Γ on $\mathscr{H}$. We have $\mathscr{S}_\Gamma = \mathscr{H}$ since for $h \in \mathscr{H}$

$$\begin{aligned} \Gamma(h) &= \int [\log \int \exp(h(t)\, x)\, d\rho(x)]\, dt \\ &\leqq \tfrac{1}{2} \|h\|^2 \cdot \log \int \exp(x^2/2)\, d\rho(x) < \infty. \end{aligned} \tag{C.8}$$

By [1; §6.1.3], Γ is weakly lower semicontinuous on $\mathscr{H}$ and is therefore a closed convex functional. By [20; Theorem, p. 45], we conclude that $I(x)$ is a closed convex functional on $\mathscr{H}$ and that $I^* = \Gamma^{**} = \Gamma$. Thus the hypotheses on $F_2 \doteq I$ in Theorem C.1 are satisfied. We now consider $F_1 \doteq \beta F$. If $\mathscr{J}$ is not non-negative semidefinite, then there exists an $h \in \mathscr{H}$, $h \neq 0$, $\lambda > 0$ such that $\mathscr{J} h = -\lambda h$. Now $0 = F_1(0) = F_1((h + (-h))/2) > (F_1(h) + F_1(-h))/2 = -\lambda\beta \|h\|^2$, which shows that F_1 is not convex. Conversely, if $\mathscr{J}$ is non-negative semidefinite, then

$$\langle \mathscr{J} h_1, h_1\rangle + \langle \mathscr{J} h_2, h_2\rangle - 2\langle \mathscr{J} h_1, h_2\rangle = \langle \mathscr{J}(h_1 - h_2), h_1 - h_2\rangle \geqq 0,$$

which implies that $\frac{1}{2}[F_1(h_1) + F_1(h_2)] \geqq F_1((h_1 + h_2)/2)$ for all $h_1, h_2 \in \mathscr{H}$. Hence F_1 is convex. It is not difficult to verify that

$$F_1^*(\alpha) = \frac{1}{2\beta} \|\mathscr{J}^{-1/2} \alpha\|^2, \qquad \alpha \in \mathscr{H}, \tag{C.9}$$

where $\mathscr{J}^{-1/2}$ is defined as follows. Define $\mathscr{H}_1$ to be the subset of $\mathscr{H}$ on which $\mathscr{J}$ is positive-definite. Then $\sqrt{\mathscr{J}}$, the unique non-negative definite, symmetric square root of $\mathscr{J}$, is invertible on a dense subset $\mathscr{H}_2$ of $\mathscr{H}_1$. In (C.9), we define

$$\mathscr{J}^{-1/2} \alpha \doteq \begin{cases} (\sqrt{\mathscr{J}})^{-1} \alpha & \text{for } \alpha \in \mathscr{H}_2, \\ +\infty & \text{for } \alpha \in \mathscr{H} \setminus \mathscr{H}_2. \end{cases} \tag{C.10}$$

[10] A translation invariant, continuous, non-negative semidefinite kernel is nuclear [24; Thm. 1.5.1]. Thus, $\mathscr{J}$ is the covariance function of a mean zero Gaussian measure on $\mathscr{H}$ [17; Theorem V.6.1].

We conclude from Theorems 1.3 and C.1 that if $\mathscr{J}$ is non-negative semi-definite, then

$$-\beta\psi(\beta)=\sup_{f\in\mathscr{H}}[\beta F(f)-I(f)]=\sup_{h\in\mathscr{H}}\left[\Gamma(h)-\frac{1}{2\beta}\|\mathscr{J}^{-1/2}h\|^2\right]. \tag{C.11}$$

The second supremum in (C.11) is the formula for $-\beta\psi(\beta)$ given in [15; Theorem 1.2] under the assumption that $J(s,t)$ be the covariance function of a mean zero Gaussian measure P on $\mathscr{C}(\mathbb{T})$. (The proof in this paper required $\mathscr{C}(\mathbb{T})$ rather than $\mathscr{H}$.) The latter assumption implies that $\mathscr{J}$ is non-negative semidefinite (cf., footnote 10)), which is the condition we needed to derive (C.11). The functional F_1^* in (C.9) is the entropy, or action, functional of the Gaussian measure P [16, 30].

Note. *Connection with the Lebowitz-Penrose Theorem.* Consider a ferromagnetic system on a subset Λ of the lattice $\mathbb{Z}^r$, some $r\in\{1,2,\dots\}$. Let the interaction strength between sites i and j be $J_{ij}\doteq J(\|i-j\|)$, where J is a function of compact support, $\int_{\mathbb{R}^r} J(\|x\|)\,dx=1$, and $\|-\|$ is the Euclidean distance. Scale the interaction by $J_{ij}^{(\varepsilon)}\doteq\varepsilon^r J(\varepsilon\|i-j\|)$, where $\varepsilon>0$ is small, and denote the corresponding specific free energy (i.e., after taking $\Lambda\uparrow\mathbb{Z}^r$) by $\psi^{(\varepsilon)}(\beta)$. The Lebowitz-Penrose theorem [28; p. 105] states that $\lim_{\varepsilon\to 0}\psi^{(\varepsilon)}(\beta)$ exists and equals the Curie-Weiss specific free energy. The difference between this set-up and the circle model is that in the latter we have $\varepsilon=1/n$ and so the thermodynamic limit $(n\to\infty)$ and the $\varepsilon\to 0$ limit are taken simultaneously. Nevertheless, Theorem 1.4 in the present paper implies that for suitable ferromagnetic interactions the circle model specific free energy and the Curie-Weiss specific free energy are equal (as are the corresponding laws of large numbers). The dichotomy occurs in the *antiferromagnetic* case, for which the circle model specific free energy and laws of large numbers are completely different from the Curie-Weiss limits.

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Received January 5, 1982; in revised and final form December 5, 1982