Z. Wahrscheinlichkeitstheorie verw. Gebiete
61, 379–387 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Random Hyperplanes Meeting a Convex Body

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We consider a convex body K with interior points in d-dimensional Euclidean space $\mathbb{R}^d$ and $n \leqq d$ independent and identically distributed random hyperplanes meeting K. About the distribution of the hyperplanes we merely assume that it is induced from a translation invariant measure on the space of hyperplanes in $\mathbb{R}^d$; if this measure is, moreover, rigid motion invariant, the hyperplanes are said to be isotropic. Let p_n denote the probability that the hyperplanes intersect inside K. In case K is a ball, R.E. Miles [10] has conjectured that p_n is maximal precisely when the random hyperplanes are isotropic. He has also conjectured that in general p_n is maximal when K is a ball and the hyperplanes are isotropic. In the following we show that the first conjecture is true, while the second holds for $d=2$, but not for $n=d>2$.

Finally we consider an arbitrary (fixed) number N of independent and identically distributed hyperplanes meeting K and we ask for the expectation of the number of k-dimensional cells in the cell-complex decomposition of the interior of K which is induced by the hyperplanes. The explicit formula giving the result involves the probabilities p_n defined above, and in the isotropic case it generalizes results obtained earlier by Santaló for the cases $d=2$ and $d=3$.

1. Preliminaries

By a random hyperplane meeting K we understand a "uniform random hyperplane in K" in the sense of Miles [10]. For the reader's convenience we repeat the definition, but we choose a slightly different approach.

By $\mathscr{H}^d$ we denote the space of hyperplanes in $\mathbb{R}^d$, topologized as usual. On $\mathscr{H}^d$ the translation and rigid motion groups act in the obvious way. We assume that we are given a measure μ on the Borel sets of $\mathscr{H}^d$ which is invariant under translations and is locally finite, i.e., finite on compact sets. Such a measure can be represented in a convenient way. To see this, let $S^{d-1} = \{u \in \mathbb{R}^d\colon \|u\| = 1\}$ be the unit sphere of $\mathbb{R}^d$, choose a vector $e \in S^{d-1}$ and put $S_e^{d-1} = \{u \in S^{d-1}\colon \langle u, e\rangle > 0\}$; here $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are, respectively, the norm

0044-3719/82/0061/0379/$01.80

and the scalar product of $\mathbb{R}^d$. We define a map $\gamma\colon S_e^{d-1} \times \mathbb{R} \to \mathscr{H}^d$ by letting $\gamma(u,t)$ be the hyperplane through te with normal vector u. Then γ is a homeomorphism onto its image $\mathscr{H}'$, which consists precisely of those hyperplanes which are not parallel to e. Let $A \subset S_e^{d-1}$ be a Borel set, and for any Borel set $B \subset \mathbb{R}$ define $\eta(B) = \mu(\gamma(A \times B))$. Clearly η is a translation invariant Borel measure on $\mathbb{R}$ which is finite on compact sets, hence it is a constant multiple of Lebesgue measure λ. The constant factor, of course, depends on A, call it $\nu(A)$. Thus we have $\mu(\gamma(A \times B)) = \nu(A)\,\lambda(B)$. If we now let A vary over the Borel subsets of S_e^{d-1}, this clearly defines a finite measure ν on S_e^{d-1}, and we may write $\mu \circ \gamma(A \times B) = \nu \otimes \lambda(A \times B)$, which in turn implies $\mu \circ \gamma = \nu \otimes \lambda$. Thus the restriction of μ to $\mathscr{H}'$ is the image measure of $\nu \otimes \lambda$ under γ. Using the translation invariance and the local finiteness of μ, one can easily choose the vector e such that $\mu(\mathscr{H}^d \setminus \mathscr{H}') = 0$. Then for any integrable function f on $\mathscr{H}^d$ it follows that

$$\int_{\mathscr{H}^d} f\,d\mu = \int_{S_e^{d-1}} \int_{\mathbb{R}} f \circ \gamma(u,t)\,d\lambda(t)\,d\nu(u).$$

It is convenient to write $\langle u, e\rangle\, t = \tau$ in the inner integral, so that τ is the oriented distance of the hyperplane $\gamma(u,t)$ from the origin, and to define an even measure φ on S^{d-1} by $d\varphi(u) = d\nu(u)/2\langle u,e\rangle$ for $\langle u,e\rangle > 0$ and $d\varphi(-u) = d\varphi(u)$. If we finally write

$$H_{u,\tau} = \{x \in \mathbb{R}^d\colon \langle x, u\rangle = \tau\},$$

we arrive at the formula

$$\int_{\mathscr{H}^d} f\,d\mu = \int_{S^{d-1}} \int_{-\infty}^{\infty} f(H_{u,\tau})\,d\tau\,d\varphi(u), \tag{1.1}$$

which gives a convenient normal form for the translation invariant, locally finite measures μ on $\mathscr{H}^d$. In fact, Miles [10] assumes right from the beginning that the measure defining his uniform random hyperplanes is of a form equivalent to (1.1), but we felt that this assumption ought to be motivated by the above simple reasoning. One could also have used an argument of Matheron [8], p. 66, but the one given here is direct and more elementary, avoiding conditional probabilities.

Now let $K \subset \mathbb{R}^d$ be a given convex body (compact, convex subset) with interior points, and let μ be given and represented as above. Then the total measure of the set of hyperplanes which meet K is

$$\int_{\mathscr{H}^d} \chi(K \cap H)\,d\mu(H) = \int_{S^{d-1}} \int_{-\infty}^{\infty} \chi(K \cap H_{u,\tau})\,d\tau\,d\varphi(u),$$

where χ is the Euler characteristic, i.e. $\chi(L) = 1$ for any nonempty convex body L and $\chi(\emptyset) = 0$. The inner integral on the right is just the width $w(K,u)$ of K in direction u, that is, the distance between the two supporting hyperplanes of K orthogonal to u. Writing

$$\mu_K(\beta) = \mu(\beta \cap \{H \in \mathscr{H}^d\colon K \cap H \neq \emptyset\}) \tag{1.2}$$

for Borel sets $\beta \subset \mathscr{H}^d$ and assuming that $\mu \neq 0$, we see that

$$\frac{\mu_K}{\int\limits_{S^{d-1}} w(K,u)\, d\varphi(u)} \tag{1.3}$$

is a probability measure on $\mathscr{H}^d$. Clearly we may assume, without loss of generality, that φ is a probability measure. To conclude with a formal definition, we now say that a *random hyperplane meeting* K is a measurable map from some probability space into the space $\mathscr{H}^d$ of hyperplanes such that the distribution of the hyperplane (the image measure of the probability) is given by (1.3), where φ is some even probability measure on the sphere S^{d-1} and μ_K is defined by (1.1) and (1.2). We shall say that φ is the *generating orientation distribution* of the random hyperplane.

We denote by ω the unique rotation invariant probability measure on S^{d-1}. A random hyperplane with generating orientation distribution ω will be called *isotropic*.

2. The Probability of Intersection in K

We consider a given convex body $K \subset \mathbb{R}^d$ with interior points, a number $n \in \{2, \ldots, d\}$, and n independent and identically distributed random hyperplanes meeting K. By $p_n(K, \varphi)$, where φ is the common generating orientation distribution of the random hyperplanes, we denote the probability that these hyperplanes have a common point in K. By the assumption of independence we may write

$$p_n(K,\varphi) = \frac{\int\limits_{\mathscr{H}^d} \cdots \int\limits_{\mathscr{H}^d} \chi(K \cap H_1 \cap \ldots \cap H_n)\, d\mu(H_1) \ldots d\mu(H_n)}{\left[\int\limits_{S^{d-1}} w(K,u)\, d\varphi(u)\right]^n}$$

with μ as in (1.1). For $u_1, \ldots, u_n \in S^{d-1}$ let $[u_1, \ldots, u_n]$ denote the n-dimensional volume of the parallelepiped spanned by $u_1, \ldots, u_n$. Further, let $\Pi_{u_1, \ldots, u_n}$ denote the orthogonal projection on to the linear subspace spanned by $u_1, \ldots, u_n$, and let λ_n denote n-dimensional Lebesgue measure. Using (1.1) one easily shows (see also Miles [10]) that

$$p_n(K,\varphi) = \frac{\int \cdots \int \lambda_n(\Pi_{u_1, \ldots, u_n} K)\,[u_1, \ldots, u_n]\, d\varphi(u_1) \ldots d\varphi(u_n)}{\left[\int w(K,u)\, d\varphi(u)\right]^n}. \tag{2.1}$$

Here, and in the following, all integrations with respect to φ are over S^{d-1}. In [10], p. 224, Miles conjectured that

$$p_n(B,\varphi) \leqq p_n(B,\omega)$$

for the unit ball B of $\mathbb{R}^d$. Up to now, only the two-dimensional case has been decided. Miles [10] himself gave an argument for $d=2$ which shows the following. If a measure φ for which $p_2(B,\varphi)$ is maximal (the existence of such measures can be shown by familiar compactness arguments) has a continuous density with respect to Lebesgue measure (on the circle), then it must be normalized Lebesgue measure. A complete (affirmative) answer for the two-dimensional case is contained in the work of Rasson [12].

As we shall see in the following, a combination of known results in the theory of convex bodies, in particular from the theory of mixed volumes, yields an affirmative answer for the general case.

Theorem 1. *The probability that n independent, identically distributed random hyperplanes meeting a ball intersect inside that ball is maximal precisely when the random hyperplanes are isotropic.*

For the proof we observe first that for the case of the unit ball B formula (2.1) reduces to

$$p_n(B,\varphi)=2^{-n}\kappa_n\int\dots\int[u_1,\dots,u_n]\,d\varphi(u_1)\dots d\varphi(u_n) \tag{2.2}$$

where κ_n is the volume of B.

Following Matheron [6, 8] in his work on Poisson hyperplanes, we consider the auxiliary convex body Z_φ defined in the following way. Writing

$$h(u)=\tfrac{1}{2}\int|\langle u,v\rangle|\,d\varphi(v) \qquad \text{for } u\in\mathbb{R}^d$$

one sees that h satisfies the conditions which are necessary and sufficient for a function to be a support function (see, e.g., Bonnesen-Fenchel [2], Sect. 17, or Leichtweiß [5], § 12), hence there exists a unique convex body Z_φ for which h is the support function. We consider the Minkowski quermassintegrals W_k of Z_φ (for a definition, see Bonnesen-Fenchel [2], Sect. 32, or Matheron [8], pp. 76–78). They can be expressed explicitly my means of the generating measure φ. In special cases such formulas occur already in Blaschke [1], p. 156; for the general case we refer to Matheron [8], Chap. 4.5, and W. Weil [17], formula (9). There one finds

$$W_k(Z_\varphi)=\frac{k!\,\kappa_k}{d!}\int\dots\int[u_1,\dots,u_{d-k}]\,d\varphi(u_1)\dots d\varphi(u_{d-k}) \tag{2.3}$$

for $k=0,\dots,d-1$, which together with (2.2) yields

$$p_n(B,\varphi)=\frac{d!\,\kappa_n}{2^n(d-n)!\,\kappa_{d-n}}\,W_{d-n}(Z_\varphi).$$

We note also that

$$W_{d-1}(Z_\varphi)=\frac{(d-1)!\,\kappa_{d-1}}{d!}\int d\varphi(u)=\frac{\kappa_{d-1}}{d}.$$

Now we make use of the Minkowski-Fenchel-Aleksandrov inequalities between the quermassintegrals (see, e.g., Busemann [3], § 7, Leichtweiß [5], § 23). A particular case is

$$W_{d-1}(Z_\varphi)^n\geqq\kappa_d^{n-1}\,W_{d-n}(Z_\varphi),$$

with equality if and only if Z_φ is a ball. We deduce that

$$p_n(B,\varphi)\leqq p_n(B,\omega)$$

with equality only if Z_φ is a ball. The latter occurs only if φ is proportional (and hence equal) to ω, as follows from a known uniqueness result. (See Theorem 1.4 in Schneider [16], where references (on p. 304) and a proof are given. A different proof was presented by Matheron [7].) This proves Theorem 1.

We return to the case of a general convex body K with interior points. Later in his paper [10] (see p. 232) Miles, after showing that $p_n(K,\omega) \leqq p_n(B,\omega)$, conjectured further that

$$p_n(K,\varphi) \leqq p_n(B,\omega).$$

We shall prove this for $d=2$, and we show that it is not true for $n=d>2$.

Without loss of generality we may assume that the generating orientation distribution φ is not concentrated on a great sphere (otherwise the problem reduces to one in a lower dimensional space). Again we construct an auxilary convex body, starting from the measure φ. Let M_φ be the solution of Minkowski's problem for the measure φ (see, e.g., Busemann [3], § 8). Thus M_φ is the convex body, unique up to a translation, for which

$$S_{d-1}(M_\varphi, \cdot) = \varphi. \tag{2.4}$$

Here $S_{d-1}(L,\cdot)$ is the area function of the convex body L, defined as follows: for a Borel set $\beta \subset S^{d-1}$, $S_{d-1}(L,\beta)$ is the area of the set of boundary points of L at which there exists an exterior unit normal vector belonging to β. Since the measure φ is even and not concentrated on a great sphere, it satisfies the conditions of Minkowski's theorem, the latter taken in its generalized form due to Aleksandrov and Fenchel and Jessen.

We use some results from the theory of mixed volumes, for which the reader is referred to Busemann [3], pp. 62, 50, or Leichtweiß [5], §§ 23, 24. In particular, because of (2.4) we have

$$V(K, M_\varphi, \ldots, M_\varphi) = \frac{1}{d}\int h(K,u)\, d\varphi(u) = \frac{1}{2d}\int w(K,u)\, d\varphi(u),$$

where $h(K,\cdot)$ is the support function of K, and

$$V(M_\varphi) = \frac{1}{2d}\int w(M_\varphi, u)\, d\varphi(u). \tag{2.5}$$

From the Fenchel-Aleksandrov inequality

$$V(K, M_\varphi, \ldots, M_\varphi)^d \geqq V(K)\, V(M_\varphi)^{d-1} \tag{2.6}$$

we deduce that

$$\frac{V(K)}{[\int w(K,u)\, d\varphi(u)]^d} \leqq \frac{1}{(2d)^d}\frac{1}{V(M_\varphi)^{d-1}} = \frac{V(M_\varphi)}{[\int w(M_\varphi,u)\, d\varphi(u)]^d}.$$

Multiplying both sides by $\int \ldots \int [u_1, \ldots, u_d]\, d\varphi(u_1) \ldots d\varphi(u_d)$ we see from (2.1) that

$$p_d(K,\varphi) \leqq p_d(M_\varphi, \varphi). \tag{2.7}$$

Equality in (2.6) and hence in (2.7) holds if and only if K and M_φ are homothetic. Thus for every even probability measure φ which is not concentrated on a great sphere, the maximum

$$\max_K p_d(K, \varphi)$$

is assumed for a convex body M_φ which is unique up to a homothety. Now the question of Miles for $n=d$ reduces to the question whether $p_d(M_\varphi, \varphi)$, which depends only on φ, is maximal for $\varphi=\omega$.

To further study this question, we rewrite (2.7) by using the body Z_φ defined earlier. By (2.1), (2.3) (observing $W_0=V$, the volume), and (2.5) we have

$$\begin{aligned} p_d(M_\varphi, \varphi) &= \frac{V(M_\varphi)\int \ldots \int [u_1, \ldots, u_d]\, d\varphi(u_1) \ldots d\varphi(u_d)}{[\int w(M_\varphi, u)\, d\varphi(u)]^d} \\ &= \frac{V(M_\varphi)\, d!\, V(Z_\varphi)}{[2d\, V(M_\varphi)]^d} = \frac{d!}{2^d d^d} \frac{V(Z_\varphi)}{V(M_\varphi)^{d-1}}. \end{aligned} \tag{2.8}$$

By the definitions of Z_φ and M_φ we have

$$h(Z_\varphi, u) = \tfrac{1}{2}\int |\langle u, v\rangle|\, dS_{d-1}(M_\varphi, v) \qquad \text{for } u \in S^{d-1},$$

and this is precisely the $(d-1)$-dimensional volume of the orthogonal projection of M_φ on to a hyperplane orthogonal to u. Thus Z_φ is what is usually called the *projection body* of M_φ (see Bonnesen-Fenchel [2], Sect. 30). In the plane we obviously have $Z_\varphi = 2R_{\pi/2} M_\varphi$ (up to a translation), where $R_{\pi/2}$ denotes a rotation by the angle $\pi/2$. Thus for $d=2$, the quotient $V(Z_\varphi)/V(M_\varphi)^{d-1} = 4$ and thus the probability $p_2(M_\varphi, \varphi) = 1/2$ is independent of φ. Hence inequality (2.7) together with the equality condition yields the following.

Theorem 2. *Let K be a convex body with interior points in the plane. The probability that two independent and identically distributed random lines meeting K intersect in K is at most $1/2$, and it is equal to $1/2$ precisely when the generating orientation distribution of the random lines is proportional to the area function $S_1(K, \cdot)$ of K.*

It should be noted that here the case of a generating orientation distribution concentrated on a pair of antipodal points need not be excluded, this case being trivial, as it gives rise to two random lines which almost surely do not intersect at all.

Collecting our results for general d, we have arrived at

$$p_d(K, \varphi) \leq p_d(M_\varphi, \varphi) = \frac{d!}{2^d d^d} \frac{V(\Pi M_\varphi)}{V(M_\varphi)^{d-1}},$$

where ΠL denotes the projection body of the convex body L. Unfortunately, for $d>2$ it is an open problem to determine all centrally symmetric convex bodies L (with interior points) for which the affine invariant $V(\Pi L)\,V(L)^{1-d}$ attains its maximum. We conjecture that

$$V(\Pi L)\,V(L)^{1-d} \leqq 2^d$$

for any such L, with equality precisely when L is a direct sum of convex bodies of dimensions $\leqq 2$. A proof of this conjecture would give the exact upper bound of the probability $p_d(K,\varphi)$. At least, it is not difficult to show that $V(\Pi L)\,V(L)^{1-d}=2^d$ if L is as described above, whereas for the ball B we have

$$V(\Pi B)\,V(B)^{1-d}=\pi^{\frac{d}{2}}\frac{\Gamma\left(\frac{d+2}{2}\right)^{d-2}}{\Gamma\left(\frac{d+1}{2}\right)^{d}}<2^d \qquad \text{for } d\geqq 3.$$

The latter inequality is immediately verified for small dimensions. As this already establishes a counterexample, it does not seem worthwhile to reproduce the proof for all $d\geqq 3$ (*I* owe an elegant proof to Peter McMullen).

Thus, choosing L as above and φ as its normalized area function, we have $p_d(L,\varphi)=p_d(M_\varphi,\varphi)>p_d(B,\omega)$ (observe that M_ω is a ball). This establishes a negative answer to the second conjecture of Miles for $n=d$.

3. The Decomposition of K by Random Hyperplanes

In this section we consider an arbitrary (but fixed) number N of independent and identically distributed random hyperplanes meeting the convex body K (which has interior points). Almost every realization of these random hyperplanes determines, in the obvious way, a decomposition of the interior of K into relatively open convex cells of dimensions $0,1,\ldots,d$ (namely, the intersections of the interior of K with the relative interiors of the faces of the hyperplane arrangement). For $k=0,\ldots,d$, the random variable v_k is defined as the number of k-dimensional cells of such a subdivision. In the following we will determine the expected value $E(v_k)$ of v_k. For $d=2,3$ and isotropic hyperplanes this has been done by Santaló [13, 14], see also Santaló [15], Sects. I.4.4, p. 54, and III.16.4, Note 7. His method is not restricted to convex bodies, but apparently it does not extend to higher dimensions. Our treatment of the d-dimensional case uses a combinatorial lemma due to Miles [9], [11]. (For the case $k=d$ of this lemma, see also Janson [4].)

Let there be given N independent and identically distributed random hyperplanes $H_1,\ldots,H_N$ meeting K, with generating orientation distribution φ. Clearly with probability 1 these hyperplanes are in general position, which means that the intersection of any $m\leqq d+1$ of them has dimension at most $d-m$. Then the lemma of Miles [11] says that

$$v_k=\sum_{n=d-k}^{d}\binom{n}{d-k}\alpha_n,$$

where α_n is the number of n-tuples among the hyperplanes whose intersection meets the interior of K. Thus we get

$$E(v_k) = \sum_{n=d-k}^{d} \binom{n}{d-k} \sum_{i_1 < \ldots < i_n} \operatorname{Prob}\left(K \cap \bigcap_{r=1}^{n} H_{i_r} \neq \emptyset\right)$$
$$= \sum_{n=d-k}^{d} \binom{n}{d-k} \binom{N}{n} p_n(K, \varphi)$$

with $p_n(K, \varphi)$ as defined in §2. This formula lends new interest to the extremum problems considered earlier and inequalities obtained there. In particular, Theorem 1 implies: If K is a ball, then the expected value of the number of k-cells is maximal precisely when the random hyperplanes are isotropic.

Of course, in the isotropic case the probabilities $p_n(K, \omega)$ can be computed by known methods of integral geometry, since the integral

$$\int \ldots \int \chi(K \cap H_1 \cap \ldots \cap H_n)\, d\mu(H_1) \ldots d\mu(H_n),$$

with μ induced from $\varphi = \omega$ according to (2.1), can be computed recursively, using the Crofton formula ([15], p. 233). One obtains

$$E(v_k) = \sum_{n=d-k}^{d} \binom{n}{d-k} \binom{N}{n} \frac{n!}{2^n} \frac{\kappa_n}{\kappa_{d-n}} \left(\frac{\kappa_{d-1}}{d}\right)^n \binom{d}{n} \frac{W_{d-n}(K)}{W_{d-1}(K)^n}.$$

For $d = 2, 3$ this coincides with the formulas given by Santaló.

The formula simplifies if expressed in terms of the "intrinsic n-volumes" $V_n(K)$ defined by

$$\kappa_{d-n} V_n(K) = \binom{d}{n} W_{d-n}(K),$$

which are independent of the dimension of the space in which K lies. We have

$$E(v_k) = \sum_{n=d-k}^{d} \binom{n}{d-k} \binom{N}{n} \frac{n!}{2^n} \kappa_n \frac{V_n(K)}{V_1(K)^n}.$$

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Received January 29, 1982; in revised form April 16, 1982