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Eigenvalue Expansions for Diffusion Hitting Times

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1. Introduction

Consider a non-singular diffusion on an interval (r_0, r_1) and let $r_0 < a < b < r_1$. Set τ_{ab} to be the first time the diffusion hits b, starting at a, with moment generating function (m.g.f.) $\phi_{ab}(\lambda) = E\{\exp(\lambda \tau_{ab})\}$. Since we shall be concerned with the behaviour of $\phi(\lambda)$ for *positive* λ, it is more convenient to work with m.g.f.s than with Laplace transforms. We shall show that all such diffusion hitting times are *generalized convolutions of mixtures of exponential distributions* (g.c.m.e.d.s).

If r_0 is not a natural boundary, more can be said; then τ_{ab} can be written as an *infinite convolution of elementary mixtures of exponential distributions*. The parameters are given by the eigenvalues of associated Sturm-Liouville expansions.

Furthermore, normalizing τ_{ab} to have mass 1 and letting $a \downarrow r_0$ leads to an *infinite convolution of exponential densities.*

Sections 2 and 3 summarize the necessary information needed about g.c.m.e.d.s and diffusion theory, respectively. The expansion for τ_{ab} when r_0 is not natural is derived in Sects. 4 and 5. A related series expansion for the density of τ_{ab} is discussed in Sect. 6, and Sect. 7 gives the general result which holds for all diffusion hitting times. An example based on the *Bessel* diffusion process is analyzed in Sect. 8. Some simple formulae for the first two moments are given in Sect. 9.

2. Some Classes of Infinitely Divisible Distributions

By a theorem of Goldie (1967) and Steutel (1967) all *mixtures of exponential distributions* (m.e.d.s) are infinitely divisible. The simplest such mixture assigns probability $p_1 > 0$ to a point mass at the origin and probability $p_2 > 0$ to an exponential density of parameter $\lambda_0 \in (0, \infty)$, where $p_1 + p_2 = 1$. It has m.g.f.

$$\begin{aligned}\phi(\lambda) &= p_1 + p_2(1-\lambda/\lambda_0)^{-1} \\ &= (1-\lambda/\lambda_1)/(1-\lambda/\lambda_0) \qquad (2.1)\end{aligned}$$

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where $\lambda_0 < \lambda_1 = \lambda_0/p_1$. We shall call (2.1) the m.g.f. of an *elementary m.e.d.* Allowing defective distributions, we can also include as limiting cases the m.g.f.s $\phi(\lambda) \equiv e^{-\alpha}$ where $\alpha \geqq 0$.

The function

$$\phi(\lambda) = e^{-\alpha} \prod_{k=1}^{\infty} \{(1-\lambda/\lambda'_k)/(1-\lambda/\lambda_k)\}$$

is a non-degenerate m.g.f. of an *infinite convolution of elementary m.e.d.s.* provided $0 < \lambda_k < \lambda'_k$ and $\sum \lambda_k^{-1} < \infty$. These m.g.f.s appear as diffusion hitting times; see Sect. 5.

It is also of interest to consider

$$\phi(\lambda) = \prod (1-\lambda/\lambda_k)^{-1}, \tag{2.2}$$

which represents the m.g.f. of a *convolution of exponential densities.* We allow the product to be finite or infinite, but an infinite product represents a non-degenerate distribution if and only if $\sum \lambda_k^{-1} < \infty$. These functions represent the m.g.f.s of all *Polyà frequency densities* with support $(0, \infty)$ (Karlin, 1968, p. 345). Further, such densities are *strongly unimodal* (Kielson, 1971).

The class of infinitely divisible distributions consisting of all weak limits of distributions represented in (2.1) has been termed by Bondesson (1978) the class of *generalized convolutions of mixtures of exponential distributions (g.c.m.e.d.s).* An important subclass of the g.c.m.e.d.s is the class of *generalized gamma convolutions (g.g.c.s)*, which includes as a special case the distributions with m.g.f.s (2.2). Further, the m.g.f. (2.1) represents a g.g.c. if and only if $\{\lambda'_k\} \subset \{\lambda_k\}$, in which case (2.1) can be reduced to the form (2.2).

3. Diffusion Theory

A non-singular diffusion on an interval $[r_0, r_1]$, $-\infty \leqq r_0 < r_1 \leqq \infty$ can be described by three Borel measures, which can be most conveniently represented by non-decreasing functions on (r_0, r_1) (Ito and McKean, 1965, and Mandl, 1968): the *speed measure* $m(x)$, *natural scale* $s(x)$, and *killing measure* $k(x)$. The underlying diffusion is unaltered if the three measures are rescaled,

$$m(dx) \to c\,m(dx), \qquad s(dx) \to c^{-1} s(dx), \qquad k(dx) \to c\,k(dx), \tag{3.1}$$

where $c > 0$. Further, associated with each non-decreasing function is an arbitrary additive constant which can be chosen at our convenience.

We shall use Mandl's terminology for the boundaries: *regular, entrance, exit,* and *natural.* For regular boundaries there is a further subclassification. The boundary r_i is called *absorbing regular* if the diffusing particle dies when it reaches r_i. Otherwise the boundary is called *reflecting regular* and the description of the diffusion must be augmented by assigning speed measure and killing measure to r_i, $0 \leqq m\{r_i\}$, $k\{r_i\} < \infty$. A reflecting boundary is called *instantaneously reflecting* if $m\{r_i\} = k\{r_i\} = 0$.

Associated with the diffusion is a generalized second order linear differential operator A, and boundary conditions at each endpoint. The relevant boundary

Table 1. Initial and boundary conditions for the initial problem

Boundary behaviour at r_0	Boundary condition	Initial condition
Absorbing regular	$u(r_0)=0$	$u^+(r_0)=1$
Reflecting regular	$u^+(r_0)=[k\{r_0\}-\lambda m\{r_0\}]\,u(r_0)$	$u(r_0)=1$
Entrance	$u^+(r_0)=0$	$u(r_0)=1$
Exit	$u(r_0)=0$	$u^+(r_0)=1$
Natural	$u(r_0)=0$	–

condition at r_0 for each type of boundary is given in Table 1. In this table $u(x)$ is a continuous function on (r_0, r_1) for which the right-hand derivative with respect to natural scale,

$$u^+(x)=\lim_{\varepsilon\downarrow 0}\{u(x+\varepsilon)-u(x)\}/\{s(x+\varepsilon)-s(x)\}, \tag{3.2}$$

exists, and we set $u(r_i)=\lim_{x\to r_i} u(x)$, $u^+(r_i)=\lim_{x\to r_i} u^+(x)$, when these limits exist.

Let $r_0<a<b<r_1$ and consider τ_{ab}, the first time the diffusion hits b, starting at a, with m.g.f. $\phi_{ab}(\lambda)$. For fixed λ, let $v(x)$ denote the solution of

$$Av+\lambda v=0 \tag{3.3}$$

together with the relevant boundary condition at r_0, given in Table 1. This solution is unique up to a multiplicative constant and can be used to find the m.g.f. of τ_{ab}; namely

$$\phi_{ab}(\lambda)=v(a)/v(b) \tag{3.4}$$

(Ito and McKean, 1965, pp. 128–130).

4. An Initial Value Problem

Unfortunately, the above approach does not give us sufficient information about the dependence of $\phi_{ab}(\lambda)$ on λ for our purposes. Therefore, we consider the solution of (3.3) which also satisfies the *initial* condition of Table 1.

Theorem 4.1. *Suppose r_0 is not a natural boundary and let $u(x,\lambda)$ denote the (unique) solution of* (3.3) *which satisfies the relevant boundary condition and initial condition of Table 1.*

Then

(a) *u is jointly continuous in x and λ for $x\in[r_0, r_1)$ and $\lambda\in C$, and u is an entire function of λ;*

(b) *$u(x,0)$ is a strictly positive non-decreasing function for $x\in(r_0, r_1)$;*

(c) *for fixed $x\in(r_0, r_1)$, the zeros of $u(x,\cdot)$ are simple and positive and form a sequence*

$$0<\lambda_{1,x}<\lambda_{2,x}<\dots;$$

(d) *for $x<y$, $\lambda_{k,x}>\lambda_{k,y}$;*

(e) *for* $x \in (r_0, r_1)$, $\sum \lambda_{k,x}^{-1} < \infty$ *and* $u(x, \lambda)$ *can be written as the canonical product*

$$u(x, \lambda) = u(x, 0) \prod_{k=1}^{\infty} (1 - \lambda/\lambda_{k,x});$$

(f) *for all fixed* $x \in (r_0, r_1)$ *and* $\varepsilon > 0$, $|u(x, \lambda)| + |u^+(x, \lambda)| = O(e^{\varepsilon|\lambda|})$ *as* $|\lambda| \to \infty$.

Remark on the Proof. For fixed a consider the boundary value problem

$$A v(x) + \lambda v(x) = 0, \qquad r_0 \leqq x \leqq a,$$

such that

$$B(v) = 0$$

and

$$v(a) = 0,$$

where $B(\cdot)$ is the boundary condition at r_0 given in Table 1. This problem has a solution if and only if $\lambda = \lambda_{k,a}$ for some k. Thus the numbers $\{\lambda_{k,a}\}_{k=1}^{\infty}$ can be considered as eigenvalues in a Sturm-Liouville expansion. Such expansions are discussed in McKean (1956) and McKean and Ray (1962).

For a classical differential operator with regular boundary, $u(x, \lambda)$ can be constructed using the method of Titchmarsh (1962), pp. 6–7. In this case $u(x, \lambda)$ is an entire function of λ of order $\frac{1}{2}$, so the product formula holds by Hadamard's factorization theorem (Titchmarsh, 1939, p. 250).

For a general diffusion with regular boundary, a similar construction can be used, and with some care, the argument can be extended to cover the entrance and exit cases also. Full details are given in Kent (1979b). □

Note that for an absorbing regular or exit boundary, $u(x, \lambda)$ is unique only up to the scaling of the natural scale described in (3.1).

If a mild regularity condition is satisfied, then it can be shown that the eigenvalues increase at rate ck^2, as $k \to \infty$ (McKean and Ray, 1962). More specifically, suppose there exists a positive function, $\Phi(x)$ monotone in a neighborhood of r_0, such that for $x \in (r_0, r_1)$,

$$\int_{(r_0, x)} \Phi(w)\, m(dw) < \infty \quad \text{and} \quad \int_{(r_0, x)} \Phi(w)^{-1} s(dw) < \infty. \tag{4.1}$$

Let $D^2(x) = m(dx)/s(dx)$ denote the Radon-Nikodym derivative of the absolutely continuous part of the speed measure with respect to the natural scale. If (4.1) holds, then

$$\lim_{k \to \infty} \lambda_{k,x}/k^2 = \left\{\pi^{-1} \int_{r_0}^{x} D(w)\, s(dw)\right\}^{-2}. \tag{4.2}$$
$$= c, \qquad \text{say.}$$

Note that (4.1) implies $c > 0$, and if $D(x) \not\equiv 0$ then $c < \infty$. Also, if r_0 is regular, then (4.1) always holds with $\Phi(x) \equiv 1$.

5. Expansions for Diffusion Hitting Times

Using Theorem 4.1 in (3.4) gives the following expression, which is the main result of this paper.

Theorem 5.1. *Suppose r_0 is not a natural boundary and let $r_0<a<b<r_1$. Then τ_{ab} has m.g.f. $\phi_{ab}(\lambda)$ given by*

$$\phi_{ab}(\lambda)=\frac{u(a,0)}{u(b,0)}\prod_{k=1}^{\infty}\frac{(1-\lambda/\lambda_{k,a})}{(1-\lambda/\lambda_{k,b})}.$$

Remarks. (1) Note that $P(\tau_{ab}<\infty)=u(a,0)/u(b,0)\leqq 1$ by part (b) of Theorem 4.1. If the killing measure vanishes and if r_0 is reflecting regular or extrance, then $u(a,0)=u(b,0)=1$.

(2) $\phi_{ab}(\lambda)$ is the m.g.f. of an infinite convolution of elementary m.e.d.s; see Sect. 2.

(3) Note that for $a\neq r_0$, $\phi_{ab}(\lambda)$ is *not* in general a g.g.c. because $\{\lambda_{k,a}\}_{k=1}^{\infty}$ is not in general a subset of $\{\lambda_{k,b}\}_{k=1}^{\infty}$.

(4) An analogue of Theorem 5.1 for birth-death processes is given in Rosenlund (1977). With state space the non-negative integers and with 0 a reflecting boundary, $\phi_{nm}(\lambda)$, $0\leqq n<m$, can be written as the m.g.f. of a convolution of n elementary m.e.d.s together with $m-n$ exponential distributions. The case $n=0$ was first given in Kielson (1971).

(5) Letting $a\to r_0$ leads to the m.g.f. of an infinite convolution of exponential densities. If r_0 is exit or absorbing regular, then τ_{ab} must first be normalized to be finite with probability 1, so we express the result in this form.

Corollary 5.1. *If r_0 is not a natural boundary, then*

$$u(b,0)\lim_{a\to r_0}\{\phi_{ab}(\lambda)/u(a,0)\}=\prod_{k=1}^{\infty}(1-\lambda/\lambda_{k,b})^{-1}.$$

Proof. Using the initial and boundary conditions of Table 1, we see that $u(a,\lambda)/u(a,0)\to 1$ as $a\to 0$, provided r_0 is not natural.

6. Series Expansions for Diffusion Hitting Time Densities

For fixed $x\in(r_0,r_1)$, write $\lambda_{k,x}=\lambda_k$ and let

$$h(\lambda)=u(x,0)^{-1}u(x,\lambda)=\prod_{k=1}^{\infty}(1-\lambda/\lambda_k). \tag{6.1}$$

Formally, we can write $h(\lambda)^{-1}$ in a partial fraction expansion

$$h(\lambda)^{-1}=\sum_{k=1}^{\infty}\{h'(\lambda_k)\}^{-1}/(\lambda-\lambda_k). \tag{6.2}$$

Note that $\{h'(\lambda_k)\}^{-1}$, the residue of $h(\lambda)^{-1}$ at $\lambda=\lambda_k$, alternates in sign as k varies. Inverting this m.g.f. term-by-term we get

$$h(\lambda)^{-1}=\int_0^{\infty}e^{\lambda t}f(t)\,dt, \qquad \lambda<0 \tag{6.3}$$

where

$$f(t)=-\sum_{k=1}^{\infty}\{h'(\lambda_k)\}^{-1}e^{-\lambda_k t}, \qquad t>0 \tag{6.4}$$

is the probability density with m.g.f. $h(\lambda)^{-1}$.

Unfortunately, (6.2) does *not* converge in general. However, subject to the following mild regularity condition, we can show that (6.4) does converge and that (6.3) is valid.

Condition A. *Suppose that for all* $x \in (r_0, r_1)$ *and* $\varepsilon > 0$,

$$\{u'(x, \lambda_{k,x})\}^{-1} = O(\exp\{\varepsilon \lambda_{k,x}\}) \quad \text{as } k \to \infty,$$

where the prime denotes differentiation with respect to λ.

The following lemmas give sufficient conditions for Condition A to hold at r_0.

Lemma 6.1. *If* r_0 *is entrance, choose the additive constant of* $m(x)$ *so that* $m(r_0) = 0$ *and set* $\rho_1(x) = \int_{(r_0, x]} m(w) s(dw)$. *Note that* $s(r_0) = -\infty$. *Then each of the following conditions is sufficient for Condition A to hold.*

(a) $\lim_{x \to r_0} \rho_1(x) \log\{-s(x)\} = 0$;

(b) *for some* $c > 0$, $\rho_1(x) = O(\{-s(x)\}^{-c})$ as $x \to r_0$;

(c) *for some* $c > 0$, $m(x) = O(\{-s(x)\}^{-c-1})$ as $x \to r_0$;

(d) $m(dx)$ *is absolutely continuous with respect to natural scale and for some* $c > 0$,

$$m(dx)/s(dx) = O(\{-s(x)\}^{-c-2}) \quad \text{as} \quad x \to r_0.$$

Lemma 6.2. *If* r_0 *is exit, choose the additive constant of* $s(x)$ *so that* $s(r_0) = 0$ *and set* $\rho_2(x) = \int_{(r_0, x]} s(w) m(dw)$. *Then each of the following conditions is sufficient for Condition A to hold.*

(a) $\lim_{x \to r_0} \rho_2(x) \log\{1/s(x)\} = 0$;

(b) *for some* $c > 0$, $\rho_2(x) = O(s(x)^c)$ *as* $x \to r_0$;

(c) *for some* $c > 0$, *and* $x_0 \in (r_0, r_1)$, $\{m(x_0) - m(x)\} = O(s(x)^{c-1})$ *as* $x \to r_0$;

(d) $m(dx)$ *is absolutely continuous with respect to natural scale and for some* $c > 0$,

$$m(dx)/s(dx) = O(s(x)^{c-2}) \quad \text{as} \quad x \to r_0.$$

Lemma 6.3. *If* r_0 *is regular then Condition A is always satisfied.*

Proofs. First, note that in Lemma 6.1 and Lemma 6.2 we may without loss of generality take $0 < c < 1$. Then clearly (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) in each lemma. Also, in Lemma 6.1, $s(x)$ is defined only up to an additive constant. In Lemma 6.2(c) we have explicitly taken account of the additive constant in $m(x)$ because $m(dx)$ might be integrable at r_0 (in which case the lemma always holds).

The proofs are based on a result which follows from the fundamental formula in Sturm's comparison theorem (Titchmarsh, 1962, pp. 107–8). If $u(x_1, \lambda_1) = 0, r_0 < x_1 < r_1$, then

$$\partial u(x_1, \lambda_1)/\partial \lambda = \Big\{ \int_{[r_0, x_1]} u^2(w, \lambda_1) m(dw) \Big\} / u^+(x_1, \lambda_1).$$

Thus, in view of Theorem 4.1(f) applied to $u^+(x_1, \lambda_1)$, we see that the verification of Condition A can be reformulated in terms of the 'size' of $u(x, \lambda)$. In the presence of the above regularity conditions, it is relatively straightforward to show that $u(x, \lambda)$ is 'not too small' near $x = r_0$ for λ large and positive, and from this bound Condition A can be established. Full details can be found in Kent (1979b). □

We can now use Condition A to derive series expansions for diffusion hitting time densities. The analogous theory for birth-death processes is discussed in Rosenlund (1977).

Theorem 6.1. *Fix $r_0 < a < b < r_1$ where r_0 is not a natural boundary, and suppose Condition A is satisfied. Then $\phi_{ab}(\lambda)$ is the m.g.f. of the density*

$$f(t) = -\sum d_k \exp(-\lambda_{k,b} t), \qquad t > 0 \tag{6.5}$$

where

$$d_k = u(a, \lambda_{k,b})/u'(b, \lambda_{k,b}), \tag{6.6}$$

and the prime denotes differentiation with respect to λ. For all $t_0 > 0$, (6.5) converges uniformly for $t > t_0$.

Proof. Write $\phi_{ab}(\lambda) = \phi(\lambda)$ and set

$$\phi_N(\lambda) = \frac{u(a,0)}{u(b,0)} \prod_{k=1}^{N} \frac{(1-\lambda/\lambda_{k,a})}{(1-\lambda/\lambda_{k,b})}$$
$$= \sum_{k=1}^{\infty} \frac{d_{k,N}}{\lambda - \lambda_{k,b}} = \int_0^{\infty} e^{\lambda t} f_N(t)\, dt$$

where

$$d_{k,N} = -\lambda_{k,b}(1-\lambda_{k,b}/\lambda_{k,a}) \frac{u(a,0)}{u(b,0)} \prod_{\substack{n=1 \\ n \neq k}}^{N} \frac{(1-\lambda_{k,b}/\lambda_{n,a})}{(1-\lambda_{k,b}/\lambda_{n,b})}, \qquad 1 \leqq k \leqq N,$$
$$= 0, \qquad k > N$$

and

$$f(t) = -\sum_{k=1}^{N} d_{k,N} \exp(-\lambda_{k,b} t).$$

The partial fraction expansion is valid here because N is finite. For $N > k$,

$$d_{k,N}/d_{k,N-1} = \frac{(1-\lambda_{k,b}/\lambda_{N,a})}{(1-\lambda_{k,b}/\lambda_{N,b})} > 1$$

since $0 < \lambda_{k,b} < \lambda_{N,b} < \lambda_{N,a}$. Thus, $|d_{k,N}| \uparrow |d_k|$ as $N \uparrow \infty$.

Given $t_0 > 0$ choose $\varepsilon < \frac{1}{2} t_0$. Then from Theorem 4.1(f) and Condition A, there exists a constant C such that for $t > t_0$,

$$|\sum d_{k,N} \exp(-\lambda_{k,b} t)| \leqq \sum |d_k| \exp(-\lambda_{k,b} t_0)$$
$$= \sum \{|u(a, \lambda_{k,b})|/|u'(b, \lambda_{k,b})|\} \exp(-\lambda_{k,b} t_0)$$
$$\leqq C \sum \exp\{-\lambda_{k,b}(t_0 - 2\varepsilon)\} < \infty$$

since $\sum \lambda_{k,b}^{-1} < \infty$. Thus for all $t_0 > 0$,

$$f_N(t) \to f(t) \quad \text{uniformly for } t \geqq t_0, \tag{6.7}$$

by dominated convergence. Let $F_N(t)$ denote the distribution function of $f_N(t)$. Since $\phi_N(\lambda) \to \phi(\lambda)$ we see that $F_N \to F$, for some distribution function F. Further, by (6.7), F has the density $f(t)$ for $t>0$. Finally, note that F has no mass at 0 because for any diffusion hitting time $P(\tau_{ab}=0)=0$. (However, F may have some mass at ∞.)

Remarks. (1) The density $f(t)$ in (6.5) can be integrated term-by-term to give

$$P(t<\tau_{ab}<\infty) = -\sum_{k=1}^{\infty} (d_k/\lambda_{k,b}) \exp(-\lambda_{k,b} t), \qquad t>0.$$

(2) In the limiting case $a \to r_0$, (6.5) still holds; the proof is in fact slightly simpler here. For the m.g.f. $h(\lambda)^{-1} = \prod_{k=1}^{\infty} (1-\lambda/\lambda_{k,b})^{-1}$ of Corollary 5.1 and of (6.1), the coefficients $\{d_k\}$ take the form given in (6.4); namely

$$d_k = \{h'(\lambda_{k,b})\}^{-1} = -\lambda_{k,b} \prod_{\substack{n=1 \\ n \neq k}}^{\infty} (1-\lambda_{k,b}/\lambda_{n,b})^{-1}.$$

7. General Diffusion Hitting Times

If r_0 is a natural boundary then the above eigenvalue expansion does not hold. However, any diffusion hitting time may be approximated by a hitting time for a diffusion with a regular boundary. Since the class of g.c.m.e.d.s is closed under weak limits, it follows that *all diffusion hitting times are g.c.m.e.d.s.*

It was noted by Bondesson (1979) that for diffusion processes of sufficient regularity, this result may be deduced by weak convergence from the corresponding result for birth-death processes. (See also Kielson, 1971, p. 396.)

For a general diffusion hitting time, there is a very close connection between the canonical measure of the hitting time and the *spectral measure* of the differential generator of the diffusion. Details are given in Kent (1979c).

8. Example: the Bessel Process

Consider the Bessel diffusion on $(0, \infty)$ with generator

$$A = \tfrac{1}{2}[d^2/dx^2 + \{(2\nu+1)x^{-1}\}\, d/dx],$$

where ν is a real-valued parameter.

If $q = 2\nu+2$ is a positive integer, then this process represents the radial part of standard Brownian motion in R^q. The speed measure and natural scale are given by

$$m(dx) = 2(\tfrac{1}{2}x)^{2\nu+1} dx, \qquad s(dx) = (\tfrac{1}{2}x)^{-2\nu-1} dx.$$

Table 2. Initial value solutions for the Bessel process

ν	Boundary behaviour at 0	Initial value solution $u(x, \lambda)$
$\nu \geqq 0$	entrance	$u_1(x, \lambda)$
$-1<\nu<0$	instantaneously reflecting	$u_1(x, \lambda)$
$-1<\nu<0$	absorbing regular	$u_2(x, \lambda)$
$\nu \leqq -1$	exit	$u_2(x, \lambda)$

The boundary behaviour at $r_0=0$ for differing values of ν is given in Table 2. For simplicity suppose that when 0 is a reflecting regular boundary, it is made instantaneously reflecting.

Let $J_\nu(x)$ denote the usual Bessel function. Then the initial value solutions $u(x, \lambda)$ of Theorem 4.1 for differing boundary conditions are listed in Table 2. Here

$$u_1(x, \lambda)=\Gamma(\nu+1)[\tfrac{1}{2}x\{2\lambda\}^{\frac{1}{2}}]^{-\nu} J_\nu(x\{2\lambda\}^{\frac{1}{2}}),$$
$$u_2(x, \lambda)=\Gamma(-\nu)[\tfrac{1}{2}x/\{2\lambda\}^{\frac{1}{2}}]^{-\nu} J_{-\nu}(x\{2\lambda\}^{\frac{1}{2}}).$$

Some of these results are included in Kent (1978, 1979a), in terms of Laplace transforms rather than m.g.f.s.

Let $\{j_{\nu,k}\}_{k=1}^{\infty}$ denote the positive zeros of $J_\nu(\cdot)$ for $\nu>-1$. Then from Table 2,

$$\lambda_{k,x}=j_{\nu,k}^2/(2x^2) \tag{8.1}$$

for $-1<\nu<0$ (instantaneously reflecting) and $\nu \geqq 0$ (entrance); and

$$\lambda_{k,x}=j_{-\nu,k}^2/(2x^2) \tag{8.2}$$

for $-1<\nu<0$ (absorbing regular) and $\nu \leqq -1$ (exit).

Setting $x=2^{-\frac{1}{2}}$ in the product formula for $u_1(x, \lambda)$, $\nu>-1$, yields the standard product formula for $J_\nu(\lambda^{\frac{1}{2}})$ (Watson, 1944, p. 498).

It is easily checked that for all boundary conditions the regularity condition (4.1) holds with $\Phi(x)\equiv 1$; hence (4.2) holds, thus verifying the standard formula

$$\lim_{k\to\infty} j_{\nu,k}/k=\pi, \qquad \nu>-1 \tag{8.3}$$

(Watson, 1944, p. 506). In particular, we see that for all boundary conditions, $u(x, \lambda)$ is an entire function of λ of order $\frac{1}{2}$.

When constructing the series expansion of Sect. 6, it is easily checked that Condition A is always satisfied, so that Theorem 6.1 is valid. Formula (6.6) for the coefficients can be simplified by noting

$$u_1'(x, \lambda_{k,x})=-\Gamma(\nu+1)\,x^2 J_{\nu+1}(j_{\nu,k})/\{2(\tfrac{1}{2}j_{\nu,k})^{\nu+1}\},$$
$$u_2'(x, \lambda_{k,x})=-2\Gamma(-\nu)\,x^{2-2\nu}(2j_{-\nu,k})^{\nu-1} J_{-\nu+1}(j_{-\nu,k}).$$

It is interesting to look at the convergence properties of the partial fraction expansion (6.2), with $h(\lambda)=u_i(x, \lambda)$, $i=1,2$. Using (8.3) and the asymptotic

Table 3. Convergence behaviour of the partial fraction expansion (6.2) for the Bessel process

ν	Boundary behaviour at 0	Convergence behaviour
$\nu \geqq \frac{1}{2}$	entrance	divergence
$-\frac{1}{2} \leqq \nu < \frac{1}{2}$	entrance or instantaneously reflecting	conditional convergence
$-1 < \nu < -\frac{1}{2}$	instantaneously reflecting	absolute convergence
$-\frac{1}{2} < \nu < 0$	absorbing regular	conditional convergence
$\nu \leqq -\frac{1}{2}$	absorbing regular or exit	divergence

formula

$$J_{\nu+1}(j_{\nu,k}) = (-1)^{k+1}\{2/(\pi^2 k)\}^{\frac{1}{2}}\{1+O(k^{-1})\} \quad \text{as } k \to \infty$$

for $\nu > -1$ (Watson, 1944, pp. 199, 506), we find that the series (6.2) has the convergence behaviour listed in Table 3. However, note that the series (6.4) *always* converges.

For $\nu = -\frac{1}{2}$, the Bessel process is just 1-dimensional Brownian motion either reflected or absorbed at the origin and the initial value solutions become

$$u_1(x, \lambda) = \cos(\lambda^{\frac{1}{2}} x), \qquad u_2(x, \lambda) = \lambda^{-\frac{1}{2}} \sin(\lambda^{\frac{1}{2}} x).$$

Formula (6.4) for the absorbing process is given in Feller (1966, p. 330).

Some of the above results for $u_1(x, \lambda)$ are included in Ismail and Kelker (1979) and Ismail and Ping (1979), but they mistakenly assert the convergence of the partial fraction expansion (6.2) of $u_1(x, \lambda)^{-1}$ for all $\nu > -1$.

9. Moments

The moments of a diffusion hitting time can be found by differentiating the m.g.f. at the origin. For the hitting times of Theorem 5.1, these moments can also be expressed using the corresponding eigenvalues.

It is easily checked that the elementary m.e.d. with m.g.f. $(1-\lambda/\lambda_1)/(1-\lambda/\lambda_0)$ has mean and variance

$$\frac{\lambda_1 - \lambda_0}{\lambda_1 \lambda_0} \quad \text{and} \quad \frac{\lambda_1^2 - \lambda_0^2}{\lambda_1^2 \lambda_0^2},$$

respectively. Since the hitting time of Theorem 5.1, conditioned to be finite with probability 1, is an infinite convolution of elementary m.e.d.'s with m.g.f. $\prod_{k=1}^{\infty} \{(1-\lambda/\lambda_{k,a})/(1-\lambda/\lambda_{k,b})\}$, it has mean and variance

$$\sum_{k=1}^{\infty} \frac{\lambda_{k,a} - \lambda_{k,b}}{\lambda_{k,a}\lambda_{k,b}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\lambda_{k,a}^2 - \lambda_{k,b}^2}{\lambda_{k,a}^2 \lambda_{k,b}^2}.$$

Further, the mean and variance for the m.g.f. of Corollary 5.1 can be obtained by letting $\lambda_{k,a} \to \infty$, $k = 1, 2, \ldots$, and are given by

$$\sum \lambda_{k,b}^{-1} \quad \text{and} \quad \sum \lambda_{k,b}^{-2}.$$

Similar calculations can be carried out to express the higher moments.

For example, consider either Brownian motion with an absorbing boundary at 0 or the radial part of Brownian motion in R^3. (These are the Bessel processes of parameters $\nu = \mp\frac{1}{2}$). In both cases the m.g.f. of Corollary 5.1 (taking $b=\pi$) is $\phi(\lambda)=\lambda^{\frac{1}{2}}\pi/\sin(\lambda^{\frac{1}{2}}\pi)$ with eigenvalues $\lambda_{k,\pi}=k^2$, $k=1,2,\ldots$. The two methods given here for calculating the mean and variance lead to the well-known formulae

$$\frac{\pi^2}{6}=\sum_{k=1}^{\infty} k^{-2} \quad \text{and} \quad \frac{\pi^4}{90}=\sum_{k=1}^{\infty} k^{-4}.$$

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References

Bondesson, L.: A class of infinitely divisible distributions. Research Report No. 110, University of Stockholm (1978, revised 1979)

Feller, W.: An Introduction to Probability Theory and its Applications. Vol. II. New York: Wiley 1966

Goldie, C.M.: A class of infinitely divisible distributions. Proc. Cambridge Phil. Soc. **63**, 1141–1143 (1967)

Ismail, M.E.H., Kelker, D.: Special functions, Stieltjes transforms and infinite divisibility. SIAM J. Math. Anal. **10**, 884–901 (1979)

Ismail, M.E.H., May, C.P.: Special functions, infinite divisibility, and transcendental equations. Math. Proc. Cambridge Phil. Soc. **85**, 453–464 (1979)

Itô, K., McKean, H.P., Jr.: Diffusion Processes and their Sample Paths. Berlin-Heidelberg-New York: Springer 1965

Karlin, S.: Total Positivity. Stanford University Press 1968

Kent, J.T.: Some probabilistic properties of Bessel functions. Ann Probability **6**, 760–770 (1978)

Kent, J.T.: Discussion following Steutel, F.W., Infinite divisibility in theory and practice. Scand. J. Statist. **6**, 57–64 (1979a)

Kent, J.T.: Eigenvalue expansions for diffusion hitting times, Research Report No. 3, Dept. of Statistics, University of Leeds (1979b)

Kent, J.T.: The spectral decomposition of a diffusion hitting time. Research Report No. 5, Dept. of Statistics, University of Leeds (1979c)

Kielson, J.: Log-concavity and log-convexity in passage time densities of diffusion and birth-death processes. J. Appl. Probability **8**, 391–398 (1971)

McKean, H.P., Jr.: Elementary solutions for certain parabolic partial differential equations. Trans. Amer. Math. Soc. **82**, 519–548 (1956)

McKean, H.P., Jr., Ray, D.B.: Spectral distribution of a differential operator. Duke Math. J. **29**, 281–292 (1962)

Mandl, P.: One-dimensional Markov Processes. Berlin-Heidelberg-New York: Springer 1968

Rosenlund, S.I.: Upwards passage times in the non-negative birth-death process. Scand. J. Statist. **4**, 90–92 (1979)

Steutel, F.W.: Note on the infinite divisibility of exponential mixtures. Ann. Math. Statist. **38**, 1303–1305 (1967)

Titchmarsh, E.C.: The Theory of Functions, second edition. Oxford: University Press 1939

Titchmarsh, E.C.: Eigenfunction Expansions Associated with Second-order Differential Equations, Part I, second edition. Oxford: University Press 1962

Watson, G.N.: A Treatise on the Theory of Bessel Functions, second edition. Cambridge: University Press (1944)

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