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On Necessary and Sufficient Conditions for an Infinitely Divisible Distribution to be Normal or Degenerate

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Introduction

Recently, Ruegg [4] has given elegant generalizations of the familiar result that a non-degenerate infinitely divisible probability measure on the line cannot have compact support (vid. [2], p. 174). Using the theory of entire functions and deep results about entire characteristic functions, he has shown that certain asymptotic conditions on the tail of an infinitely divisible distribution are sufficient to force it to be either normal or degenerate. In this note we obtain conditions which are both necessary and sufficient, and we do so with quite elementary methods, viz., the Lévy-Khinchine representation formula and an elementary inequality for Laplace transforms. Our principal result is

Theorem 1. *Let $F(x)$ denote the cumulative distribution function of a probability measure on the line, so that $1-F(x)+F(-x)=O(\exp(-xM(x)))$ as $x\to\infty$, where $M(x)$ is a non-negative measurable function. Assume that the measure is infinitely divisible.*

(a) *The measure is normal (possible degenerate) if and only if $M(x)$ can be chosen such that $M(x)/\ln x\to\infty$ as $x\to\infty$ and such that $M(x)$ is continuous and strictly monotone increasing for all sufficiently large x.*

(b) *The measure is degenerate if and only if, in addition to the conditions in* (a), *$M(x)/x\to\infty$ as $x\to\infty$.*

In this context we can summarize Ruegg's results very simply: he drew the appropriate conclusion about the distribution in the special cases $M(x)=a\,x^{\alpha}$ and $M(x)=a(\ln x)^{\alpha}$ where a, $\alpha>0$.

With the elementary methods we develop to prove this theorem we are able to obtain certain one-sided results as well. One result leads to a sufficient condition for an infinitely divisible random variable to be bounded from above or below, while in the spirit of [5] another result gives a sharp necessary condition on the one-sided asymptotic behavior of an infinitely divisible distribution.

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A Lower Bound

The characteristic function ϕ of an infinitely divisible cumulative distribution function F can be represented uniquely by the Lévy-Khinchine formula ([3], p. 76)

$$\begin{aligned}\phi(t) &= \int_{-\infty}^{\infty} e^{ixt}\, dF(x) \\ &= \exp\left[-i\alpha t - \beta^2 t^2 + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) x^{-2}(1+x^2)\, d\mu(x)\right]\end{aligned} \tag{1}$$

where α, β, and t are real and $d\mu$ is a non-negative bounded measure with no point mass at zero. The distribution F will be normal or degenerate if and only if $d\mu \equiv 0$; it will be degenerate if and only if $\beta = 0$ as well.

Under suitable conditions on the rate at which $F(-x) \to 0$ or $1 - F(x) \to 0$ as $x \to \infty$ the function ϕ will be analytic in the upper or lower half plane (or both) and hence in the appropriate domain the Laplace transform of F will converge and will have the unique representation

$$\begin{aligned}\phi(iy) &= \int_{-\infty}^{\infty} e^{-xy}\, dF(x) \\ &= \exp\left[\alpha y + \beta^2 y^2 + \int_{-\infty}^{\infty} \left(e^{-xy} - 1 + \frac{xy}{1+x^2}\right) x^{-2}(1+x^2)\, d\mu(x)\right]\end{aligned} \tag{2}$$

with α, β, $d\mu$ as in (1) and y real. For our purposes it will be convenient to utilize a device of [5] and to modify this formula as follows: for $\varepsilon > 0$ divide the domain of integration in (2) into the interval $[-\varepsilon, \varepsilon]$ and its complement and in the former utilize the identity $xy(1+x^2)^{-1} = xy - x^3 y(1+x^2)^{-1}$. If we define

$$A_\varepsilon \equiv -\int_{|x|>\varepsilon} x^{-2}(1+x^2)\, d\mu(x)$$

$$B_\varepsilon \equiv \alpha - \int_{-\varepsilon}^{\varepsilon} x\, d\mu(x) + \int_{|x|>\varepsilon} x^{-1}\, d\mu(x)$$

$$\Psi_\varepsilon(y) \equiv \int_{-\varepsilon}^{\varepsilon} (e^{-xy} - 1 + xy)\, x^{-2}(1+x^2)\, d\mu(x),$$

then formula (2) may be written in the form

$$\phi(iy) = \exp\left[A_\varepsilon + B_\varepsilon y + \beta^2 y^2 + \Psi_\varepsilon(y) + \int_{|x|>\varepsilon} e^{-xy} x^{-2}(1+x^2)\, d\mu(x)\right], \tag{3}$$

but since $\Psi_\varepsilon(y) \geqq 0$ for all $\varepsilon > 0$ and all real y we obtain the inequality

$$\phi(iy) \geqq \exp\left[A_\varepsilon + B_\varepsilon y + \beta^2 y^2 + \int_{|x|>\varepsilon} e^{-xy} x^{-2}(1+x^2)\, d\mu(x)\right]. \tag{4}$$

Now let $c < 0$ and choose $0 < \varepsilon < |c|$. If $d\mu$ has mass $m > 0$ in the interval $(-\infty, c)$ then for all $y \geqq 0$ we derive from (4) the lower bound

$$\begin{aligned}\phi(iy) &\geqq \exp\left[A_\varepsilon + B_\varepsilon y + \beta^2 y^2 + \int_{-\infty}^{c} e^{-xy} x^{-2}(1+x^2)\, d\mu(x)\right] \\ &\geqq \exp\left[A_\varepsilon + B_\varepsilon y + \beta^2 y^2 + m c^{-2}(1+c^2)\, e^{-cy}\right].\end{aligned}$$

Similarly, the latter inequality holds for all $y \leqq 0$ if $d\mu$ has mass $m>0$ in (c, ∞) for some $c>0$. Thus, if $d\mu$ has positive mass in the interval $(-\infty, c)$ for some $c<0$ [in the interval (c, ∞) for some $c>0$], then there exist real numbers A, B, γ with $\gamma \neq 0$ such that

$$\phi(i y) \geqq \exp[A+B y+\beta^2 y^2+\gamma^2 e^{|c y|}] \tag{5}$$

for all $y \geqq 0$ [all $y \leqq 0$].

Using this simple estimate, we conclude that if $a \geqq 0$ and if $e^{-a^2 y^2} \phi(i y)$ remains bounded as $y \to \infty$, then $d\mu$ cannot have positive mass on $(-\infty, c)$ for any $c<0$ and, furthermore, $\beta^2 > a^2$ is impossible. By the same reasoning, if $\exp(-e^{a y}) \phi(i y)$ remains bounded as $y \to \infty$, then $d\mu$ cannot have positive mass on $(-\infty, -a)$. Similar considerations using (5) as $y \to -\infty$ complete the proof of

Proposition 1. *Let ϕ denote the characteristic function of an infinitely divisible probability measure and let ϕ be represented by the Lévy-Khinchine formula* (1).

(i) *Suppose $\phi(z)$ is analytic in the upper [lower] half plane and let $a \geqq 0$. If $\phi(i y)=\mathrm{O}(e^{a^2 y^2})$ as $y \to \infty$ [as $y \to -\infty$] then $d\mu$ has no mass in $(-\infty, 0)$ [in $(0, \infty)$] and $\beta^2 \leqq a^2$. If $\phi(i y)=\mathrm{O}(\exp(e^{a|y|}))$ as $y \to \infty$ [as $y \to -\infty$], then $d\mu$ has no mass in $(-\infty, -a)$ [in (a, ∞)].*

(ii) *Suppose $\phi(z)$ is an entire function. If $\phi(i y)=\mathrm{O}(e^{a^2 y^2})$ as $y \to \pm\infty$ for all $a>0$ then $d\mu \equiv 0$ and $\beta=0$, i.e., the original distribution is degenerate. If $\phi(i y)=\mathrm{O}(\exp(e^{a|y|}))$ as $y \to \pm\infty$ for all $a>0$ then $d\mu \equiv 0$, i.e., the original distribution is either normal or degenerate.*

An Upper Bound

Now let $\phi(z)=\int_{-\infty}^{\infty} e^{izx}\, dF(x)$ denote the characteristic function of a cumulative distribution function F. It is well known that this integral is convergent for all z in the upper half plane if and only if $F(-x)=\mathrm{O}(e^{-ax})$ as $x \to \infty$ for all $a>0$ ([7], pp. 39–40 and 237–240). Under this condition, integration by parts is permissible and for $y>0$ we obtain

$$\begin{aligned} 0 \leqq \phi(i y) &= \int_{-\infty}^{\infty} e^{-yx}\, dF(x) = y \int_{-\infty}^{\infty} e^{-yx} F(x)\, dx \\ &= y \int_0^{\infty} [e^{yx} F(-x) + e^{-yx} F(x)]\, dx \\ &\leqq y \int_0^{\infty} [e^{yx} F(-x) + e^{-yx}]\, dx = 1 + y \int_0^{\infty} e^{yx} F(-x)\, dx. \end{aligned} \tag{6}$$

Similar considerations lead to a similar inequality in the lower half plane. If we introduce the tail probability function $T(x) \equiv 1 - F(x) + F(-x)$ we can unite these results in

Proposition 2. *Let ϕ denote the characteristic function of a probability measure on the line with cumulative distribution function F.*

(i) *The function* $\phi(z)$ *is analytic in the upper half plane if and only if* $F(-x)=O(e^{-ax})$ *as* $x\to\infty$ *for all* $a>0$, *and in this event*

$$0\leqq\phi(iy)\leqq 1+y\int_0^\infty e^{yx}F(-x)\,dx$$

for all $y\geqq 0$. *Similarly,* $\phi(z)$ *is analytic in the lower half plane if and only if* $1-F(x)=O(e^{-ax})$ *as* $x\to\infty$ *for all* $a>0$, *and in this event*

$$0\leqq\phi(iy)\leqq 1-y\int_0^\infty e^{-yx}[1-F(x)]\,dx$$

for all $y\leqq 0$.

(ii) *The function* $\phi(z)$ *is entire if and only if* $T(x)=O(e^{-ax})$ *as* $x\to\infty$ *for all* $a>0$, *and in this event*

$$0\leqq\phi(iy)\leqq 1+|y|\int_0^\infty e^{|y|x}T(x)\,dx \tag{7}$$

for all real y.

Proof of Theorem 1

We shall use the upper bound (7) to deduce the theorem via the asymptotic conditions given in Proposition 1 (ii), but in order to do so we need the following simple inequality for Laplace transforms.

Lemma 1. *Let* $M(x)$ *be a non-negative measurable function on* $[0,\infty)$ *such that* $M(x)$ *is continuous and strictly monotone increasing on* $[A,\infty)$ *for some* $A>0$ *and such that* $M(x)\to\infty$ *as* $x\to\infty$. *Then for all* $y>M(A)$,

$$\int_0^\infty e^{yx-xM(x)}\,dx\leqq 1+y^{-1}e^{yM^{-1}(y+1)}.$$

Proof. The inverse function M^{-1} is well defined on $[M(A),\infty)$ and one computes

$$\begin{aligned}\int_0^\infty e^{yx-xM(x)}\,dx&=\int_0^{M^{-1}(y+1)}e^{-x(M(x)-y)}\,dx+\int_{M^{-1}(y+1)}^\infty e^{-x(M(x)-y)}\,dx\\&\leqq\int_0^{M^{-1}(y+1)}e^{xy}\,dx+\int_{M^{-1}(y+1)}^\infty e^{-x}\,dx=y^{-1}[e^{yM^{-1}(y+1)}-1]+e^{-M^{-1}(y+1)}\\&\leqq 1+y^{-1}e^{yM^{-1}(y+1)}.\quad\text{q.e.d.}\end{aligned}$$

Corollary 1. *Let* $M(x)$ *be as in Lemma* 1 *but assume that* $M(x)/x\to\infty$ *as* $x\to\infty$. *Then*

$$y\int_0^\infty e^{xy-xM(x)}\,dx=O(e^{ay^2})\qquad\text{as } y\to\infty$$

for all $a>0$.

Proof. Using the lemma, we have for all large y

$$y\,e^{-ay^2}\int_0^\infty e^{xy-xM(x)}\,dx\leqq y\,e^{-ay^2}+e^{-y[ay-M^{-1}(y+1)]},$$

so it is sufficient to have $ay\geqq M^{-1}(y+1)$ for all sufficiently large y for each $a>0$. Since $M(x)$ is monotone increasing this would follow from the inequality $M(ay)\geqq y+1$, but $M(ay)/y\to\infty$ as $y\to\infty$ and so this is surely the case.

Corollary 2. *Let $M(x)$ be as in Lemma 1 but assume that $M(x)/\ln x \to \infty$ as $x \to \infty$. Then*

$$y \int_0^\infty e^{xy - xM(x)}\, dx = \mathrm{O}(\exp(e^{ay})) \qquad \text{as } y \to \infty$$

for all $a > 0$.

Proof. Using the lemma, we have for all large y

$$y \exp(-e^{ay}) \int_0^\infty e^{xy - xM(x)}\, dx \leqq y \exp(-e^{ay}) + \exp[-y(y^{-1} e^{ay} - M^{-1}(y+1))]$$

so it is sufficient to have $y^{-1} e^{ay} \geqq M^{-1}(y+1)$ for all sufficiently large y for each $a > 0$. As before, this would follow from $M(y^{-1} e^{ay}) \geqq y + 1$. But $\frac{1}{2} a y > \ln y$ for all sufficiently large y and hence

$$M(y^{-1} e^{ay})/y \geqq \tfrac{1}{2} a [M(y^{-1} e^{ay})/\ln(y^{-1} e^{ay})] \to \infty$$

as $y \to \infty$. q.e.d.

We can now prove Theorem 1 easily. The asymptotic condition in (b) of the theorem ensures that F has an entire characteristic function ϕ by the first part of Proposition 2 (ii). If we successively apply Proposition 2 (ii), the hypothesis on $T(x)$, and Corollary 1, we find that

$$\begin{aligned} \phi(iy) &\leqq 1 + |y| \int_0^\infty e^{x|y|}\, T(x)\, dx \\ &= 1 + \mathrm{O}\left[|y| \int_0^\infty e^{x|y| - xM(x)}\, dx\right] \\ &= \mathrm{O}(e^{ay^2}) \end{aligned}$$

as $|y| \to \infty$ for all $a > 0$. Finally, Proposition 1(ii) shows that F must be degenerate. The sufficiency of the asymptotic condition in (a) of the theorem is established by parallel reasoning using Corollary 2.

The necessity of these asymptotic conditions is proved in each case by direct computation. If F is degenerate then $T(x) \equiv 0$ for all large x and hence $T(x) = \mathrm{O}(e^{-x^3})$. But then $M(x) = x^2$ meets all the required conditions and $M(x)/x \to \infty$ as $x \to \infty$. In the normal case, a simple change of variable shows that it suffices to prove that

$$\int_x^\infty e^{-t^2}\, dt = \mathrm{O}(e^{-xM(x)})$$

for some function $M(x)$ meeting the required conditions. If we set

$$M(x) \equiv \max\left[-x^{-1} \ln \int_x^\infty e^{-t^2}\, dt, 0\right], \qquad x > 0$$

then we shall be done if we can show that $M(x)/\ln x \to \infty$ as $x \to \infty$ and that $M'(x) > 0$ for all sufficiently large x. Both of these statements follow from repeated applications of de l'Hôpital's rule for indeterminant limits. The computations are straightforward but tedious, and we content ourselves with the following sketch of the steps involved: First show that $x^{-2}\ln\int_x^\infty e^{-t^2}\,dt\to -1$ as $x\to\infty$ and deduce from this that $M(x)/\ln x\to\infty$ and $M(x)\to\infty$ as $x\to\infty$. Now use this to show that the derivative of the numerator of $M'(x)$ is ultimately strictly negative. But since the numerator of $M'(x)$ tends to zero as $x\to\infty$ and the denominator is always positive, one concludes that ultimately $M'(x)>0$. The proof of the theorem is complete.

Since the asymptotic conditions in (a) and (b) are necessary and sufficient conditions, it follows that an infinitely divisible distribution is a non-degenerate normal distribution if and only if (a) is satisfied but (b) is not. Thus, within the family of non-degenerate infinitely divisible distributions the normal distributions are characterized as those whose tail probabilities decrease most rapidly.

Random Variables Bounded from above or below

The half of Theorem 1 dealing with degenerate laws says, in effect, that if $T(x)\to 0$ fast enough then $T(x)\equiv 0$ for all large x. If part (i) of Propositions 1 and 2 is used in the same way as part (ii) was used in the above proof, one sided versions of our theorem result. For notation, we refer to the Lévy-Khinchine formula (1).

Theorem 2. *Let $F(x)$ denote the cumulative distribution function of a probability measure on the line, so that $F(-x)=O(\exp(-xM(x)))$ as $x\to\infty$, where $M(x)$ is a non-negative measurable function. Assume that the measure is infinitely divisible.*

(a) *If $M(x)$ can be chosen such that $M(x)/\ln x\to\infty$ as $x\to\infty$ and such that it is continuous and strictly monotone increasing for all sufficiently large x, then the measure $d\mu$ has no mass in $(-\infty,0)$.*

(b) *If, in addition to the conditions in* (a), *$M(x)/x\to\infty$ as $x\to\infty$, then $\beta=0$ and the measure $d\mu$ has no mass in $(-\infty,0)$.*

There is of course a similar result involving the asymptotic properties of $1-F(x)$ and the mass of $d\mu$ on $(0,\infty)$.

In the spirit of our introductory remarks for this section, it is interesting to express these results about the Lévy-Khinchine measure $d\mu$ in terms of the original distribution F. Baxter and Shapiro [1] have given necessary and sufficient conditions for an infinitely divisible random variable to be bounded from below in terms of the quantities β and $d\mu$ in (1). From their Theorem 2 one obtains easily the following

Lemma 2. *Let an infinitely divisible probability measure on the line have cumulative distribution function F and let its characteristic function ϕ have the Lévy-Khinchine representation* (1). *Assume that $d\mu$ has no mass in $(-\infty,0)$, that $\beta=0$, and that $\int_{+0}^{1}x^{-1}\,d\mu(x)<\infty$. Then*

$$F(-x)\equiv 0 \quad \text{if} \quad x>\alpha+\int_0^\infty x^{-1}\,d\mu(x).$$

When this is united with the above remarks we obtain

Theorem 3. *Let $F(x)$ denote the cumulative distribution function of an infinitely divisible probability measure on the line whose characteristic function has the Lévy-Khinchine representation* (1). *Assume that $F(-x)=O(\exp(-x M(x)))$ as $x\to\infty$ where $M(x)$ is a non-negative measurable function.*

If (i) $\int_{+0}^{1} x^{-1}\, d\mu(x)<\infty$, *and*

(ii) *$M(x)$ may be chosen to be continuous and strictly monotone increasing for all sufficiently large x and such that $M(x)/x\to\infty$ as $x\to\infty$,*

then $F(-x)\equiv 0$ for all sufficiently large x.

Thus, under suitable conditions we see that if $F(-x)\to 0$ fast enough as $x\to\infty$ then ultimately $F(-x)\equiv 0$, and we can even give an explicit estimate for the support of F. This should be compared with the familiar situation for non-negative random variables discussed in [2], p. 539. There is, of course, a similar theorem for random variables which are bounded from above. Finally we remark that if (ii) is weakened to require that $M(x)/\ln x\to\infty$ as $x\to\infty$, then we can conclude from (a) of Theorem 2 that $F=F_1 * F_2$ where $F_1(-x)\equiv 0$ for all sufficiently large x and F_2 is a normal distribution.

A One-Sided Necessary Condition

Although Theorem 3 gives a precise sufficient condition, hypothesis (i) makes it impossible to apply if one has knowledge only of the asymptotic behavior of the distribution F. However, this behavior does contain additional useful information and we shall show, roughly, that if $F(-x)\to 0$ faster than $e^{-x\ln x}$, then it must actually decrease as fast as e^{-x^2}. The gap between these two rates thus excludes certain types of one sided asymptotic behavior for unbounded infinitely divisible distributions. We shall need a simple lower bound for Laplace transforms in the same spirit as Lemma 1.

Lemma 3. *Let $m(x)$ be a non-negative measurable function on $[0,\infty)$ such that $m(x)$ is continuous and strictly monotone increasing on $[A,\infty)$ for some $A>0$ and such that $m(x)\to\infty$ as $x\to\infty$. Then there exists some $\theta>0$ such that for all $y>3m(A)$,*

$$\int_0^\infty e^{yx-xm(x)}\,dx\geqq \theta\, y^{-1}\, e^{\frac{1}{2}y m^{-1}(\frac{1}{2}y)}.$$

Proof. The inverse function m^{-1} is well defined on $[m(A),\infty)$ and one computes for $y>3m(A)$

$$\begin{aligned}\int_0^\infty e^{yx-xm(x)}\,dx &\geqq \int_A^{m^{-1}(\frac{1}{2}y)} e^{yx-xm(x)}\,dx \geqq \int_A^{m^{-1}(\frac{1}{2}y)} e^{\frac{1}{2}yx}\,dx\\ &=2y^{-1}\left[e^{\frac{1}{2}ym^{-1}(\frac{1}{2}y)}-e^{\frac{1}{2}yA}\right]\\ &\geqq 2y^{-1}\left[e^{\frac{1}{2}ym^{-1}(\frac{1}{2}y)}-\rho\, e^{\frac{1}{2}ym^{-1}(\frac{1}{2}y)}\right]\\ &=2(1-\rho)\,y^{-1}\,e^{\frac{1}{2}ym^{-1}(\frac{1}{2}y)},\end{aligned}$$

where $\rho=\exp\{\frac{3}{2}m(A)[A-m^{-1}(\frac{3}{2}m(A))]\}<1$ and one may set $\theta=2(1-\rho)>0$. q.e.d.

Corollary 3. *Let* $m(x)$ *be as in Lemma* 3 *but assume that* $\liminf m(x)/x=0$ *as* $x\to\infty$. *Then for all* $K>0$,

$$\limsup \left\{e^{-Ky^2}\, y\int_0^\infty e^{yx-xm(x)}\,dx\right\}=\infty \qquad as\ y\to\infty.$$

In particular, $y\int_0^\infty e^{yx-xm(x)}\,dx\neq O(e^{Ky^2})$ *for any* $K>0$ *as* $y\to\infty$.

Proof. Using the lemma, for all large y we have

$$e^{-Ky^2}\, y\int_0^\infty e^{yx-xm(x)}\,dx\geqq\theta\exp[-Ky^2+\tfrac{1}{2}y\,m^{-1}(\tfrac{1}{2}y)]$$

$$=\theta\exp\left\{\frac{1}{4}y^2\left[\frac{m^{-1}(\frac{1}{2}y)}{\frac{1}{2}y}-4K\right]\right\}.$$

But if $\liminf m(x)/x=0$ as $x\to\infty$ then $\limsup m^{-1}(\frac{1}{2}y)/\frac{1}{2}y=\infty$ and we are done.

Theorem 4. *Let* $F(x)$ *denote the cumulative distribution function of an infinitely divisible probability measure on the line. Assume that there are positive constants* C_1, C_2 *such that*

$$C_1\,e^{-xm(x)}\leqq F(-x)\leqq C_2\,e^{-xM(x)}$$

for all $x\geqq 0$, *where* $m(x)$ *and* $M(x)$ *are non-negative measurable functions which are continuous and strictly monotone increasing for all sufficiently large x. If* $\lim M(x)/\ln x=\infty$ *as* $x\to\infty$, *then necessarily* $\liminf m(x)/x>0$.

Proof. Let the characteristic function ϕ of F have the representation (1). If $\lim M(x)/\ln x=\infty$ as $x\to\infty$ then we conclude from Theorem 2(a) that the Lévy-Khinchine measure $d\mu$ has no mass in $(-\infty,0)$ and hence from (3) we have for $y>0$ and $\varepsilon=1$

$$\phi(iy)=\exp\left[A_1+B_1\,y+\beta^2\,y^2+\int_0^1(e^{-xy}-1+xy)\,x^{-2}(1+x^2)\,d\mu(x)\right.$$

$$\left.+\int_1^\infty e^{-xy}\,x^{-2}(1+x^2)\,d\mu(x)\right]$$

$$\leqq\exp\left[A_1+2\int_1^\infty d\mu(x)+B_1\,y+y^2\left(\beta^2+\tfrac{1}{2}\int_0^1(1+x^2)\,d\mu(x)\right)\right].$$

In particular, $\phi(iy)=O(e^{Ky^2})$ for some $K>0$ as $y\to\infty$. But for $y>0$ we have from (6) the lower bound

$$\phi(iy)\geqq y\int_0^\infty e^{xy}\,F(-x)\,dx\geqq C_1\,y\int_0^\infty e^{xy-xm(x)}\,dx,$$

and hence by Corollary 3 it is impossible to have $\phi(iy)=O(e^{Ky^2})$ for any $K>0$ if $\liminf m(x)/x=0$.

Thus, if $F(x)$ is an infinitely divisible distribution function and $F(-x)\sim a\exp(-b\,x^{1+\varepsilon})$, with $a, b>0$ then if $\varepsilon>0$ it must be at least 1; $0<\varepsilon<1$ is not possible. Also, $F(-x)\sim a\exp(-b\,x(\ln x)^\alpha)$ is not possible for any $\alpha>1$, nor is $F(-x)\sim a\exp(-b\,x\ln x\ln\ln x)$. There is, of course, a similar theorem restricting the asymptotic behavior of $1-F(x)$ as $x\to\infty$. These results generalize and simplify results of Ruegg [5] which he obtained with deeper function-theoretic methods.

Notice that Theorem 4 requires information about the asymptotic behavior of $F(-x)$ (or of $1-F(x)$) alone and disregards the behavior of $1-F(x)$ (or of $F(-x)$) entirely. On the other hand, Theorem 1 requires a simultaneous estimate of both these quantities. It is worth observing that Theorem 2 leads easily to a useful intermediate result.

Theorem 5. *Let $F(x)$ be the cumulative distribution function of an infinitely divisible probability measure on the line. Assume that $F(-x)=O(\exp(-xM(x)))$ and $1-F(x)=O(\exp(-xN(x)))$ as $x\to\infty$, where $M(x)$ and $N(x)$ are non-negative measurable functions which are continuous and strictly monotone increasing for all sufficiently large x. Finally, assume that $N(x)/\ln x\to\infty$ as $x\to\infty$. Then the measure is degenerate if and only if $M(x)$ can be chosen such that $M(x)/x\to\infty$ as $x\to\infty$.*

In particular, if $F(x)$ is the cumulative distribution function of a non-negative non-degenerate infinitely divisible random variable then $F(x)\equiv 0$ for all $x<0$ and the above assumption about $M(x)$ is trivially satisfied. In this event it is not possible to have $1-F(x)=O(\exp(-xN(x)))$ with $N(x)/\ln x\to\infty$ as $x\to\infty$. In the special case of a discrete distribution with $N(x)=x^{\varepsilon}$, $\varepsilon>0$, this result was first announced by Steutel in his thesis [6]; his methods are quite different from ours.

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