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Stability diagrams for coupled Mathieu-equations

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Summary: A method is described for determining stability diagrams for coupled Mathieu-equations. The boundary curves are found by searching for those almost-periodic solutions of the differential equations for which the real part of the characteristic exponent changes from zero to a non-zero value. The method derives analytic expressions for the involved determinants and is able to find the transition curves even for parameters that cannot be considered small.

Stabilitätsdiagramme für gekoppelte Mathieu-Gleichungen

Übersicht: Es wird eine Methode zur Ermittlung der Stabilitätsdiagramme für gekoppelte Mathieu-Gleichungen beschrieben. Die Grenzkurven der Stabilitätsbereiche werden erhalten, indem man nach jenen fastperiodischen Lösungen der Differentialgleichungen sucht, für die der Realteil des charakteristischen Exponenten von Null zu einem nichtverschwindenden Wert wechselt. Analytische Ausdrücke für die auftretenden Determinanten werden hergeleitet. Die Methode kann auch bei Parametern angewendet werden, die nicht klein sind.

1 Introduction

The stability problem for the n coupled Mathieu-equations

$$\mathbf{y}'' + (\mathbf{A}^* + 2q\mathbf{Q}^* \cos \omega t)\, \mathbf{y} = \mathbf{0} \tag{1}$$

has been studied by many authors, Cesari [1], Bolotin [2], Hsu [3—4], Fu and Nemat-Nasser [5—7]. In (1) $\mathbf{A}^*$ is the $n \times n$ stiffness matrix, $2q\mathbf{Q}^* \cos \omega t$ the parametric excitation matrix with q giving the excitation level, and $(\ )'$ is $\mathrm{d}(\ )/\mathrm{d}t$. It has been shown theoretically, [3, 7], and experimentally, Yamamoto and Saito [8], that instability may occur for ω near twice the natural frequencies and their subharmonics, $2\omega_j/s$, $j = 1, 2, \ldots, n$, s integer, and also close to the so-called combination frequencies and their subharmonics, $|\pm\omega_j \pm \omega_k|/s$, $j, k = 1, 2, \ldots, n$, $j \neq k$. The squared frequencies ω_j^2 are the eigenvalues of $\mathbf{A}^*$. The boundary curves for the instability regions must therefore emerge from some of these frequencies. We also know that for the instability domains corresponding to the natural frequencies, the boundary curves can be found by searching for the periodic solutions of (1) and that on the boundary curves corresponding to the combination frequencies we have almost-periodic solutions, [7]. These solutions may, however, also exist elsewhere in the parameter plane, Lindh and Likins [9]. So even though we are able to determine almost-periodic solutions to (1), e.g., by the use of infinite determinants, we cannot be certain that all these solutions will determine a stability boundary. In other words, we can give necessary conditions for being on the stability curves, but it is more difficult to give the sufficient conditions. In the literature, this is only carried out with very few terms of either infinite determinants, [7], Meirovitch and Wallace [10], or perturbation expansions, Nayfeh and Mook [11], since it leads to very cumbersome calculations. Furthermore it can be very difficult to determine the "right" number of terms to include in this type of computation, as shown in

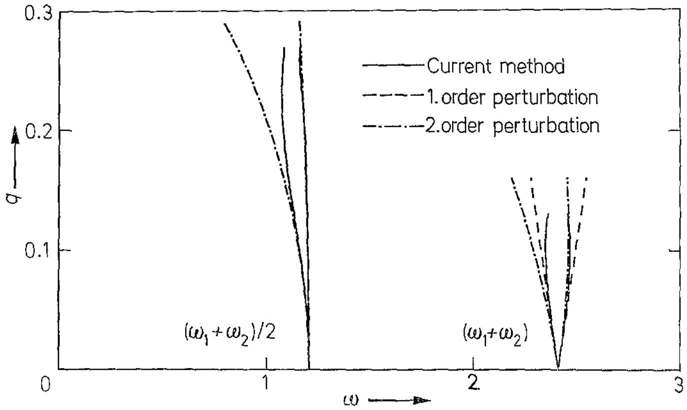

Fig. 1. Curves for the unstable domains corresponding to $\omega = (\omega_1 + \omega_2)/s$, $s = 1, 2$ generated by the method to be described and first and second order perturbation method as described in [11]

Fig. 1. This figure is based on (25) and clearly illustrates some of the disadvantages of the perturbation methods. We see that including first order terms will give part of the curves corresponding to the combination frequency $\omega_1 + \omega_2$, whereas no indication of the curves belonging to $(\omega_1 + \omega_2)/2$ is found. On the other hand, taking second order terms into account will give us part of these latter curves, but will return a poorer estimate for the curves corresponding to the main combination frequency. Also, these methods are only applicable for parameters that can be considered small.

In this paper, we shall use a method that involves infinite determinants, but employ computerised symbolic manipulation, in a manner similar to that used in Pedersen [12], to rewrite the determinants as polynomials in the desired variables. We can thereby find zeros for the determinants in a simple and fast way by means of the Newton-Raphson method. This we use together with a condition on the characteristic exponent for the solutions on the boundary curves. With these tools we can make a computer code that can determine a boundary curve from the frequency found analytically, combination as well as natural, and as far up in the parameter plane as we desire. Thus we shall be able to generate stability diagrams that are not confined to small parameters only.

We will present stability diagrams for two coupled Mathieu-equations and verify them, both by direct numeric simulation as well as combined with Floquet-theory.

2 The infinite determinant

We look at coupled Mathieu-equations in the form of (1). By substituting

$$\omega t = 2\tau \tag{2}$$

we transform (1) into the standard form

$$\omega^2 \ddot{\mathbf{y}} + (\mathbf{A} + 2q\mathbf{Q} \cos 2\tau)\, \mathbf{y} = \mathbf{0} \tag{3}$$

with

$$\mathbf{A} = 4\mathbf{A}^* \quad \text{and} \quad \mathbf{Q} = 4\mathbf{Q}^* \tag{4}$$

and $(\dot{\ })$ as $\mathrm{d}(\ )/\mathrm{d}\tau$. This is the system for which we want to find stability diagrams.

According to Floquet-theory, [9], the solutions of (3) can be written as

$$\mathbf{y} = \mathrm{e}^{h\tau} \boldsymbol{\phi}(\tau), \tag{5}$$

where $\boldsymbol{\phi}(\tau)$ is π-periodic,

$$\boldsymbol{\phi}(\tau+\pi)=\boldsymbol{\phi}(\tau) \tag{6}$$

and h is a complex quantity,

$$h=\alpha+\mathrm{i}\beta. \tag{7}$$

Eq. (5) shows us that we must have Re (h) greater than zero in order to make $\mathbf{y}$ unstable and, since we have no damping, Re (h) equal to zero in the stable domains. This fact will be used to actually determine the boundary curves.

To proceed from (5) we make the classical expansion, [9], of $\boldsymbol{\phi}(\tau)$ in its Fourier-series to obtain

$$\boldsymbol{\phi}(\tau)=\sum_{j=1}^{\infty}\mathbf{b}_j v_j, \qquad v_j=\begin{cases}1 & j=1\\ \sin\dfrac{j}{2}\tau & j=2,4,\ldots\\ \cos\dfrac{j-1}{2}\tau & j=3,5,\ldots\end{cases} \tag{8}$$

Combining (3), (5) and (8) brings us to

$$\mathrm{e}^{h\tau}\sum_{j=1}^{\infty}\left(\omega^2\mathbf{I}\ddot{v}_j+2h\omega^2\mathbf{I}\dot{v}_j+(\omega^2h^2\mathbf{I}+\mathbf{A}+2q\mathbf{Q}\cos 2\tau)\,v_j\right)\mathbf{b}_j=\mathbf{0}, \tag{9}$$

where $\mathbf{I}$ is the identity matrix. Cutting after N terms and dividing by $\mathrm{e}^{h\tau}$ gives

$$\sum_{j=1}^{N}\left(\omega^2\mathbf{I}\ddot{v}_j+2h\omega^2\mathbf{I}\dot{v}_j+(\omega^2h^2\mathbf{I}+\mathbf{A}+2q\mathbf{Q}\cos 2\tau)\,v_j\right)\mathbf{b}_j=\mathbf{0}. \tag{10}$$

We insert the v_j's from (8) and use the relations

$$\dot{v}_j=\begin{cases}0 & j=1\\ \dfrac{j}{2}\,v_{j+1} & j=2,4,\ldots;\\ -\dfrac{j-1}{2}\,v_{j-1} & j=3,5,\ldots\end{cases} \qquad \ddot{v}_j=\begin{cases}0 & j=1\\ -\left(\dfrac{j}{2}\right)^2 v_j & j=2,4,\ldots\\ -\left(\dfrac{j-1}{2}\right)^2 v_j & j=3,5,\ldots\end{cases} \tag{11}$$

and

$$v_j\cos 2\tau=\begin{cases}v_{j+4} & j=1\\ \dfrac{1}{2}\,(v_{j+4}-v_j) & j=2\\ \dfrac{1}{2}\,(v_{j+4}+v_j) & j=3\\ \dfrac{1}{2}\,v_{j+4} & j=4\\ \dfrac{1}{2}\,(v_{j+4}+v_{j-4}) & j>4\end{cases} \tag{12}$$

If we then denote the k'th component of $\mathbf{b}_j$ by b_{kj} and write (10) in its component form we will get n equations in the $n\times N$ unknowns b_{kj}. We then argue that in each of the n equations, the coefficients of each v_j must equal zero in order to make the equations zero for all τ. Thus we have $n\times N$ equations for the $n\times N$ unknowns.

3 Reducing the determinant to polynomials

Ordering the b_{kj}'s as $b_{11}, b_{12}, \ldots, b_{1N}, b_{21}, \ldots, b_{nN}$ in the vector $\mathbf{b}$ and assembling the coefficients in the matrix $\mathbf{D}$, we have

$$\mathbf{D}\mathbf{b} = \mathbf{0} \tag{13}$$

with

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \ldots & \mathbf{D}_{1n} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \ldots & \mathbf{D}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{D}_{n1} & \mathbf{D}_{n2} & \ldots & \mathbf{D}_{nn} \end{bmatrix}. \tag{14}$$

In (14) $\mathbf{D}_{kj}$ are $N \times N$ matrices with

$$\mathbf{D}_{kk} = \begin{bmatrix} \frac{1}{2}\omega^2 h^2 + \frac{1}{2} a_{kk} & 0 & 0 & 0 & q q_{kk} & \ldots \\ 0 & \omega^2 h^2 - \omega^2 + a_{kk} - q q_{kk} & 2\omega^2 h & 0 & 0 & \ldots \\ 0 & -2\omega^2 h & w^2 h^2 - \omega^2 + a_{kk} + q q_{kk} & 0 & 0 & \ldots \\ 0 & 0 & 0 & \omega^2 h^2 - 4\omega^2 + a_{kk} & 4\omega^2 h & \ldots \\ q q_{kk} & 0 & 0 & -4\omega^2 h & \omega^2 h^2 - 4\omega^2 + a_{kk} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix} \tag{15}$$

$$\mathbf{D}_{kj} = \begin{bmatrix} \frac{1}{2} a_{kk} & 0 & 0 & 0 & q q_{kj} & \ldots \\ 0 & a_{kj} - q q_{kj} & 0 & 0 & 0 & \ldots \\ 0 & 0 & a_{kj} + q q_{kj} & 0 & 0 & \ldots \\ 0 & 0 & 0 & a_{kj} & 0 & \ldots \\ q q_{kj} & 0 & 0 & 0 & a_{kj} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix} \tag{16}$$

for $k \neq j$. Here, a_{kj} and q_{kj} are the kj'th elements of the $\mathbf{A}$- and $\mathbf{Q}$-matrices from (3), and all the equations corresponding to the b_{1j}'s are divided by 2. This division ensures that $\mathbf{D}_{kj}$, $k \neq j$, is symmetric and that $\mathbf{D}_{kk}$ can be written as the sum of a symmetric and an antimetric matrix where the only non-zero elements of the antimetric one are the elements $d_{k,k+1}$ and $d_{k+1,k}$, $k = 2, 4, \ldots$ In (15) that means the elements $\pm 2\omega h^2$, $\pm 4\omega h^2$, …

In order for $\mathbf{b}$ to have a solution different from zero, the determinant

$$\Delta = |\mathbf{D}| \tag{17}$$

must be zero. An examination of the matrices shows that the equations $2, 3, 6, 7, \ldots, N + 2$, $N + 3, N + 6, \ldots$, i.e. the ones corresponding to the π-periodic expansion functions, decouple from the remaining ones, and we can write

$$\Delta = 0 \tag{18}$$

as

$$\Delta_1 \Delta_2 = 0 \tag{19}$$

as in [12], thereby reducing the size of the determinants with which we shall work. Assuming the elements of **A** and **Q** to be given constants, the determinants will be polynomials in q, ω and h. We therefore apply the same idea as in [12] and use some Fortran written symbolic manipulation to rewrite the determinants as

$$\Delta_p = \sum P^{\cdot}_{klm} \omega^k q^l h^m, \qquad p = 1, 2. \tag{20}$$

Now we use the fact that we are looking for solutions with α equal to zero, for the natural frequencies as well as for the combination frequencies. We also use the fact that the determinants are independent of the sign of h. This is due to the symmetric/antimetric pattern of (15) and (16) and the fact that the determinant of a matrix is equal to the determinant of its transpose. Therefore we shall only have even powers of h in (20) and so

$$h^m = (\mathrm{i}\beta)^{2r} = (-1)^r \beta^{2r}. \tag{21}$$

Altogether, this means we can write (20) as

$$\Delta_p = \sum P^{\cdot}_{klr} \omega^k q^l (-1)^r \beta^{2r} \tag{22}$$

or

$$\Delta_p = \sum P_{klr} \omega^k q^l \beta^{2r} \tag{23}$$

with

$$P_{klr} = (-1)^r P^{\cdot}_{klr}. \tag{24}$$

The polynomial Δ_p is then a pure real expression. It should be noted that in (23) only some of the combinations of klr will result in non-zero P_{klr}'s and the summation is only carried out for the non-zero elements.

4 Determination of boundary curves

We now have the determinants in a suitable form for our purpose. First, we want to determine the boundary curves corresponding to the natural frequencies. Here, we can take advantage of the fact that these curves are characterized by periodic solutions and simply put β equal to zero. That brings the polynomial in a state similar to the one in [12], and we can find the zeros of (18) up along a curve by the Newton-Raphson method.

The instability regions corresponding to the combination frequencies, however, must be handled in a more complex manner because here, for given q, we have to determine two variables, ω and β. As already stated, we must have α equal to zero in the stable domains and greater than zero in the unstable domains. Or, expressed in h, h must be imaginary in the stable domains and complex in the unstable domains. Since the polynomial is real, β must, at the limit, i.e. on the boundary, be at least a double root, and it is then known, Wilkinson [13], that we must have

$$|\partial\alpha/\partial\omega| = \infty \tag{25}$$

on the boundary. That in turn means that if we are at a point on a boundary curve, say (ω_0, q_0) and β_0, we can determine whether a nearby point (ω, q) belongs to the stable domain or not by trying to determine the corresponding β by the Newton-Raphson method. It it fails to converge, the point belongs to an unstable domain, if it succeeds, we are in a stable domain. This follows from the fact that we are only dealing with real quantities, and if the point (ω, q) belongs to a stable region, we can find a real β close to β_0, and if it does not this will not be possible, due to (25). So, to proceed from a once determined point on a boundary curve, we move q further up in the parameter plane and find ω by bisection, with the criterion for moving left/right being the success/failure of the Newton-Raphson procedure in the determination of β.

To start this procedure we observe that for q equal to zero, ω is known to be the combination frequency itself, so that β is the only unknown to be solved by (18).

In order to save computer time, it is possible to approximate some of the β's by quadratic extrapolation from earlier determined β's for some steps and only use the abovementioned procedure for every third or fourth step in q.

The procedure will be stopped either when we reach the desired parameter limit, i.e. when q is large enough, or when the curve crosses another curve, indicating that the solution will be unstable for all q's greater than the last found.

5 Examples

First we consider the Equations.

$$\omega^2 \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \left(\begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} + 2q \begin{bmatrix} 12 & 4 \\ 4 & 16 \end{bmatrix} \cos 2\tau \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{26}$$

This set of equations has a diagonal **A**-matrix, and the perturbation method as decribed in e.g. [11] can be used directly as comparison.

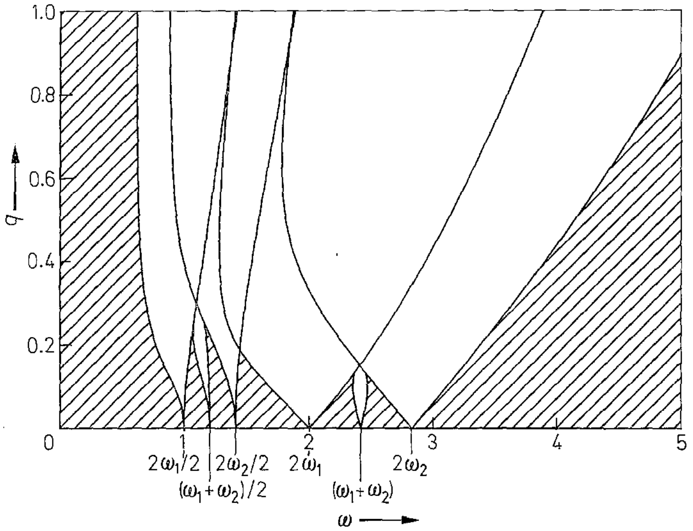

Fig. 2. First and second order instability domains for Eq. (26)

Applying the method described using 9 expansion functions will give two determinants of order 8 and 10, respectively. Searching only for the main stability domains and the ones corresponding to the first subharmonics, i.e. $s = 1, 2$, will result in the stability diagram shown in Fig. 2.

We observe in Fig. 2 that the curves corresponding to the natural frequencies can be computed in the complete parameter plane, whereas those corresponding to the combination frequencies end when they cross the other curves. This is because the crossing means entering a domain where the solutions will be unstable on both sides of the curve. Thus, it is not possible to decide upon stability/instability any more. But since we are already in an unstable domain, it is not necessary to continue this curve.

For the curves arising from $(\omega_1 + \omega_2)$, the values of β have been plotted against q in Fig. 3. From this figure we see clearly β's dependence upon q and note that the linear approximation, as suggested in e.g. [7], will only be valid for very small q's. In Fig. 3 we also see the shape of some y's of (5) as function of β.

The curves corresponding to the combination frequencies have already been presented in Fig. 1, where they are compared with those found by the method of multiple scales as described in [11].

In order to verify the curves obtained by the method described here, a direct Floquet-analysis has been carried out in a region around one of the instability domains. Using the method, as described in the appendix, we can determine whether the solutions are stable or unstable *for a given q and ω*. But since it demands $2 \times n$ time-integrations over one period *and* an eigen-

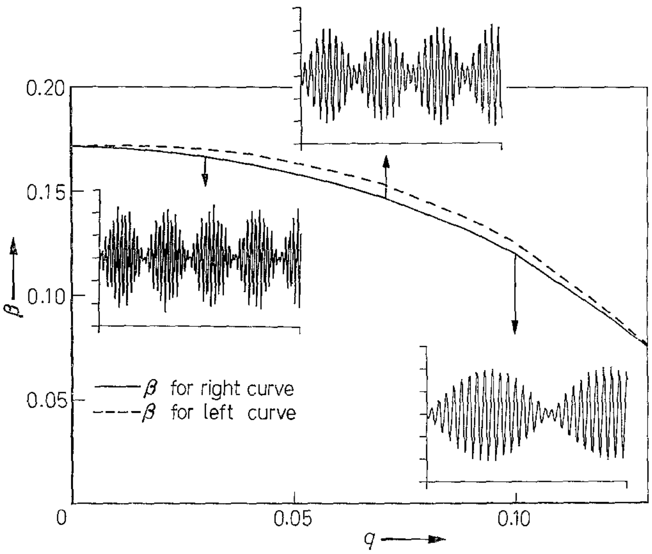

Fig. 3. A plot of β against q for the curves corresponding to $(\omega_1 + \omega_2)$ from Fig. 2. The solid line belongs to the right-hand curve, the dashed one to the left-hand curve. Also, some plots of certain y's of Eq. (5) are shown

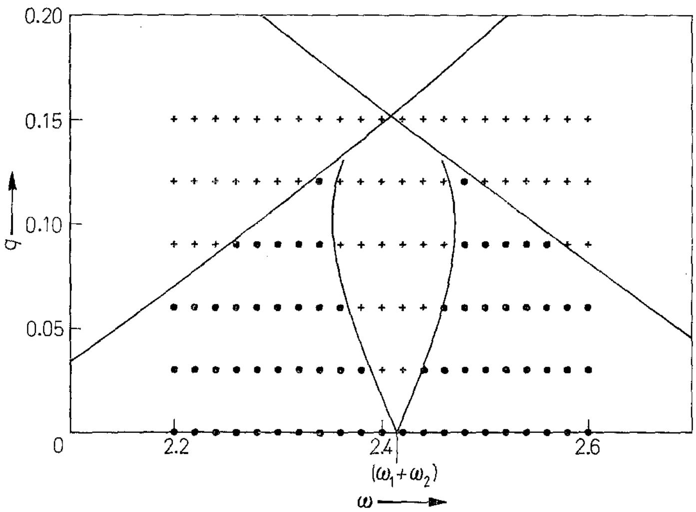

Fig. 4. Part of Fig. 2 and Floquet-analysis applied to certain points in the plane. The symbol • means that the point is stable and the symbol + means that the point is unstable

value determination for each point, this method is here used only for verification purposes. In Fig. 4 we have taken the part of Fig. 2 around the instability domain corresponding to $\omega = \omega_1 + \omega_2$ and applied the Floquet-analysis to each point in a grid in this region. From this figure it seems evident that if we made up a finer mesh for the grid, we would get the same curves as the ones in Fig. 2.

As a second example, consider the equations

$$\omega^2 \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \left(\begin{bmatrix} 12 & 4 \\ 4 & 20 \end{bmatrix} + 2q \begin{bmatrix} 12 & 4 \\ 2 & 8 \end{bmatrix} \cos 2\tau \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{27}$$

The stability diagram for (27) is shown in Fig. 5. Here, we have used 13 expansion functions, meaning determinants of order 12 and 14, respectively. In Fig. 5 it is worth noting that the curves corresponding to $\omega = (\omega_1 + \omega_2)/2$ continue up to $q \simeq 1$, i.e. the excitation level is in no way small any more, but can be compared in magnitude to the other terms of the equations. To verify that these curves actually determine the boundary between stability and instability, some numerical simulation have been carried out for $q = .86$ and $q = .96$, as shown in Table 1 and Figs. 6 and 7. It follows from Table 1 that for ω near 2, the curves are determined within ± 0.001, even when $q \simeq 1$.

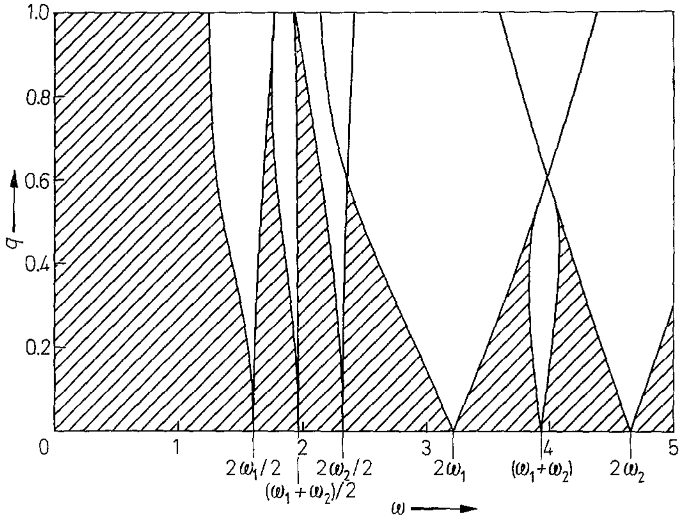

Fig. 5. Stability diagram for first and second order domains for Eq. (27)

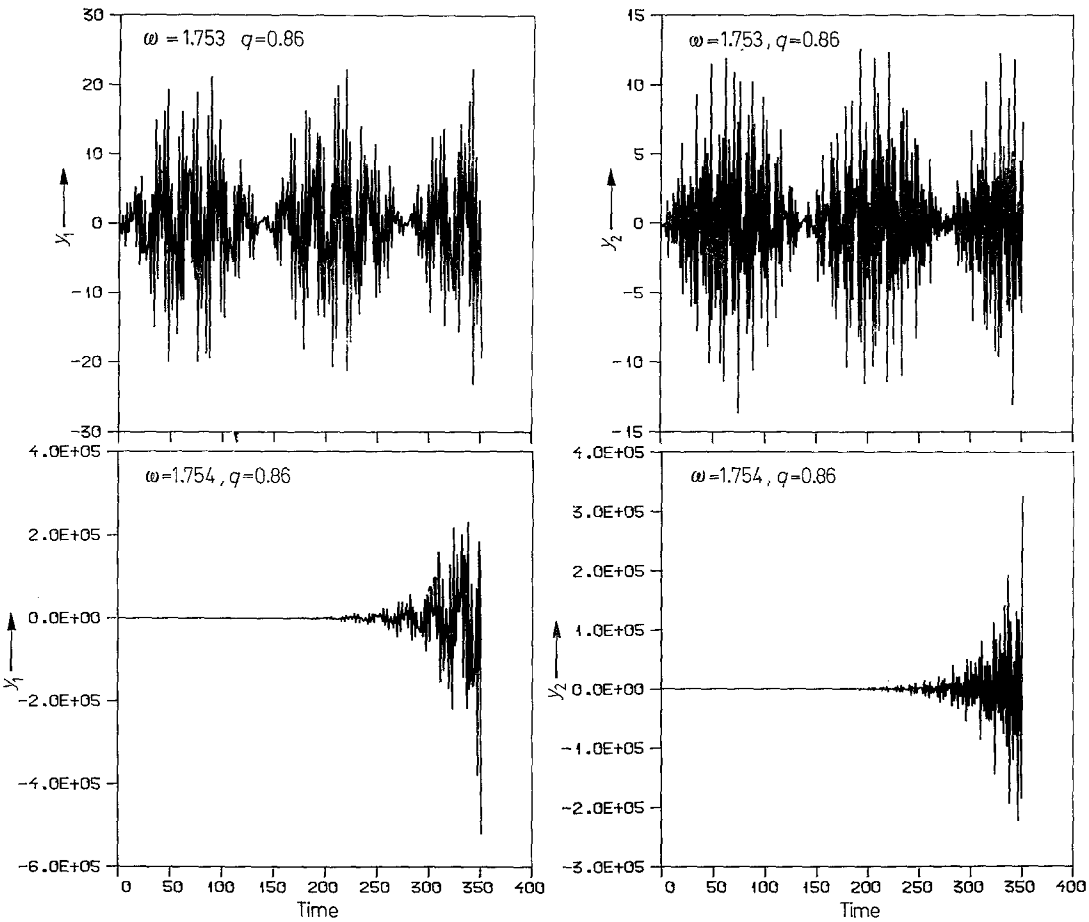

Fig. 6a–d. Numerical integration for some of the points listed in Table 1

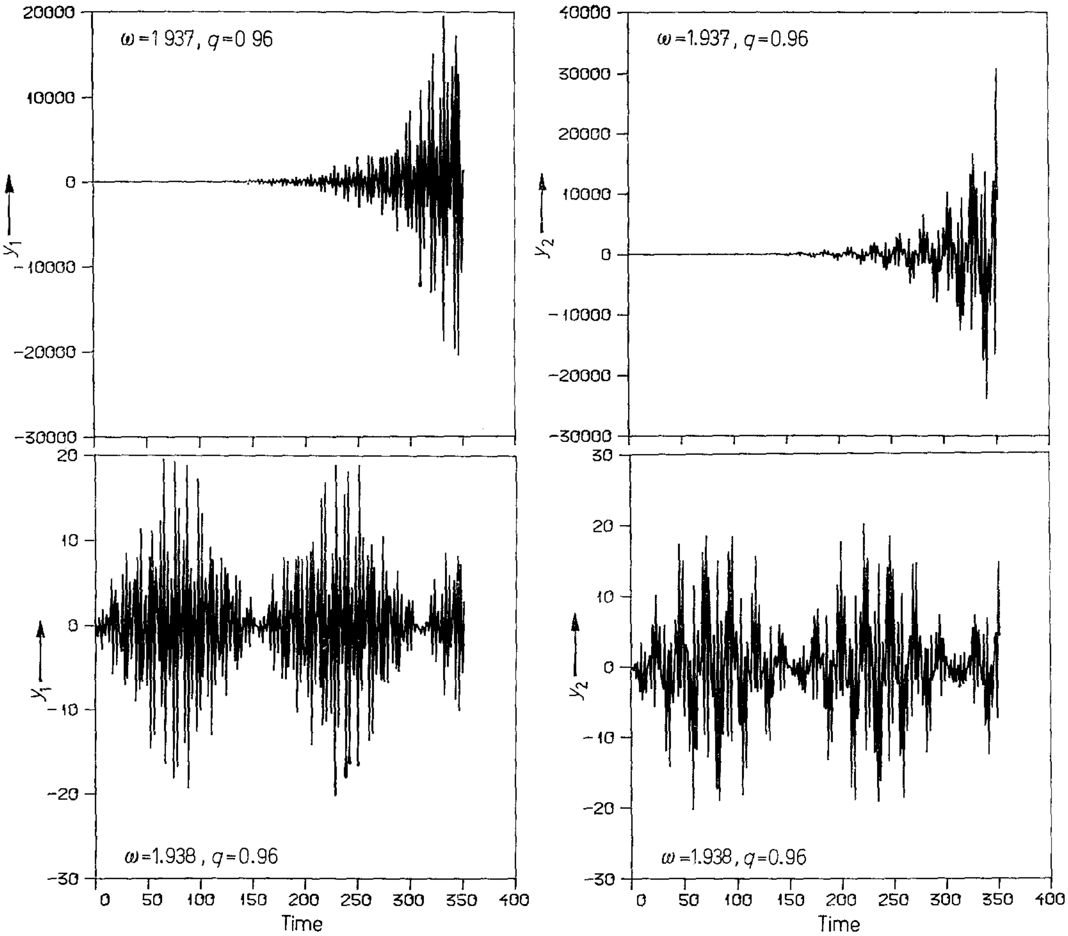

Fig. 7a—d. Numerical integration for some of the points listed in Table 1

Table 1. Result of certain numerical simulations in the plane for Eq. (27). Cf. with Fig. 5

q	ω	description		result
0.86	1.7440	inside	ω_1	unstable
0.86	1.7445	on	ω_1	
0.86	1.7450	outside	ω_1	stable
0.86	1.7530	outside	$(\omega_1 + \omega_2)/2$	stable
0.86	1.7533	on	$(\omega_1 + \omega_2)/2$	
0.86	1.7540	inside	$(\omega_1 + \omega_2)/2$	unstable
0.86	1.9470	inside	$(\omega_1 + \omega_2)/2$	unstable
0.86	1.9473	on	$(\omega_1 + \omega_2)/2$	
0.86	1.9480	outside	$(\omega_1 + \omega_2)/2$	stable
0.86	2.0180	outside	ω_2	stable
0.86	2.0189	on	ω_2	
0.86	2.0190	inside	ω_2	unstable
0.96	1.9370	inside	$(\omega_1 + \omega_2)/2$	unstable
0.96	1.9376	on	$(\omega_1 + \omega_2)/2$	
0.96	1.9380	outside	$(\omega_1 + \omega_2)/2$	stable
0.96	1.9650	outside	ω_2	stable
0.96	1.9655	on	ω_2	
0.96	1.9660	inside	ω_2	unstable

6 Conclusion

In this paper we have demonstrated a method by which we can generate stability diagrams for coupled Mathieu-equations. With the method we are able to deal with instability domains corresponding to both natural and combination frequencies and subharmonics. Also, as shown in the examples, we have here a method that does not restrict itself to small parameters in any way.

Looking at the involved matrices, it may also be noted that an extension to the more general Hill-equations will be straightforward, as will also, with some extensions, the inclusion of damping terms. But it must be realized as well that the method is restricted to two, three or perhaps four coupled equations. That is because the size of the determinants increases with the number of equations, and the necessary computer-time required for the analytical derivation of the determinants thereby increases rapidly.

We have not made any attempt to deal with cases in which we have equal eigenvalues of the **A**-matrix.

Appendix

In order to examine whether the solutions of (3) are stable or unstable *for a given set of parameters*, the Floquet-theory as used in [9] will be applied.

To do that we write (3) as a set of $2n$ first order differential equations

$$\dot{x} = \mathbf{B}(\tau)\,\mathbf{x} \tag{28}$$

with $\mathbf{B}(\tau)$ a function of $\mathbf{A}$ and $\mathbf{Q}$. The matrix $\mathbf{B}(\tau)$ is thereby periodic, due to the term $\mathbf{Q}\cos 2\tau$, with period π. From (28) a matrix $\mathbf{Z}(\tau)$ is evaluated by

$$\dot{\mathbf{Z}}(\tau) = \mathbf{B}(\tau)\,\mathbf{Z}(\tau), \tag{29}$$

$$\mathbf{Z}(0) = \mathbf{I}. \tag{30}$$

We then generate the matrix $\mathbf{Z}(\pi)$. This is done by numerical integration of (29), (30) up to π. For this matrix we find the $2n$ eigenvalues, λ_i, and, according to the Floquet-theory, we will have

$$\max\left(|\lambda_i|\right) > 1 \Leftrightarrow \text{solutions unstable}, \tag{31}$$

$$\max\left(|\lambda_i|\right) \leqq 1 \Leftrightarrow \text{solutions stable}. \tag{32}$$

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