Ingenieur-Archiv 55 (1985) 440—449

Ingenieur-Archiv

Boundary elements and symmetry

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Summary: This paper discusses the deficiencies of boundary element stiffness matrices, gives an account of the methods proposed to circumvent these defects and proposes a new procedure to obtain symmetric stiffness matrices.

Randelemente und Symmetrie

Übersicht: In diesem Aufsatz werden die Fehler diskutiert, die bei Steifigkeitsmatrizen, die mit Randelementen gewonnen wurden, auftreten, und es werden die Methoden vorgestellt, die vorgeschlagen wurden, um diese Fehler zu beheben. Ferner wird eine neue Methode vorgestellt, symmetrische Steifigkeitsmatrizen zu erhalten.

1 Introduction

Boundary elements are frequently coupled with finite elements to make use of the advantages both numerical techniques offer. The coupling is usually done by transforming the boundary element region into an 'equivalent' finite element. An operation which yields a stiffness matrix that, in general, is neither symmetric nor does it enjoy the other properties a standard FE-stiffness matrix has.

In this paper we want to give an account of these difficulties trace them back to their mathematical origin and describe the mathematical context of the methods proposed in the literature, [1, 2], to overcome these difficulties.

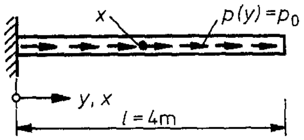

Fig. 1. A bar with a distributed load

2 Compatibility on the boundary

To calculate the displacement $u(x)$ of a bar as in Fig. 1 at a point x we load an infinite bar at the same point with a concentrated force $P = 1$, separate the portion of the infinite bar which coincides with the real bar from the infinite bar, multiply the end-forces and end-displacements we observe at the two cuts $y = 0$ and $y = l$

$$\mathbf{f}[x]^T = \{x - 1, -x\}, \qquad \boldsymbol{\delta}[x]^T = \{1, (1 - l)\, x + 1\}$$

with the conjugated quantities, the end-forces and end-displacements of the real bar,

$$\mathbf{f} = \{f_1, f_2\}, \qquad \boldsymbol{\delta}^T = \{\delta_1, \delta_2\}$$

and add to it the scalar product (integral) of the distributed load $p(y)$ and the displacement $u(y, x)$

$$u(y, x) = \frac{1}{EA}\begin{cases}(1-x)\,y + 1\,, & y \le x\\ (1-y)\,x + 1\,, & x \le y\end{cases}$$

caused by the single force within the infinite bar. Arranging the result according to Betti's principle we obtain

$$W_{1,2} = 1u(x) + \mathbf{f}[x]^T\,\boldsymbol{\delta} = \int_0^l p(y)\,u(y, x)\,dy + \mathbf{f}^T\boldsymbol{\delta}[x] = W_{2,1} \tag{1}$$

or

$$\begin{aligned} u(x) &= -\mathbf{f}[x]^T\,\boldsymbol{\delta} + \boldsymbol{\delta}[x]^T\,\mathbf{f} + \int_0^l p(y)\,u(y, x)\,dy = (1-x)\,\delta_1 + x\delta_2\\ &\quad + \frac{1}{EA}\left\{1f_1 + [(1-l)\,x + 1]\,f_2 + \int_l^x p(y)\,[(1-x)\,y + 1]\,dy\right.\\ &\quad \left. + \int_x^l p(y)\,[(1-y)\,x + 1]\,dy\right\}. \end{aligned} \tag{2}$$

With this influence-function we can calculate the end-displacements $\delta_1 = u(0)$ and $\delta_2 = u(4)$ of the bar as well. Choosing once $x = 0$ and once $x = 4$ we obtain

$$\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ -3 & 4\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} + \frac{1}{EA}\begin{bmatrix}1 & 1\\ 1 & -11\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} + \begin{bmatrix}p_1\\ p_2\end{bmatrix}, \tag{3}$$

where

$$p_1 = \int_0^l p(y)\,u(y, 0)\,dy = \frac{p_0}{EA}\int_0^4 1dy = \frac{4}{EA}\,p_0\,,$$

$$p_2 = \int_0^l p(y)\,u(y, 4)\,dy = \frac{p_0}{EA}\int_0^4 [(1-4)\,y + 1]\,dy = -\frac{20}{EA}\,p_0\,.$$

As the δ_i on the left-hand side are now the same δ_i as on the right-hand side, we conclude that these two equations constitute two compatibility conditions for the boundary data of the bar:

Given a distributed load p, two vectors $\boldsymbol{\delta} = \{\delta_i\}$ and $\mathbf{f} = \{f_i\}$ are the end-displacements and end-forces resp. of the bar if and only if they satisfy (3).

If we put all the δ-terms on one side, (3) becomes

$$\begin{bmatrix}0 & 0\\ 3 & -3\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} = \frac{1}{EA}\begin{bmatrix}1 & 1\\ 1 & -11\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} + \frac{p_0}{EA}\begin{bmatrix}4\\ -20\end{bmatrix} \tag{4}$$

or in a matrix notation

$$\mathbf{H}\boldsymbol{\delta} = \mathbf{G}\mathbf{f} + \mathbf{p}\,. \tag{5}$$

Multiplying this equation from the left with the inverse of the matrix on the right-hand side we obtain

$$EA\begin{bmatrix}0.25 & -0.25\\ -0.25 & 0.25\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} = \begin{bmatrix}f_1\\ f_2\end{bmatrix} + p_0\begin{bmatrix}2\\ 2\end{bmatrix} \tag{6}$$

or

$$\mathbf{K}\boldsymbol{\delta} = \mathbf{f} + \mathbf{r}\,, \tag{7}$$

where the vector $\mathbf{r}$ is the vector of the (negative) support reactions of the bar when both ends are fixed and where $\mathbf{K}$ is the stiffness matrix of the bar

$$\mathbf{K} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

In case the distributed forces are zero, $p(y) = 0$, the vector $\mathbf{r}$ in (7) drops out and we obtain the familiar expression

$$\mathbf{K\delta} = \mathbf{f} \tag{8}$$

which formulates a compatibility condition between the boundary values δ_i and f_i of homogeneous ($p = 0$) displacement functions.

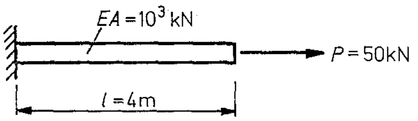

Fig. 2. A bar loaded with a single force

Let us illustrate this with a simple problem. Assume that on loading the bar in Fig. 2 at its free end with a single force, $P = 50$ kN, we observed an elongation of 0.18 m. If our observation is correct then the boundary-data, the two end-displacements, the one on the left-hand side and the one on the right-hand side

$$\delta_1 = 0, \qquad \delta_2 = 0.18 \text{ m}$$

and the two associated end-forces

$$f_1 = -50 \text{ kN}, \qquad f_2 = 50 \text{ kN},$$

should satisfy the compatibility condition (8):

$$\begin{bmatrix} 250 & -250 \\ -250 & 250 \end{bmatrix}\begin{bmatrix} 0 \\ 0.18 \end{bmatrix} = \begin{bmatrix} -50 \\ 50 \end{bmatrix}.$$

But as, obviously, 250 times 0.18 is 45 and not 50 our observation must be wrong. The data δ_i and f_i do not match, they do not satisfy the compatibility conditions.

Our usage of the word compatibility in this context might seem a little bit strange but it can be justified as follows: If one would substitute the (erroneous) data $\delta_1 = 0$, $\delta_2 = 0.18$, $f_1 = -50$ and $f_2 = 50$ into the influence function (2) then these numbers would not be the boundary values of the function so constructed. We would have not fit at the boundary (and no overall equilibrium). This is only guaranteed if the data satisfy the compatibility conditions. In which case the data are also the boundary values of the influence function.

3 Elastic plates

Let us turn now to the two-dimensional analogue of a bar, an elastic plate. The vector-valued displacement field $\mathbf{u} = \{u_1, u_2\}$ of such a plate is governed by the Navier equations

$$-L_{ij}u_j := -\mu u_{i,jj} - \frac{\mu}{1-2\nu}\, u_{j,ji} = p_i \tag{9}$$

and subject to geometric boundary conditions

$$u_i = \bar{u}_i$$

on a part Γ_1 of the boundary and static boundary conditions

$$\tau_i(\mathbf{u}) = \sigma_{ij}n_j = \bar{t}_i$$

on the complementary part Γ_2.

To calculate the horizontal and vertical displacements at a point $\mathbf{x}$ of a plate with boundary Γ we load an infinite plate with a concentrated force $\mathbf{P} = \mathbf{e}_1$ and $\mathbf{P} = \mathbf{e}_2$ resp. at $\mathbf{x}$ and apply Betti's principle to the two displacement fields, the field of the infinite plate and the field of the real plate in the domain Ω:

$$W_{1,2} = C_{ij}u_j(\mathbf{x}) + \int_\Gamma T_{ij}(\mathbf{y}, \mathbf{x})\, u_j(\mathbf{y})\ ds_\mathbf{y}$$
$$= \int_\Omega U_{ij}(\mathbf{y}, \mathbf{x})\, b_j(\mathbf{y})\, d\Omega_\mathbf{y} + \int_\Gamma U_{ij}(\mathbf{y}, \mathbf{x})\, t_j(\mathbf{y})\, ds_\mathbf{y} = W_{2,1}.$$

The terms U_{ij} in this expression are the components, $j = 1, 2$, of the fundamental solutions, $i = 1, 2$, i.e. the displacements we observe at a point $\mathbf{y}$ if at some distant point $\mathbf{x}$ acts a concentrated force $\mathbf{P}_i = \mathbf{e}_i$ and the terms T_{ij} are the components of the associated traction vector on the boundary. The values of the matrix C_{ij} which accompanies the free terms $u_j(\mathbf{x})$ depend on the position of the source point $\mathbf{x}$

$$C_{ij}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega \\ 1/2\delta_{ij}, & \mathbf{x} \in \Gamma \qquad \text{(at a smooth point)} \\ 0, & \mathbf{x} \in \Omega^c \end{cases}$$

In the case of a bar, which is a one-dimensional structural element, the compatibility conditions are algebraic equations between numbers, the boundary values of the bar. Now in the case of a plate, which is a two-dimensional structural element, the same conditions are integral equations on the boundary

$$C_{ij}(\mathbf{x})\, u_j(\mathbf{x}) + \int_\Gamma T_{ij}(\mathbf{y}, \mathbf{x})\, u_j(\mathbf{y})\, ds_\mathbf{y} = \int_\Gamma U_{ij}(\mathbf{y}, \mathbf{x})\, t_j(\mathbf{y})\, ds_\mathbf{y} + \int_\Omega U_{ij}(\mathbf{y}, \mathbf{x})\, p_j(\mathbf{y})\, d\Omega_\mathbf{y} \quad (10)$$

between the functions $u_i(\mathbf{x})$ and $t_i(\mathbf{x})$, i.e. between symbols with infinitely many degrees of freedom.

If we let the boundary functions u_i and t_i

$$u_i = u_{ij}\varphi_j(\mathbf{x}), \qquad t_i = t_{ij}\varphi_j(\mathbf{x}) \quad (11)$$

and apply a point collocation process, (10) becomes a linear system of equations

$$\mathbf{Hu} = \mathbf{Gt}$$

whose columns list the influence the boundary layers associated with the nodal values u_i and t_i resp. have on the collocation points distributed along the boundary. As each collocation point on the boundary influences each other point $\mathbf{H}$ and $\mathbf{G}$ are fully populated and because 'influence' (in general) is not a symmetric relation both $\mathbf{H}$ and $\mathbf{G}$ are unsymmetric. This is easily understood if we consider Γ to be a material wire, see Fig. 3.

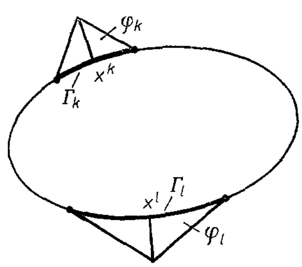

Fig. 3. A wire with a local density φ_k and φ_l, respectively

The attractive force the element Γ_k with 'mass' φ_k exerts on the collocation point $\mathbf{x}^l$, in general, is not equal to the force the element Γ_l with 'mass' φ_l exerts on the point $\mathbf{x}^k$ because form, length and, hence, also 'mass' of the two elements differ. Only in the limit, if the elements shrink to mere points does $\mathbf{G}$ become symmetric but not so $\mathbf{H}$ because its elements, the tractions $T_{ij}(\mathbf{y}, \mathbf{x})$

depend on the normals at the boundary point in question. But as the two normals at two collocation points do (in general) not coincide so do not the elements of $\mathbf{H}$ and $\mathbf{H}^T$.

In the case of a bar the exact symmetric stiffness matrix

$$\mathbf{K\delta} = \mathbf{f}$$

is obtained by multiplying the compatibility condition

$$\mathbf{H\delta} = \mathbf{Gf}$$

from the left with $\mathbf{G}^{-1}$.

In the case of a plate we must, in addition, multiply this result

$$\mathbf{G}^{-1}\mathbf{Hu} = \mathbf{t} \tag{12}$$

with the matrix

$$\mathbf{F} = \left[\int_\Gamma \varphi_i \varphi_j \, ds\right]$$

to obtain the vector of the equivalent nodal forces, $\mathbf{f}$, on the right-hand side. (The element F_{ij} of the matrix $\mathbf{F}$ is the virtual work done by the traction-component φ_j acting through the displacement-component φ_i):

$$\mathbf{FG}^{-1}\mathbf{Hu} = \mathbf{Ft} = \mathbf{f}.$$

But this result, the stiffness matrix,

$$\mathbf{K} = \mathbf{FG}^{-1}\mathbf{H} \tag{13}$$

has none of those properties exactly

(Kernel) $\mathbf{K\delta}^0 = \mathbf{0}$ ($\mathbf{\delta}^0$ = vector of a rigid-body-motion)

(Equ.) $\mathbf{\delta}^{0T}\mathbf{K\delta} = 0$

(Sym.) $\mathbf{\delta}^T\mathbf{K}\hat{\mathbf{\delta}} = \hat{\mathbf{\delta}}^T\mathbf{K\delta}$

(Pos. Def.) $\mathbf{\delta}^T\mathbf{K\delta} > 0, \quad \mathbf{\delta} \neq \mathbf{\delta}^0$

which a standard FE-stiffness matrix has. The reason is that the functions

$$u_i = u_{ij}\varphi_j(\mathbf{x}), \qquad t_i = t_{ij}\varphi_j(\mathbf{x}) \tag{14}$$

though they solve (12), strictly speaking, are not compatible, are not the boundary values of the same displacement field. If that would be true, i.e. if two pairs of vectors $\{\mathbf{u}, \mathbf{t}\}$ and $\{\hat{\mathbf{u}}, \hat{\mathbf{t}}\}$ which satisfy the discrete compatibility condition $\mathbf{Hu} = \mathbf{Gt}$ would render two pairs of vector-valued ($i = 1, 2$) boundary functions

$$u_i = u_{ij}\varphi_j, \qquad t_i = t_{ij}\varphi_j, \qquad \hat{u}_i = \hat{u}_{ij}\varphi_j, \qquad \hat{t}_i = \hat{t}_{ij}\varphi_j$$

which are the boundary values of two homogeneous ($p_i = 0$) displacement fields then Betti's principle should apply

$$\int_\Gamma u_i \hat{t}_i \, ds = \int_\Gamma t_i \hat{u}_i \, ds.$$

But as the left-hand side reads

$$\int_\Gamma u_i \hat{t}_i \, ds = \mathbf{u}^T\mathbf{F}\hat{\mathbf{t}} = \mathbf{u}^T\mathbf{FG}^{-1}\mathbf{H}\hat{\mathbf{u}} = \mathbf{u}^T\mathbf{K}\hat{\mathbf{u}}$$

and the right-hand side

$$\int_\Gamma t_i \hat{u}_i \, ds = \mathbf{tF}\hat{\mathbf{u}} = \mathbf{u}^T\mathbf{H}^T\mathbf{G}^{-1T}\mathbf{F}\hat{\mathbf{u}} = \mathbf{u}^T\mathbf{K}^T\hat{\mathbf{u}}$$

the two sides do only match if $\mathbf{K} = \mathbf{K}^T$.

4 The energy approach

At this point one is tempted to simply continue with the matrix

$$\tilde{\mathbf{K}} = \frac{1}{2}(\mathbf{K} + \mathbf{K}^T). \tag{15}$$

A manipulation which, seemingly, even can be 'justified':

The potential energy of a plate loaded with boundary tractions $\bar{t}_i$ but with zero internal forces, $p_i = 0$, reads

$$\Pi_1(\mathbf{u}) = \frac{1}{2}\int_\Omega C_{ijkl}\varepsilon_{ij}(\mathbf{u})\,\varepsilon_{kl}(\mathbf{u})\,d\Omega - \int_\Gamma \bar{t}_i u_i\,ds.$$

Because of Green's first identity

$$\int_\Omega C_{ijkl}\varepsilon_{ij}(\mathbf{u})\,\varepsilon_{kl}(\mathbf{u})\,d\Omega = \int_\Omega -L_{ij}u_j\,d\Omega + \int_\Gamma t_i u_i\,ds$$

and $L_{ij}u_j = 0$, $i = 1, 2$, the potential energy can also be expressed as

$$\Pi_1(\mathbf{u}) = \frac{1}{2}\int_\Gamma t_i u_i\,ds - \int_\Gamma \bar{t}_i u_i\,ds.$$

If we replace t_i und u_i in this expression by (14) then $\Pi_1(\mathbf{u})$ becomes

$$\Pi_1(\mathbf{u}) = \frac{1}{2}\,\mathbf{u}^T\mathbf{F}\mathbf{G}^{-1}\mathbf{H}\mathbf{u} - \bar{\mathbf{t}}^T\mathbf{u}$$

and the condition

$$\frac{\partial \Pi_1}{\partial \delta_i} = 0 \quad i = 1, 2, \ldots n$$

renders the very same Eq. (15), as before:

$$(\mathbf{K} + \mathbf{K}^T)\,\mathbf{u} = \mathbf{F}\bar{\mathbf{t}}.$$

The error we commit, also, is the same as before. The functions u_i and t_i are not the boundary values of the same displacement field. A displacement field with trace $u_i = u_{ij}\varphi_j$ does not have the tractions $t_i = t_{ij}\varphi_j$ and vice versa.

Unlike FE-methods where the right-hand side, $\mathbf{f}$, of a stiffness matrix

$$\mathbf{K}\boldsymbol{\delta} = \mathbf{f}$$

is the vector of the equivalent nodal forces of the very same function $u = \delta_i\varphi_i$ whose termwise energy products constitute the elements of $\mathbf{K}$, the right-hand side, $\mathbf{f}$, of a BE-stiffness matrix in no way — at least mathematically speaking — is associated with the $\boldsymbol{\delta}$-vector on the left hand side. It is a separate independent quantity. No differentiation or integration by parts will get us from u_i to t_i.

In this context we should also see that the functions u_i and t_i which solve the discrete compatibility conditions are not the boundary values of the BE-solution, the function we employ to calculate stresses or displacementes in the interior. The BE-solution is the function

$$u_i^h(\mathbf{x}) = \int_\Gamma [U_{ij}(\mathbf{y}, \mathbf{x})\,t_j(\mathbf{y}) - T_{ij}(\mathbf{y}, \mathbf{x})\,u_j(\mathbf{y})]\,ds_{\mathbf{y}} + \int_\Omega U_{ij}(\mathbf{y}, \mathbf{x})\,p_j(\mathbf{y})\,d\Omega_{\mathbf{y}} \tag{16}$$

and its boundary values differ by terms $\varepsilon_i(\mathbf{x})$ and $\eta_i(\mathbf{x})$ from the functions u_i and t_i

$$\lim_{\mathbf{x}\to\Gamma} u_i^h(\mathbf{x}) = u_i(\mathbf{x}) + \varepsilon_i(\mathbf{x}), \qquad \lim_{x\to\Gamma} t_i^h(\mathbf{x}) = t_i(\mathbf{x}) + \eta_i(\mathbf{x}).$$

Only at the collocation points do the ε_i vanish but not so the η_i. If we use nonconforming (= discontinuous) elements these latter even become infinite at the element interfaces.

5 Galerkin's method

True symmetric relations are obtained — following an idea of the third author — if we, firstly, complete the set of compatibility conditions, secondly, choose among the then complete set our conditions wisely and, thirdly, solve these conditions, so chosen, with Galerkin's method.

To understand this approach better let us denote the boundary values of a Kirchhoff plate, this will be our model structural element, by

$$\partial^0 w = w, \quad \partial^1 w = \frac{\partial w}{\partial n}, \quad \partial^2 w = M_n, \quad \partial^3 w = V_n .$$

Betti's principle (we neglect the corner forces, let Γ be smooth) reads in this notation

$$\int_\Omega K\Delta\Delta w\hat{w}\, d\Omega + \int_\Gamma [\partial^3 w \partial^0 \hat{w} - \partial^2 w \partial^1 \hat{w} + \partial^1 w \partial^2 \hat{w} - \partial^0 w \partial^3 \hat{w}]\, ds$$
$$- \int_\Omega w K\Delta\Delta\hat{w}\, d\Omega = 0$$

and the formulae for the four fundamental solutions $g_i(\mathrm{y}, \mathrm{x})$, $i = 0, 1, 2, 3$, of the Kirchhoff plate simply

$$g_i(\mathrm{y}, \mathrm{x}) = \partial_\mathrm{x}^i g_0(\mathrm{y}, \mathrm{x}), \qquad g_0(\mathrm{y}, \mathrm{x}) = \frac{1}{8\pi K}\, r^2 \ln r .$$

These solutions correspond to a concentrated load ($i = 0$), a single couple ($i = 1$), a bent in the slope ($i = 2$) and a discontinuity in the deflection ($i = 3$).

If we formulate with these four solutions and the real deflection, w, consecutively Betti's principle then we obtain four integral equations on the boundary, four compatibility conditions, for the four boundary values $\partial^i w$ associated with the fourth-order operator $K\Delta\Delta w$

$$\frac{1}{2}\begin{bmatrix}\partial^0 w\\ \partial^1 w\\ \partial^2 w\\ \partial^3 w\end{bmatrix} = \int_\Gamma \begin{bmatrix} \partial_\mathrm{y}^0\partial_\mathrm{x}^0 g_0 & \partial_\mathrm{y}^1\partial_\mathrm{x}^0 g_0 & \partial_\mathrm{y}^2\partial_\mathrm{x}^0 g_0 & \partial_\mathrm{y}^3\partial_\mathrm{x}^0 g_0\\ \partial_\mathrm{y}^0\partial_\mathrm{x}^1 g_0 & \partial_\mathrm{y}^1\partial_\mathrm{x}^1 g_0 & \partial_\mathrm{y}^2\partial_\mathrm{x}^1 g_0 & \partial_\mathrm{y}^3\partial_\mathrm{x}^1 g_0\\ \partial_\mathrm{y}^0\partial_\mathrm{x}^2 g_0 & \partial_\mathrm{y}^1\partial_\mathrm{x}^2 g_0 & \partial_\mathrm{y}^2\partial_\mathrm{x}^2 g_0 & \partial_\mathrm{y}^3\partial_\mathrm{x}^2 g_0\\ \partial_\mathrm{y}^0\partial_\mathrm{x}^3 g_0 & \partial_\mathrm{y}^1\partial_\mathrm{x}^3 g_0 & \partial_\mathrm{y}^2\partial_\mathrm{x}^3 g_0 & \partial_\mathrm{y}^3\partial_\mathrm{x}^3 g_0 \end{bmatrix} \begin{bmatrix}\partial^3 w\\ \partial^2 w\\ \partial^1 w\\ \partial^0 w\end{bmatrix} ds_\mathrm{y} + \int_\Omega \begin{bmatrix}\partial_\mathrm{x}^0 g_0\\ \partial_\mathrm{x}^1 g_0\\ \partial_\mathrm{x}^2 g_0\\ \partial_\mathrm{x}^3 g_0\end{bmatrix} p\, d\Omega_\mathrm{y} . \tag{17}$$

The index x or y at ∂^i is to indicate that differentiation is done with respect to the coordinates x_i of the source point or the coordinates y_i of the field point. So $\partial_\mathrm{y}^3\partial_\mathrm{x}^1 g_0(\mathrm{y}, \mathrm{x})$, e.g., is the Kirchhoff-shear (∂_y^3) at the integration point y caused by a single couple (∂_x^1) acting at the boundary point x.

(In more general terms does (17) express the fact that with the $2m$ fundamental solutions of a linear self-adjoint operator of order $2m$ an equal number of compatibility conditions between the $2m$ boundary values of such an operator can be formulated on the boundary; m of these are linear independent, see [3] p. 216).

Assume the plate is clamped

$$K\Delta\Delta w = p \quad \text{in } \Omega, \quad w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma.$$

This leaves M_n, the bending moment, and V_n, the Kirchhoff-shear as unknowns. To determine these two functions we have now the choice among four equations. If we opt for the first two in (17)

$$\frac{1}{2}\begin{bmatrix}0\\0\end{bmatrix} = \int_\Gamma \begin{bmatrix}\partial_\mathrm{y}^0\partial_\mathrm{x}^0 g_0 \partial^3 w + \partial_\mathrm{y}^1\partial_\mathrm{x}^0 g_0 \partial^2 w\\ \partial_\mathrm{y}^0\partial_\mathrm{x}^1 g_0 \partial^3 w \quad \partial_\mathrm{y}^1\partial_\mathrm{x}^1 g_0 \partial^2 w\end{bmatrix} ds_\mathrm{y} + \int_\Omega \begin{bmatrix}\partial_\mathrm{x}^0 g_0\\ \partial_\mathrm{x}^1 g_0\end{bmatrix} p\, d\Omega_\mathrm{y}$$

and solve these two with Galerkin's method, $(Lu - f, \varphi) = 0$, — that is if we multiply the equations from the right with test functions φ_i, the same functions we use in the expansions

$$M_n = \delta_i \varphi_i(\mathrm{x}) \qquad V_n = \varepsilon_i \varphi_i(\mathrm{x})$$

and integrate once more over Γ, — then the two equations render the symmetric system

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\delta} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \end{bmatrix},$$

where

$$a_{ij} = \int_\Gamma \int_\Gamma \partial_{\mathrm{y}}^0 \partial_{\mathrm{x}}^0 g_0(\mathrm{y}, \mathrm{x})\, \varphi_i(\mathrm{y})\, ds_{\mathrm{y}} \varphi_j(\mathrm{x})\, ds_{\mathrm{x}},$$

$$b_{ij} = \int_\Gamma \int_\Gamma \partial_{\mathrm{y}}^1 \partial_{\mathrm{x}}^0 g_0 \varphi_i\, ds_{\mathrm{y}} \varphi_j\, ds_{\mathrm{x}} = \int_\Gamma \int_\Gamma \partial_{\mathrm{y}}^0 \partial_{\mathrm{x}}^1 g_0 \varphi_j\, ds_{\mathrm{y}} \varphi_i\, ds_{\mathrm{y}},$$

$$c_{ij} = \int_\Gamma \int_\Gamma \partial_{\mathrm{y}}^1 \partial_{\mathrm{x}}^1 g_0 \varphi_i\, ds_{\mathrm{y}} \varphi_j\, ds_{\mathrm{x}}\ f_i^j = \int_\Gamma \int_\Gamma \partial_{\mathrm{x}}^i g_0(\mathrm{y}, \mathrm{x})\, p(\mathrm{y})\, d\Omega_{\mathrm{y}} \varphi_j(\mathrm{x})\, ds_{\mathrm{x}}.$$

If, instead, we had chosen the last two equations in (17) then, regardless of Galerkin or not, an unsymmetric system of equations would result because then the sum of the indices of the off-diagonal kernels in the two equations would not be equal, i.e. the resulting off-diagonal matrices would not be adjoint. An adjoint formulation is always possible as long as the problem is regular, that is as long as on every part of the boundary of two conjugated quantities one is known and the other is unknown.

The disadvantage of this approach is that we need to integrate two times and the fact that the deeper we move in the scheme (17), and this we must do, sometimes, to retain the symmetry the more singular the kernels become.

The integral operators which constitute the compatibility conditions can be viewed as operators which shift functions from Sobolev spaces $H^{r+2\alpha}(\Gamma)$ into spaces $H^r(\Gamma)$, [4]. If 2α is positive the operator differentiates the function and if 2α is negative it integrates a function. In a symbolic notation we may, therefore, replace the kernels by the shifts, they effect, i.e., $2\alpha = -1$ (the kernel integrates once), $2\alpha = 0$ (it neither differentiates nor integrates) etc.

The boundary values of a plate are, roughly said, zero-th, first, second and third derivatives. So, in a symbolic notation, we may replace them by the numbers 0, 1, 2 and 3 resp. The four compatibility conditions, thus, assume the following format:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 & -2 & -1 & 0 \\ -2 & -1 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \cdots.$$

The first equation is the influence function for $w \triangleq 0$. In this equation all boundary terms are mapped onto w, the function on the left-hand side, that is they are integrated (besides w, of course), the third derivative, the Kirchhoff-shear, even three times. The last equation is the influence function of the Kirchhoff-shear $V_n = 3$. In this equation all the boundary functions (besides V_n) are differentiated; the deflection w even three times. But a kernel which differentiates a function three times is very, very singular, actually it is of the order r^{-3}, and the numerical taming and handling of such a kernel is a delicate affair.

6 Green's function

Closely related to the idea to solve the compatibility conditions by Galerkin's method is the idea to derive stiffness matrices by way of Green's functions, [5], [6]. Here too, a double integration process of a symmetric kernel renders a symmetric stiffness matrix as we shall explain next.

In the absence of distributed loads the potential energy of a Kirchhoff plate which is subject to a certain deflection, w, and slope, $\frac{\partial w}{\partial n}$, conditions on the boundary

$$\Pi_1(w) = \frac{1}{2} E(w, w)$$

can be expressed in terms of boundary integrals alone

$$\Pi_1(w) = \frac{1}{2} E(w, w) = \frac{1}{2} \int_\Gamma \left[V_n w - M_n \frac{\partial w}{\partial n} \right] ds. \tag{18}$$

If we formulate the compatibility conditions (17) with Green's function $G_0 = g_0 + w_R$, the deflection of the clamped plate loaded with a concentrated force $P = 1$, the 3rd and 4th equation in (17) simplify to

$$\frac{1}{2} M_n = \int_\Gamma [\partial_y^3 \partial_x^2 G_0 \partial^0 w + \partial_y^2 \partial_x^2 G_0 \partial^1 w] \, ds_y,$$

$$\frac{1}{2} V_n = \int_\Gamma [\partial_y^3 \partial_x^3 G_0 \partial^0 w + \partial_y^2 \partial_x^3 G_0 \partial^1 w] \, ds_y.$$

Substituting these right-hand sides into (18) and letting the displacement terms on the boundary

$$w = \delta_i \varphi_i, \qquad \frac{\partial w}{\partial n} = \varepsilon_i \varphi_i$$

the potential energy becomes a quadratic form of the nodal values

$$\Pi_1(w) = \frac{1}{2} [\boldsymbol{\delta}, \boldsymbol{\epsilon}] \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\epsilon} \end{bmatrix}$$

i.e. the stiffness matrix

$$a_{ij} = \int_\Gamma \int_\Gamma \partial_y^3 \partial_x^3 G_0 \varphi_i \, ds_y \varphi_j \, ds_x,$$

$$b_{ij} = \int_\Gamma \int_\Gamma \partial_y^3 \partial_x^2 \, G_0 \varphi_i \, ds_y \varphi_j \, ds_x = \int_\Gamma \int_\Gamma \partial_y^2 \partial_x^3 \, G_0 \varphi_j \, ds_y \varphi_i \, ds_x,$$

$$c_{ij} = \int_\Gamma \int_\Gamma \partial_y^2 \partial_x^2 \, G_0 \varphi_i \, ds_y \varphi_j \, ds_x$$

is symmetric.

The obvious difficulty with this approach is that we need to know Green's function of the structural element in question. Only in a limited number of cases do we possess analytical expression of these functions, [6].

To close, let us return to the elastic plate and the difficulty we had with the stiffness matrix. If, following the example of the Kirchhoff plate, we denote by

$$\partial^0 \mathbf{u} = \mathbf{u}, \qquad \partial^1 \mathbf{u} = \boldsymbol{\tau}(\mathbf{u})$$

the boundary operators of the elastic plate Betti's principle reads

$$\int_\Omega -\mathbf{L}\mathbf{u} \cdot \hat{\mathbf{u}} \, d\Omega + \int_\Gamma \partial^1 \mathbf{u} \cdot \partial^0 \hat{\mathbf{u}} \, ds = \int_\Gamma \partial^0 \mathbf{u} \cdot \partial^1 \hat{\mathbf{u}} \, ds + \int_\Omega \mathbf{u} \cdot (-\mathbf{L}\hat{\mathbf{u}}) \, d\Omega$$

and the complete set of compatibility conditions for the boundary values now assumes the form

$$\frac{1}{2} \begin{bmatrix} \partial^0 \mathbf{u} \\ \partial^1 \mathbf{u} \end{bmatrix} = \int_\Gamma \begin{bmatrix} (\partial_y^0 \partial_x^0 \mathbf{U})^T & (\partial_y^1 \partial_x^0 \mathbf{U})^T \\ (\partial_y^0 \partial_x^1 \mathbf{U})^T & (\partial_y^1 \partial_x^1 \mathbf{U})^T \end{bmatrix} \begin{bmatrix} \partial^1 \mathbf{u} \\ \partial^0 \mathbf{u} \end{bmatrix} ds_y + \cdots, \tag{19}$$

where $\mathbf{U} = [U_{ij}]$ is the Somigliana matrix whose columns are the fundamental solutions corresponding to concentrated unit forces $\mathbf{e}_1$ and $\mathbf{e}_2$ acting at $\mathbf{x}$ and where the columns of the matrix $\partial_{\mathbf{x}}^1\mathbf{U}$ are the fundamental solutions corresponding to horizontal and vertical displacement discontinuities.

The first matrix equation is identical with the two ($i = 1, 2$) compatibility conditions (10).

If we would add to $\mathbf{U}$ a matrix $\mathbf{U}_R$ such that the columns of $\mathbf{U} + \mathbf{U}_R$ comply with homogeneous displacement conditions on the boundary then in the process of the derivation of the second equation the first integral in (19) would vanish and we would obtain an influence function for the traction vector $\mathbf{t}(\mathbf{x}) = \partial^1\mathbf{u}$ in terms of the boundary displacement $\mathbf{u}$ alone

$$\mathbf{t}(\mathbf{x}) = \int_\Gamma (\partial_{\mathbf{y}}^1 \partial_{\mathbf{x}}^1 \mathbf{U})^T \mathbf{u} \, ds_{\mathbf{y}}$$

and as the kernel of this influence function is symmetric Galerkin's method

$$\int_\Gamma t_j \varphi_i \, ds = \int_\Gamma \int_\Gamma (\partial_{\mathbf{y}}^1 \partial_{\mathbf{x}}^1 \mathbf{U})_{ij}^T u_j \, ds_{\mathbf{y}} \varphi_i \, ds_{\mathbf{x}}$$

would, thus, render a symmetric stiffnes matrix.

Comparing this result with our previous result, $\mathbf{K} = \mathbf{F}\mathbf{G}^{-1}\mathbf{H}$, we recognize that what we did by multiplying the compatibility condition from the left with $\mathbf{G}^{-1}$ was that we approximated the symmetric kernel $\partial_{\mathbf{y}}^1 \partial_{\mathbf{x}}^1 \mathbf{U}$ by the discrete and unsymmetric kernel $\mathbf{G}^{-1}\mathbf{H}$.

For completeness we should mention that Mustoe, [2], gave the energy approach an additional twist. Instead of the potential energy he uses the basic functional (= potential energy + additional boundary terms, s. [3]), and lets $\mathbf{u}$ and $\mathbf{t}$, as well as the virtual displacements $\hat{\mathbf{u}}$, the boundary values of a potential of the first kind. This requires actually three integrations, one for $\mathbf{u}$, one for $\mathbf{t}$ and one for the evaluation of the variation of the basic functional.

References

1. Brebbia, C. A.; Georgiou, P.; Combination of boundary and finite elements for elastostatics. Appl. Math. Modelling 3 (1979) 212–220
2. Zienkiewicz, O. C.; Kelly, D. W.; Bettes, P.: The coupling of the finite element method and boundary solution procedures. Int. J. Num. Meth. Eng. 11 (1977) 355–375
3. Hartmann, F.: The mathematical foundation of structural mechanics. Berlin, Heidelberg, New York: Springer 1985
4. Wendland, W. L.: On Galerkin collocation methods for integral equations of elliptic boundary value problems. In Numerical Treatment of Integral Equations, Ed.: J. Albrecht and L. Collatz. ISNM 53, Basel: Birkhäuser 1980, pp. 244–275
5. Margulies, M.: Combination of the boundary element and finite element methods. In Progress in Boundary Element Methods, Vol. 1, C. A. Brebbia, Ed. London: Pentech; New York: Halstead Press 1981
6. Katz, C.: The use of Green's functions in the numerical analysis of potential, elastic and plate bending problems. In Boundary Element Methods, C. A. Brebbia, Ed., Berlin, Heidelberg, New York: Springer 1981

Received December 12, 1984

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