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Constructions of Strictly Ergodic Systems

I. Given Entropy

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Recent results of Jewett [2] and Krieger [3] assert, that every ergodic measure theoretic dynamical system on a Lebesgue space is isomorphic to a strictly ergodic dynamical system which is embedded in a shift space with an alphabet, the minimum length of which is determined by the entropy of the transformation. However, the corresponding strictly ergodic system is not given in a constructive way. Therefore there remains the problem of constructing strictly ergodic systems. It turns out that shift spaces with finite alphabet are particularly well suited for such constructions because of their simple topological properties. In this way, Hahn and Katznelson ([1]) have found strictly ergodic systems with arbitrarily large entropies in shift spaces. In this paper, we obtain the same result by a new and considerably simpler construction. Moreover, for constructively given $h<\log k$, we construct a system with entropy h in a shift space with alphabet of length k. Also, in a shift space with compact infinite alphabet, we construct a system with infinite entropy. For the constructions we use only permutations of block systems. A considerable simplification of the proofs is possible by the result of Parry [5], that in our case topological and measure theoretic entropy are the same.

In a further paper we shall give a construction for strictly ergodic K-systems.

§ 1. Preliminaries

1. Compact Dynamical Systems

Notation. Let $\mathbb{Z}$ denote the set of integers, $\mathbb{N}=\{x\in\mathbb{Z}\mid x>0\}$. For $a\leqq b\in\mathbb{Z}$, $\langle a,b\rangle=\{x\in\mathbb{Z}\mid a\leqq x\leqq b\}$. $M\subseteq\mathbb{Z}$ is said to be *dense in* $\mathbb{Z}$, if

$$\exists L\in\mathbb{N}\ \forall t\in\mathbb{Z}\colon \langle t,t+L\rangle\cap M\neq\emptyset.$$

A sequence $(a_n)_{n\in\mathbb{Z}}$ with $a_n\in\mathbb{R}$ (the reals) is *uniform Cesàro*, if there exists $a\in\mathbb{R}$ such that $\lim_j \frac{1}{j}\sum_{i=t+1}^{t+j} a_i=a$ uniformly for all $t\in\mathbb{Z}$. $M\subseteq\mathbb{Z}$ is uniform Cesàro if $1_M=(1_M(n))_{n\in\mathbb{Z}}$ is uniform Cesàro.

For a compact space K, $C(K)=\{f\colon K\to\mathbb{R}\mid f \text{ continuous}\}$ with the maximum norm. $F\subseteq C(K)$ is *total in* $C(K)$ if the linear space it generates is dense.

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** Parts of the results are contained in the thesis of the author (Erlangen 1970).

Definition 1.1. (Ω_0, T) is a *compact dynamical system*, if Ω_0 is a compact metric space and $T: \Omega_0 \to \Omega_0$ a topological automorphism.

For an invariant subset $\Omega \subseteq \Omega_0 (T\Omega = \Omega)$, the restriction of T to Ω is again denoted by T. For $\omega \in \Omega_0$, $O(\omega) = \{T^n \omega \mid n \in \mathbb{Z}\}$ is called the *orbit* and $\overline{O(\omega)}$ the *orbit closure* of ω.

Definition 1.2. a) A closed invariant subset $\Omega \neq \emptyset$ of Ω_0 is *minimal invariant*, if

$$\emptyset \neq \Omega' \subseteq \Omega, \qquad \Omega' \text{ closed invariant} \Rightarrow \Omega' = \Omega;$$

uniquely ergodic, if there exists a unique T-invariant probability measure μ on Ω;

strictly ergodic, if it is minimal invariant and uniquely ergodic.

(Ω, T) is then called a *minimal* resp. *uniquely ergodic* resp. *strictly ergodic dynamical system.*

b) $\omega \in \Omega_0$ is *almost periodic*, if for every neighbourhood U of ω $\{t \in \mathbb{Z} \mid T^t \omega \in U\}$ is dense in $\mathbb{Z}$;

strictly transitive, if for each f in a total subset of $C(\Omega_0)$ the sequence $(f \circ T^n(\omega))_{n \in \mathbb{Z}}$ is uniform Cesàro;

strictly ergodic, if it is strictly transitive and almost periodic.

Remarks. 1. By Zorn's lemma, one can show that Ω_0 contains at least one minimal invariant subset.

2. On every non-empty, closed, invariant $\Omega \subseteq \Omega_0$ there exists at least one invariant probability measure. If this is unique, it is of course ergodic.

3. If Ω is uniquely ergodic and μ is the invariant measure, then the support of μ is minimal. On the other hand, there are minimal invariant sets which are not strictly ergodic (see [4]).

The relation between minimality properties of sets and regularity properties of points is given by the following results, which can be found in [4].

Theorem 1.3 (Gottschalk). *For a non-empty, closed, invariant $\Omega \subseteq \Omega_0$, the following are equivalent:*

a) *Ω is minimal invariant.*

b) $\forall \omega \in \Omega$: $\overline{O(\omega)} = \Omega$.

c) $\exists \omega \in \Omega$: $\overline{O(\omega)} = \Omega$, *$\omega$ is almost periodic.*

a) *implies: Each $\omega \in \Omega$ is almost periodic.*

Theorem 1.4 (Oxtoby). *For $\Omega \subseteq \Omega_0$ non-empty, closed, invariant, the following are equivalent:*

a) *Ω is uniquely ergodic.*

b) *For $f \in C(\Omega)$ (or $f \in C(\Omega_0)$) there exists $\bar{f} \in \mathbb{R}$ such that*

$$\lim_t \frac{1}{t} \sum_{i=1}^{t} f \circ T^i(\omega) = \bar{f}$$

uniformly for all $\omega \in \Omega$.

a) *is true, if $\Omega = \overline{O(\omega)}$ for a strictly transitive $\omega \in \Omega$.*

Corollary 1.5. *For $\Omega \subseteq \Omega_0$ the following are equivalent:*

a) *Ω is strictly ergodic.*

b) *$\Omega = \overline{O(\omega)}$ for a strictly ergodic $\omega \in \Omega$.*

c) *$\omega \in \Omega \Rightarrow \overline{O(\omega)} = \Omega$ and ω is strictly ergodic.*

Notation. If Ω is a strictly ergodic set and $\omega \in \Omega$, the unique invariant measure is denoted by μ_Ω or μ_ω.

2. *Strictly Ergodic Dynamical Systems in a Finite Shift Space and Their Entropy*

The space Ω_0 in which we construct strictly ergodic points will in general be a finite shift space.

Let A be a finite set (the alphabet),

$$\Omega_A = A^{\mathbb{Z}} = \{\omega = (\omega_n)_{n\in\mathbb{Z}} \mid \forall n \in \mathbb{Z}: \omega_n \in A\}$$

with the product topology (A is a discrete topological space), and T the shift transformation $T: (\omega_n)_{n\in\mathbb{Z}} \to (\omega_{n+1})_{n\in\mathbb{Z}}$. If $A = \langle 0, k-1 \rangle$ for some $k \in \mathbb{N}$, we write Ω_k instead of $\Omega_{\langle 0, k-1 \rangle}$.

We use the following notation to characterize subsets of Ω_A: $\Omega_A^f = \bigcup_{r\in\mathbb{N}} A^r$, the set of non-empty finite words or *blocks* over A. For $P \in A^r$ we call $l(P) = r$ the *length of P*. If $P = (p_1, \dots, p_{l(P)})$, $Q = (q_1, \dots, q_{l(Q)})$, let

$$PQ = P \cdot Q = (p_1, \dots, p_{l(P)}, q_1, \dots, q_{l(Q)}) \in \Omega_A^f.$$

For $\omega \in \Omega_A$, $s \leqq t \in \mathbb{Z}$ $\omega(\langle s, t \rangle) = (\omega_s, \dots, \omega_t) \in \Omega_A^f$. For $s \leqq t \in N$ and $Q \in A^t$, we define a probability vector μ_Q on A^s by the relative frequencies

$$\mu_Q(P) = \frac{1}{t-s+1} \left|\{j \leqq t-s+1 \mid (q_j, \dots, q_{j+s-1}) = P\}\right|.$$

For $r \in \mathbb{Z}$, $P \in \Omega_A^f$, the set

$${}_r[P] = \{\omega \in \Omega_A \mid \omega(\langle r, r+l(P)-1 \rangle) = P\}$$

is called a *finite cylinder*. The system of finite cylinders is a base for the topology in Ω_A, consisting of closed and open sets, and preserved under T. Hence T is an automorphism of Ω_A. The indicator functions of the finite cylinders are by the Stone-Weierstraß theorem a total set in $C(\Omega_A)$. It is therefore easy to characterize minimality properties of points in Ω_A. Almost periodic, strictly transitive and strictly ergodic elements of Ω_A will be called almost periodic sequences etc.

Lemma 1.6. a) *$\omega \in \Omega_A$ is almost periodic iff for each $P \in \Omega_A^f$ the set $\{r \in \mathbb{Z} \mid \omega \in {}_r[P]\}$ is either empty or dense in $\mathbb{Z}$.*

b) *ω is strictly transitive iff for each $P \in \Omega_A^f$ the set above is uniform Cesàro in $\mathbb{Z}$.*

Proof. Immediate consequence of Definition 1.2. $\lrcorner$

Let $P, Q \in \Omega_A^f$, $\omega \in \Omega_A$, $M \subseteq \Omega_A$. We write

$P \prec Q$ if $l(P) \leqq l(Q)$ and P is a subblock of Q;

$P \prec \omega$ if $\exists r \in \mathbb{Z}: \omega \in {}_r[P]$;

$P \prec M$ if $\exists \eta \in M: P \prec \eta$.

Especially important sub-σ-algebras of the Borel σ-algebra B on Ω_A are the following: $B_j=\sigma({}_j[x]\,|\,x\in A)(j\in\mathbb{Z})$, the finite algebra generated by the j-th coordinate mapping;

$$B_{\langle s,t\rangle}=\bigvee_{s\leqq j\leqq t} B_j=\bigvee_{s\leqq j\leqq t} T^{-j}B_0=\sigma({}_s[P]\,|\,P\in A^{t-s+1});$$

$$B_{\langle s,\infty\rangle}=\bigvee_{s\leqq j\in\mathbb{Z}} B_j.$$

Measure Theoretic and Topological Entropies in Ω_A

If μ is a T-invariant probability measure on Ω_A, the entropy $h(\mu,T)=h(\mu)$ can be computed equivalently by the formulae

$$h(\mu)=\lim_n \frac{1}{n} H_\mu(B_{\langle 0,n-1\rangle})=\lim_n H_\mu(B_0\,|\,B_{\langle 1,n\rangle}),$$

where for finite σ-algebras C, D with atoms $(c_1,\dots,c_r)$ resp. $(d_1,\dots,d_s)$

$$H_\mu(C)=\sum_i z(\mu(c_i)),$$

$$H_\mu(C\,|\,D)=\sum_j \mu(d_j)\sum_i z\left(\frac{\mu(c_i\cap d_j)}{\mu(d_j)}\right),$$

and

$$z(0)=0,\qquad z(x)=-x\log x\quad (0<x\leqq 1).$$

It is an elementary fact that $h(\mu)\leqq\log|A|$, and equality is possible only if μ is the product of equidistributions on A. Hence, for every strictly invariant $\Omega\subseteq\Omega_k$: $h(\mu_\Omega)<\log k$. We shall show that $\log k$ can be arbitrarily approximated by the entropies of such measures. For this purpose we use essentially the topological entropy, which in Ω_A can be defined as follows: Let $\Omega\subseteq\Omega_A$ be a non-empty, closed, invariant subset and

$$\theta_n=\theta_n(\Omega)=|\{P\in A^n\,|\,P\prec\Omega\}|.$$

Then $1\leqq\theta_{n+m}\leqq\theta_n\cdot\theta_m$; hence, $\lim\limits_n \frac{1}{n}\log\theta_n$ exists.

Definition 1.7. $h(\Omega)=\lim \frac{1}{n}\log\theta_n(\Omega)$ is called the *topological entropy of* Ω. If $\Omega=\overline{O(\omega)}$, $h(\omega)=h(\Omega)$.

It is again easy to see that $h(\Omega_k)=\log k$ and $\Omega\subsetneqq\Omega_k\Rightarrow h(\Omega)<\log k$. The following theorem relates the two concepts of entropy.

Theorem 1.8 (Parry). a) *Let $\emptyset\neq\Omega\subseteq\Omega_A$ be closed invariant and μ an invariant probability measure on Ω. Then $h(\mu)\leqq h(\Omega)$.*

b) *If $\Omega\subseteq\Omega_A$ is minimal invariant, there is an invariant probability measure μ on Ω with $h(\mu)=h(\Omega)$. In particular, if Ω is strictly ergodic, $h(\Omega)=h(\mu_\Omega)$.*

Proof. a) comes from the elementary relation

$$H_\mu(B_{\langle 0,n-1\rangle})\leqq\log\theta_n(\Omega).$$

b) is a special case of Theorem 3 in [5]. $\lrcorner$

Substitutions of Constant Length in Strictly Ergodic Sequences

Let $\eta \in \Omega_A$, $r \in \mathbb{N}$ and $L\colon A \to A^r$ be an injection. For $\eta \in \Omega_A$ define $\eta^L \in \Omega_A$ by $\eta^L(\langle r\,n, r(n+1)-1\rangle) = L(\eta_n)$ $(n \in \mathbb{Z})$.

Lemma 1.9. *If η is strictly ergodic, then so is η^L, and $h(\eta^L) = \frac{1}{r}\,h(\eta)$.*

Proof. a) Strict ergodicity: Almost periodicity is trivial. For strict transitivity, let $Q \prec \eta^L$, $\varepsilon > 0$. We choose $n \in \mathbb{N}$ such that $r\,n > l(Q)$. For $S \in A^{rn}$, the set

$$\{u \in \mathbb{Z} \mid \eta^L \in_{ur}[S]\}$$

is uniform Cesàro by the strict transitivity of η. We denote its density by $\tau(S)$. Let for $s \in \mathbb{Z}$, $t > n$

$$\tau_{s,t}(S) = \frac{1}{t-n+1}\,\big|\{u \in \langle 0, t-n\rangle \mid \eta^L \in_{(s+u)r}[S]\}\big|.$$

We count for each $S \in A^{rn}$ the number of times it occurs in $\eta(\langle s\,r, (s+t)\,r-1\rangle)$ at places $(s+u)\,r$ $(0 \leqq u < t)$ and in each S the frequency of Q. Since each occurence of Q at places $s\,r+k$ $\big(r\,n \leqq k < (s+t-n)\,r\big)$ is thus counted between $\big(n-(l(Q)+2r)\big)$ and n times,

$$\Big|n\,\mu_{\eta^L(\langle sr,(s+t)r-1\rangle)}(Q)\cdot t\,r - \sum_{S \in A^{rn}} (t-n+1)\,\tau_{s,t}(S)\cdot\big(n\,r - l(Q)+1\big)\,\mu_s(Q)\Big|$$
$$< 2n^2\,r + \big(l(Q)+2r\big)\,t\,r.$$

If n is so large that $\frac{1}{n}\big(l(Q)+2r\big) < \varepsilon$, then for each t with $\frac{2n}{t} < \varepsilon$ and

$$|\tau_{s,t}(S) - \tau(S)| < \varepsilon\,|A|^{-rn} \qquad (S \in A^{rn}),$$

$$\Big|\mu_{\eta^L(\langle sr,(s+t)r-1\rangle)}(Q) - \sum_{S \in A^{rn}} \tau(S)\,\mu_S(Q)\Big| < 3\,\varepsilon \qquad \text{for all } s \in \mathbb{Z}.$$

This implies strict ergodicity.

b) Entropy: The result follows immediately from

$$\theta_n(\eta) \leqq \theta_{nr}(\eta^L) \leqq r\,\theta_{n+1}(\eta). \quad \lrcorner$$

§ 2. Strictly Ergodic Sequences with Given Finite Entropy

Let the two sequences of functions

$$m_j\colon \mathbb{N} \to \mathbb{N}, \qquad \lambda_j\colon \mathbb{N} \to \mathbb{R}^+$$

be defined inductively by

$$m_1 = 1, \qquad e^{m_1(k)\,\lambda_1(k)} = k$$
$$m_{j+1} = m_j\,e^{m_j\lambda_j}, \qquad e^{m_{j+1}\lambda_{j+1}} = (e^{m^j\lambda^j})\,!$$

and $\lambda = \lim \lambda_j$.

We omit the argument k whenever there can be no confusion. Using the notation $\overset{\leftarrow 0 \rightarrow}{k!\dots!}=k$, $\overset{\leftarrow j \rightarrow}{k!\dots!}=(\overset{\leftarrow j-1 \rightarrow}{k\,!\dots!}\,)!$, we obtain, for $j \geqq 2$, the explicit expressions

$$e^{m_j \lambda_j}=\overset{\leftarrow j-1 \rightarrow}{k\,!\dots!}\,, \qquad m_j=\prod_{i=0}^{j-2} \overset{\leftarrow i \rightarrow}{k!\dots!}, \qquad \text{and } \lambda_j=m_j^{-1} \log \overset{\leftarrow i-1 \rightarrow}{k\,!\dots!}\,.$$

The existence of λ is granted by

Lemma 2.1. *$(\lambda_j)_{j \in \mathbb{N}}$ is a decreasing sequence. $\lambda(1)=\lambda(2)=0$, $\lambda(k)>0$ $(k \geqq 3)$ and $\lim_k(\log k-\lambda(k))=1$.*

Proof. $k=1$: $m_j=1$, $\lambda_j=0$ for $j \in \mathbb{N}$.
$k=2$: $e^{m_j \lambda_j}=2$, $m_j=2^{j-1}$, hence $\lambda_j=2^{-j+1} \log 2$.
$k \geqq 3$: First we see that $e^{m_j \lambda_j}>m_j+1$ and $m_{j+1}>m_j^2$; for $e^{m_1 \lambda_1}=k>2=m_1+1$,

$$e^{m_{j+1} \lambda_{j+1}}=(e^{m_j \lambda_j})!>(e^{m_j \lambda_j}-2)\, e^{m_j \lambda_j}+1 \geqq m_j\, e^{m_j \lambda_j}+1=m_{j+1}+1.$$

Monotonicity of λ_j is seen from $n!<n^n$:

$$e^{m_{j+1} \lambda_{j+1}}<e^{m_j \lambda_j e^{m_j \lambda_j}}=e^{m_{j+1} \lambda_j}.$$

Using Stirling's formula $1 \leqq n!\,(2\pi n)^{-\frac{1}{2}}\, n^{-n}\, e^n \leqq e^{\frac{1}{12n}}$, we estimate

$$0 \leqq m_{j+1} \lambda_{j+1}-\tfrac{1}{2}(\log 2\pi+m_j \lambda_j)-m_{j+1} \lambda_j+e^{m_j \lambda_j} \leqq \tfrac{1}{12}\, e^{-m_j \lambda_j} \leqq \tfrac{1}{36};$$

$$0<\lambda_{j+1}-\lambda_j+\frac{1}{m_j} \leqq \frac{1}{2 m_{j+1}}\left(\frac{1}{18}+\log 2\pi+m_j \lambda_j\right)<2 m_{j+1}^{-1} \cdot m_j \lambda_j \leqq 2 \log k\, e^{-m_j \lambda_j}$$

$$0<\lambda(k)-\log k+\sum_{j \in \mathbb{N}} \frac{1}{m_j} \leqq 2 \log k \sum_{j \in \mathbb{N}} e^{-m_j \lambda_j}.$$

Since $m_1=1$, $m_2=k$, we have:

$$1<\sum_{j \in \mathbb{N}} \frac{1}{m_j}<\sum_j\left(\frac{1}{k}\right)^j=\frac{k}{k-1}$$

and

$$\lambda(k)>\lambda_1(k)-\sum_j \frac{1}{m_j(k)} \geqq \log k-\frac{k}{k-1}>0 \qquad \text{for } k \geqq 4,$$

$$\lambda(3)=\tfrac{1}{3}\, \lambda(3!)>0.$$

From

$$2 \log k \sum_{j \in \mathbb{N}} e^{-m_j \lambda_j}<2(k-1)^{-1} \log k \rightarrow 0$$

and $\dfrac{k}{k-1} \rightarrow 1$ it follows that $\log k-\lambda(k) \xrightarrow[k \rightarrow \infty]{} 1$. $\lrcorner$

Suppose $M \subseteq \Omega_A^f$ is a finite set of blocks *(block system)* of constant length r. M can be given an order, e.g. the lexicographic one: $M=\{Q_1, \dots, Q_{|M|}\}$.

Definition 2.2. $\tilde{M}=\{Q_{\sigma(1)} \cdot \dots \cdot Q_{\sigma(|M|)} \mid \sigma \in \mathfrak{S}_{|M|}\}$. ($\mathfrak{S}_i$ is the permutation group of $\langle 1, i\rangle$.)

Remark. $|\tilde{M}|=|M|!$ and the elements of $\tilde{M}$ have the constant length $r\,|M|$.

Lemma 2.3. *For each $k \in \mathbb{N}$ there is a strictly ergodic sequence $\eta \in \Omega_k$ with $h(\eta) = \lambda(k)$.*

Proof. The case $k=1$ is trivial. For $k>1$ put

$$M_1 = \langle 0, k-1 \rangle$$
$$M_{j+1} = \tilde{M}_j \qquad (j \in \mathbb{N}).$$

By induction one shows that $|M_j| = e^{m_j \lambda_j}$ and the elements of M_j have the constant length m_j. So,

$$\frac{1}{m_j} \log |M_j| = \lambda_j \to \lambda(k).$$

In M_j we choose two blocks L_j, F_j such that

$$\begin{aligned} &L_j \quad \text{is the tail of} \quad L_{j+1}, \\ &F_j \quad \text{is the head of } F_{j+1} \qquad (j \in \mathbb{N}) \end{aligned}$$

and put

$$\eta(\langle -m_j, m_j - 1 \rangle) = L_j \cdot F_j.$$

This defines a sequence $\eta \in \Omega_k$.

η is strictly ergodic: To prove almost periodicity, let $Q \in \Omega_A^f$ with $Q \prec \eta$. Then there is a $j \in \mathbb{N}$ such that $L_j \neq F_j$ and $Q \prec L_j \cdot F_j$, and a $P \in M_{j+1}$ with $L_j \cdot F_j \prec P$. Since $P \prec S$ for each $S \in M_{j+2}$, Q occurs in each block $\eta(\langle t m_{j+2}, (t+1) m_{j+2} - 1 \rangle)$ $(t \in \mathbb{Z})$. Strict transitivity of η can be shown as for η^L in Lemma 1.9, since for each $P \in M_j$ the set $\{t \in \mathbb{Z} \mid \eta \in_{t m_j} [P]\}$ is uniform Cesàro.

Now we show $h(\eta) = \lambda(k)$. That $h(\eta) \geqq \lambda(k)$ is trivial, because $(P \in M_j \Rightarrow P \prec \eta)$ implies $\theta_{m_j}(\eta) \geqq |M_j|$. On the other hand, let $l(P) = m_{j+1}$, $P \prec \eta$. Then there exists

$$Q = Q_1 \cdot \dots \cdot Q_{|M_j|+1} \qquad (Q_j \in M_j \text{ for } 1 \leqq i \leqq |M_j| + 1)$$

with $P \prec Q$. Therefore

$$\theta_{m_{j+1}}(\eta) \leqq m_j \, |M_j|^{|M_j|+1} = m_{j+1} \, |M_j|^{|M_j|}$$

and

$$\begin{aligned} h(\eta) &\leqq \frac{1}{m_{j+1}} \log \theta_{m_{j+1}}(\eta) \leqq \frac{\log m_{j+1}}{m_{j+1}} + \frac{|M_j|}{m_{j+1}} \log |M_j| \\ &= \frac{\log m_{j+1}}{m_{j+1}} + \frac{1}{m_j} \log |M_j| \to \lambda(k). \quad \lrcorner \end{aligned}$$

Remark. For $k=2$, in case $L_1 = F_1 = 0$, the sequence defined is the Morse sequence μ of Gottschalk-Hedlund, *Topological Dynamics*, 12.28. The sequences of 12.37 ibidem are obtained by varying the choices of L_1 and F_1: μ' for $L_1 = F_1 = 1$; ν for $L_1 = 1$, $F_1 = 0$; ν' for $L_1 = 0$, $F_1 = 1$.

Lemma 2.4. *For $k \geqq 3$, $0 < h < \lambda(k)$, there exists an $\eta \in \Omega_k$ with $h(\eta) = h$.*

Proof. Let $j_0 = \min \left\{ j \in \mathbb{N} \mid j \geqq 2,\ \lambda_j(k) \geqq h + \frac{2}{m_j} \right\}$. We construct a sequence of block systems $(N_j)_{j \in \mathbb{N}}$, each of constant length n_j, with $N_{j+1} \subseteq \tilde{N}_j$, and such that in each N_j there are three different elements L_j, F_j, S_j (S_j only for $j \geqq 2$) with the properties

$$L_j \quad \text{is the tail of} \quad L_{j+1},$$
$$F_j \quad \text{is the head of}\, F_{j+1}, \tag{$*$}$$
$$L_{j-1}\cdot F_{j-1} \prec S_j.$$

This is achieved by putting

$$N_1 = \langle 0, k-1\rangle,$$
$$N_j = \tilde{N}_{j-1} \qquad (1<j<j_0)$$

and by selecting, for each $j \geqq j_0$, $N_j \subseteq \tilde{N}_{j-1}$ such that $(*)$ is fulfilled and

$$n_j h + 3 > \log |N_j| \geqq n_j h + 2. \tag{$**$}$$

The choice is possible, since $(**)$ implies $|N_j| > e^2 > 4$ and

$$\log |\tilde{N}_j| = \log(|N_j|\,!) \geqq |N_j|(\log |N_j| - 1) \geqq |N_j|(n_j h + 1) > n_{j+1} h + 2.$$

We define again $\eta \in \Omega_k$ by

$$\eta(\langle -n_j, n_j - 1\rangle) = L_j \cdot F_j.$$

Strict ergodicity of η follows as in Lemma 2.3. By $(**)$, for $j \geqq j_0$:

$$h + \frac{3}{n_j} > \frac{1}{n_j} \log |N_j| \geqq h + \frac{2}{n_j}.$$

Therefore

$$h(\eta) \geqq \lim_j \frac{1}{n_j} \log |N_j| = h.$$

$\theta_{n_{j+1}}(\eta) \leqq n_j\, |N_j|^{|N_j|+1} = n_{j+1}\, |N_j|^{|N_j|}$ furnishes again

$$h(\eta) \leqq \frac{1}{n_{j+1}} \log \theta_{n_{j+1}}(\eta) \leqq \frac{\log n_{j+1}}{n_{j+1}} + \frac{1}{n_j} \log |N_j| \to h. \quad \lrcorner$$

Theorem 2.5. *For $k \geqq 2$ and $0 < h < \log k$, there exists a strictly ergodic sequence $\omega \in \Omega_k$ with $h(\omega) = h$.*

Proof. Since $\lambda(k^n) > \log k^n - 2 = n \log k - 2$, we can choose n so large that $n h < \lambda(k^n)$. We now find a strictly ergodic sequence $\eta \in \Omega_{k^n}$ with $h(\eta) = n h$ and a bijective mapping L: $\langle 0, k^n - 1\rangle \to \langle 0, k-1\rangle^n$. By Lemma 1.9, we have for the strictly ergodic sequence $\eta^L \in \Omega_k$: $h(\eta^L) = h$. $\lrcorner$

§ 3. A Sequence with Infinite Entropy

1. Permutation Sequences with Repetitions

Let $k \geqq 2$ and $r = (r_j)_{j \in \mathbb{N}}$ be a sequence of natural numbers. Construct a sequence $(N_j)_{j \in \mathbb{N}}$ of block systems N_j, each of constant length n_j, by induction as follows. Set $N_1 = \langle 0, k-1\rangle$. If N_j is constructed, let

$$\mathfrak{S}^j_{k,r} = \{\sigma\colon \langle 1, r_j\, |N_j|\rangle \to N_j \,\big|\, |\sigma^{-1}(P)| = r_j \; (P \in N_j)\}$$

and

$$N_{j+1} = \{\sigma(1) \cdot \cdots \cdot \sigma(r_j\, |N_j|) \,|\, \sigma \in \mathfrak{S}^j_{k,r}\}.$$

Then we have the relations

$$|N_1| = k, \qquad n_1 = 1$$

$$|N_{j+1}| = \frac{(r_j\,|N_j|)!}{r_j!^{|N_j|}}, \qquad n_{j+1} = n_j\, r_j\, |N_j|.$$

Let again L_j, $F_j \in N_j$ be such that

$$L_j \quad \text{is the tail of} \quad L_{j+1},$$
$$F_j \quad \text{is the head of } F_{j+1},$$

and define $\eta \in \Omega_k$ by

$$\eta(\langle -n_j, n_j - 1\rangle) = L_j \cdot F_j.$$

We call r the repetition frequency of η.

Lemma 3.1. *η is a strictly ergodic sequence, and $h(\eta) \geqq \frac{1}{2}\log k$ if $k \geqq 10$.*

Proof. Strict ergodicity is shown as for all other sequences considered before. To estimate the entropy, set $|N_j| = e^{n_j \tau_j}$. By Stirling's formula,

$$|N_{j+1}| > \frac{(r_j\,|N_j|)^{r_j |N_j|}\, e^{-r_j |N_j|}}{r_j^{r_j |N_j|}} = \left(\frac{|N_j|}{e}\right)^{r_j |N_j|},$$

and thus

$$e^{n_{j+1}\tau_{j+1}} > e^{(n_j \tau_j - 1)\, r_j |N_j|} = e^{n_{j+1}\left(\tau_j - \frac{1}{n_j}\right)}; \qquad \tau_{j+1} > \tau_j - \frac{1}{n_j}.$$

Now, since for $k \geqq 6$: $e \cdot \sqrt[k]{k^2} < 6$, and $1 = n_1 < |N_1| = k$, we can show by induction:

$$n_{j+1} = n_j\, r_j\, |N_j| < (r_j\,|N_j|)^2 < \left(\frac{6}{e}\right)^{r_j |N_j|} \leqq \left(\frac{|N_j|}{e}\right)^{r_j |N_j|} < |N_{j+1}|,$$

and hence, for $k > 6$: $n_{j+1} > n_j^2$. This implies

$$\tau_j > \log k - \frac{k}{k-1} > \frac{1}{2}\log k \qquad (k \geqq 10,\ j \in \mathbb{N}).$$

Since $\theta_{n_j}(\eta) \geqq |N_j|$, we have $\dfrac{1}{n_j}\log\theta_{n_j}(\eta) \geqq \tau_j > \frac{1}{2}\log k$ and $h(\eta) \geqq \frac{1}{2}\log k$ for $k \geqq 10$. $\lrcorner$

2. A Shift Space with Infinite Alphabet

Let $(A^j)_{j \in \mathbb{N}}$ be a strictly increasing sequence of finite sets with $A^1 = \{0\}$ and let $A = \bigcup A^j$ be topologized in such a way that 0 is the Alexandroff point of $A \setminus \{0\}$. Define Ω_A, Ω_A^f and T as for finite alphabets.

Let π^j: $\Omega_A \cup \Omega_A^f \to \Omega_{A^j} \cup \Omega_{A^j}^f$ be the mapping induced on the sequences and blocks by

$$x \mapsto \begin{cases} x & x \in A^j \\ 0 & x \in \complement A^j \end{cases}$$

$$A \to A^j.$$

π^j: $\Omega_A \to \Omega_{A^j}$ is continuous and commutes with T.

Lemma 3.2. *$\omega\in\Omega_A$ is almost periodic (strictly transitive) iff each $\pi^j\omega\in\Omega_{A^j}$ is almost periodic (strictly transitive).*

Proof. a) If ω is almost periodic, $\pi^j\,\overline{O(\omega)}$ is invariant and closed by the properties of π^j, and for $\eta\in\overline{O(\omega)}$, $O(\pi^j(\eta))$ is dense in $\pi^j\,\overline{O(\omega)}$. Therefore $\pi^j\omega$ is almost periodic.

If ω is strictly transitive, we have for $f\in C(\overline{O(\pi^j\omega)})$:

$$\left(f\circ T^n(\pi^j\omega)\right)_{n\in\mathbb{Z}}=\left(f\circ\pi^j(T^n\omega)\right)_{n\in\mathbb{Z}}$$

is uniform Cesàro, because $f\circ\pi^j\in C(\overline{O(\omega)})$. Therefore $\pi^j\omega$ is strictly transitive.

b) Assume now each $\pi^j\omega$ to be almost periodic. A base for the topology in Ω_A is provided by the class of special finite cylinders of the form

$$\bigcap_{n\in N_1}{}_n[a_n]\cap\bigcap_{n\in N_2}{}_n[A^j]',$$

where $N_1,N_2\subseteq\mathbb{Z}$ are finite sets, $j\in\mathbb{N}$, $a_n\in A^j\setminus\{0\}$ $(n\in N_1)$, and

$${}_n[A^j]'=\{\eta\in\Omega_A\,|\,\eta_n\in(A\setminus A^j)\cup\{0\}\}.$$

Now, the cylinder above is equal to

$$(\pi^j)^{-1}\Big(\bigcap_{n\in N_1}{}_n[a_n]\cap\bigcap_{n\in N_2}{}_n[0]\Big).$$

The last set is open in Ω_{A^j} and is visited by $\pi^j\omega$ at an empty or dense set of times.

By the Stone-Weierstraß theorem, the indicator functions of the special finite cylinders are a total set in $C(\Omega_A)$. They can be written in the form $g=f\circ\pi^j$, where f is the indicator function of a cylinder in Ω_{A^j}. Therefore, if $\pi^j\omega$ is strictly transitive, the sequence $\left(g\circ T^n(\omega)\right)_{n\in\mathbb{Z}}=\left(f\circ T^n(\pi^j\omega)\right)_{n\in\mathbb{Z}}$ is uniform Cesàro. ┘

Let $\Omega\subseteq\Omega_A$ be strictly ergodic, $\omega\in\Omega$. We avoid the definition of topological entropy for general compact metric spaces and consider $h(\mu_\Omega)=h(\mu_\omega)$.

Lemma 3.3. $h(\mu_\omega)=\sup_{j\in\mathbb{N}}h(\pi^j\omega)$.

Proof. We only use and show the simpler direction "$\geqq$". The other inequality is proved by an approximation argument.

Set $\mu_j=\mu_{\pi^j\omega}$. Then $h(\pi^j\omega)=h(\mu_j)$, $\pi^j\mu_\omega=\mu_j$ by the uniqueness of the measure, and so: $h(\mu_\omega)\geqq h(\mu_j)$ $(j\in\mathbb{N})$. ┘

3. *Construction*

Let $(k_j)_{j\in\mathbb{N}}$, $(r_j)_{j\in\mathbb{N}}$, $(n_j)_{j\in\mathbb{N}}$ be the sequences of natural numbers defined by

$$k_1=1,\qquad n_j=\prod_{i\leqq j}k_j$$

$$r_j=[e^{3jn_j}],\qquad k_{j+1}=k_j!\,r_j,$$

(notice that $k_2!=r_1!=20!>10$) and $(A_j)_{j\in\mathbb{N}}$ a sequence of finite sets with

$$A_1=\{0\},\qquad |A_j|=k_j,\qquad A_i\cap A_j=\{0\}\qquad (i\neq j).$$

Set $A^j=\bigcup_{i\leqq j}A_i$, $A=\bigcup A^j$ as in the last section. Construct a sequence $(N_j)_{j\in\mathbb{N}}$ of block systems $N_j \subseteq \Omega^f_{A^j}$, each of constant length n_j, such that $|N_j| = k_j!$ and $\mu_P(0) = \frac{1}{n_j}$ for $P \in N_j$ (i.e. 0 occurs exactly once in each $P \in N_j$), as follows.

Let $N_1 = \{0\}$. If N_j is constructed, let $\sigma: A_{j+1} \to N_j$ be a mapping with $|\sigma^{-1}(P)| = r_j$ $(P \in N_j)$. σ is onto, because $|A_{j+1}| = r_j |N_j|$. For $a \in A_{j+1}$ let P_a be the block in $\Omega^f_{A^{j+1}}$, obtained when 0 in $\sigma(a)$ is replaced by a, and

$$\overline{N}_j = \{P_a \mid a \in A_{j+1}\}, \qquad N_{j+1} = \tilde{\overline{N}}_j.$$

$|\overline{N}_j| = k_{j+1}$, $|N_{j+1}| = k_{j+1}!$, and 0 occurs once in each element of N_{j+1}, because there is only one element of $\overline{N}_j$ in which it occurs. $P \in N_{j+1}$ has the length $k_{j+1}\, n_j = n_{j+1}$.

Let π^j be as before. Choose $L_j, F_j \in N_j$ such that

$$L_j \quad \text{is the tail of} \quad \pi^j L_{j+1},$$

$$F_j \quad \text{is the head of} \quad \pi^j F_{j+1},$$

and define $\eta \in \Omega_A$ by

$$\pi^j \eta(\langle -n_j, n_j - 1\rangle) = L_j \cdot F_j.$$

Theorem 3.4. *η is strictly ergodic and $h(\mu_\eta) = \infty$.*

Proof. We show that $\pi^j \eta$ is strictly ergodic and eventually $h(\pi^j \eta) > j$.

$\pi^j \eta$ may be constructed in the following way: $\pi^j N_j = N_j$. $\pi^j N_{j+1}$ is the set of blocks of length n_{j+1}, composed of N_j-blocks, where each block occurs exactly r_j times. More generally, $\pi^j N_{j+n+1}$ is the set of blocks of length n_{j+n+1}, composed of $\pi^j N_{j+n}$-blocks, where each block occurs $\frac{k_{j+n+1}}{|\pi^j N_{j+n}|} = \frac{n_{j+n+1}}{n_{j+n}\, |\pi^j N_{j+n}|}$ times. Further, for $n \geqq j$,

$$\pi^j L_n \quad \text{is the tail of} \quad \pi^j L_{n+1},$$

$$\pi^j F_n \quad \text{is the head of} \quad \pi^j F_{n+1}.$$

This means, one could construct a permutation sequence with alphabet N_j and repetition frequencies $\left(\frac{k_{j+n}}{|\pi^j N_{j+n-1}|}\right)_{n \in \mathbb{N}}$, and then substitute the block $P \in N_j$ for the element P of the alphabet. By the Lemmas 1.9 and 3.1, the resulting sequence $\pi^j \eta$ is strictly ergodic and

$$h(\pi^j \eta) \geqq \frac{1}{2n_j} \log |N_j| = \frac{\log k_j!}{2n_j} \geqq \frac{k_j(\log k_j - 1)}{2k_j n_{j-1}}$$
$$= (2n_{j-1})^{-1} (\log k_{j-1}! - 1 + \log r_{j-1}) > j \qquad (j \geqq 3). \quad \lrcorner$$

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