Z. Wahrscheinlichkeitstheorie verw. Gebiete
37, 145–155 (1976)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Note on Pseudo-Metrics on the Plane

R. V. Ambartzumian
Institute of Mathematics, Armenian Academy of Sciences, Erevan, USSR

Every c-finite* measure μ on the set G of the lines on the plane such that

(0) $\mu(\{g \in G: P \in g\}) = 0$

for every point $P \in R^2$ generates a pseudo-metric F on the plane when one puts $F(P_1, P_2) = \frac{1}{2}\mu(\{g \in G: g$ separates the points P_1 and $P_2\})$. The pseudo-metrics which are generated in this way possess the property of linear additivity, that is $F(P_1, P_3) = F(P_1, P_2) + F(P_2, P_3)$ for P_1, P_2, P_3 on a line, P_2 between P_1 and P_3, and are continuous with respect to the Euclidean topology in $R^2 \times R^2$. In this paper we prove the converse: every linear additive and continuous pseudo-metric F is generated as above by some c-finite measure μ on G for which (0) holds.

The method of proof shows that values of linearly additive and continuous pseudo-metric F inside every bounded convex polygon C are determined completely by the values of F on $(\partial C)^2$.

The representation of pseudo-metrics by measures is useful in derivation of inequalities for the former.

1. The Theorem

Denote by G the set of lines on R^2. Every $g \in G$ may be parametrized by a pair (p, φ), $p \geqq 0$, $0 \leqq \varphi < 2\pi$, which is the pair of polar coordinates of the foot of the perpendicular from the origin on g. The set $\{(p, \varphi)\}$ is naturally endowed with the topology of the semi-infinite cylinder in R^3. We will consider the topology on G which is induced by the bijection $(p, \varphi) \to g$.

Denote by $\mathscr{B}$ the class of Borel subsets of G. Note, that $\mathscr{B}$ may be equivalently defined as the minimal σ-algebra of subsets of G which contains the sets $\{g \in G: g$ separates P_1 and $P_2\}$, P_1 and $P_2 \in R^2$.

We also introduce the notations Δ for the class of finite linear segments (either closed, open or semi-open) on R^2,

$[X] = \{g \in G: g \cap X \neq \emptyset\}$ for $X \subset R^2$.

* A measure is called c-finite if it is finite on compact sets

Assume that μ is a c-finite measure on $\mathscr{B}$ with the property that

(1) $\mu[P]=0$ for every $P\in R^2$,

$\mu[X]$ is the shortening of usual notation $\mu([X])$.

By virtue of (1), for every segment $\delta\in\Delta$ with P_1 and P_2 for its endpoints

$$\mu[\delta]=\mu\ (\{g\colon g \text{ separates } P_1 \text{ and } P_2\}).$$

Define a function $F\colon R^2\times R^2\to[0,\infty)$

(2) $F(P_1,P_2)=\frac{1}{2}\mu[\delta]$.

Proposition 1. *The validity of* (1) *implies the following (obvious) properties of* F:

I) $F(P_1,P_2)=F(P_2,P_1)$;

II) F *is linearly additive: for* P_1,P_2,P_3 *on a line,* P_2 *between* P_1 *and* P_3

$$F(P_1,P_3)=F(P_1,P_2)+F(P_2,P_3);$$

III) F *satisfies the triangle inequality:*

$$F(P_1,P_2)\le F(P_2,P_3)+F(P_1,P_3)$$

for every $P_1,P_2,P_3\in R^2$;

IV) F *is continuous with respect to the Euclidean topology in* $R^2\times R^2$.

Taken separately, the condition I)–IV) imply

$$F\geqq 0 \quad\text{and}\quad F(P,P)=0$$

and therefore they define a continuous and linearly additive pseudo-metric on the plane.

The question arises naturally whether the converse of Proposition 1 is true, and the answer is given by the following theorem.

Theorem. 1. *Every function* $F(P_1,P_2)$ *satisfying the conditions* I)–IV) *is generated in the way of* (2) *by some (unique) measure* μ *on* $\mathscr{B}$ *possessing the property* (1).

2. *Let* F_1 *and* F_2 *be two functions satisfying* I)–IV). *If for a bounded convex polygon* $C\subset R^2$

$$F_1(P_1,P_2)=F_2(P_1,P_2),\qquad P_1,P_2\in\partial C$$

then

$$F_1(P_1,P_2)\equiv F_2(P_1,P_2),\qquad P_1,P_2\in C.$$

In the sequel we use when suitable the notation

$$F(\delta)=F(P_1,P_2).$$

Before giving a detailed proof in the next paragraph, we would like to say a few words on the nature of the problem.

The class $\{[\delta]\colon \delta\in\Delta\}$ is neither a ring nor a semi-ring, so it fails to be of a standard type for which the procedure of extension of a measure upon the containing minimal σ-algebra is well developed.

A reduction to a standard case may be achieved by considering the semi-ring

$$H=\left\{\bigcap_{i=1}^{n}[\delta_i]:\ \delta_i\in\Delta,\ i=1,\dots,n,\ n=0,1,2,\dots\right\}.$$

Here the problem arises whether for any c-finite measure m on $\mathscr{B}$ satisfying $m([P])=0$, $P\in R^2$, the values of m on the class $\{[\delta]:\ \delta\in\Delta\}$ determine the values of m on H. The answer is yes, and this in principle enables one to carry out the reduction (it was in fact the starting point of the present study).

The combinatorial way of reasoning which led to the solution of an analogous problem in [2] (see also [1]) works without any substantial changes in the present situation and provides the formulae for calculating $m(B)$, $B\in H$ in terms of $m[\delta]$, $\delta\in\Delta$:

$$2m\left(\bigcap_{i=1}^{n}[\delta_i]\right)=2\sum m[\delta_i]\,I_{n-1}(\delta_i)+\sum m[d_i]\,I_{n-2}(d_i)-\sum m[s_i]\,I_{n-2}(s_i). \qquad (*)$$

Here $d_i\in\Delta$ $(s_i\in\Delta)$ are of the following type: one endpoint is an endpoint of δ_l and the other is an endpoint of δ_j, $l\neq j$; δ_l and δ_j lie in different (in the same) halfplanes with respect to continuation of d_i (of s_i); $I_k(\delta)=1$ if the continuation of $\delta\in\Delta$ separates the endpoints of exactly k segments from the set $\{\delta_i\}_1^n$.

The Equation $(*)$ is valid if no three terminations of $\{\delta_i\}_1^n$ lie on a line.

Therefore it is natural to define (in analogy with $(*)$) the mapping μ: $H\to R$ by the formula

$$\mu\left(\bigcap_{1}^{n}[\delta_i]\right)=2\sum F(\delta_i)\,I_{n-1}(\delta_i)+\sum F(d_i)\,I_{n-2}(d_i)-\sum F(s_i)\,I_{n-2}(s_i). \qquad (**)$$

The first assertion of the theorem now may be equivalently reformulated as follows: *for every F satisfying* I)–IV) *the mapping μ in fact acts from H into* $[0,\infty)$ *and is σ-additive on H.*

Fortunately, by means of a rather special construction the proof of this statement can be reduced to considering $(**)$ only in the cases of $n=2$ (Eq. (3) below) and $n=3$ (Eq. (4)).

The second assertion of the theorem also becomes clear from this construction.

Note that the proof which follows is self-contained and does not rely on $(*)$ in its general form.

2. The Proof

We divide the proof into three parts.

I. Construction of the Measure μ_C. Let $C\subset R^2$ be a bounded, convex, open polygon, and $\{a_i\}$ be the set of the sides of C. Obviously it is possible to define each a_i as a half-open segment in such a way that

$$[C]=\bigcup_{i<j}([a_i]\cap[a_j])$$

with disjoint sets under the sign of union.

With each pair $i<j$ we associate the class H_{ij} of subsets of $[a_i]\cap[a_j]$ such that

$$H_{ij}=\{[\delta_1]\cap[\delta_2]:\ \delta_1\subset a_i,\ \delta_2\subset a_j,\ \delta_1,\delta_2\in\Delta\}.$$

Obviously H_{ij} is a semi-algebra generating the σ-algebra of Borel subsets of $[a_i]\cap[a_j]$.

Therefore every c-finite measure on $[C]$ is determined uniquely by its values on the class

$$H_C=\bigcup_{i<j} H_{ij}.$$

Moreover, well known criteria may be applied to ensure that a mapping

$$H_C\to[0,\infty)$$

may be extended to a measure on the Borel subsets of $[C]$. We apply this possibility to the mapping

$$\mu_C\colon\ H_C\to[0,\infty)$$

which is defined as follows.

In the case of δ_1 and δ_2 situated in Figure 1 (no three endpoints of δ_1 and δ_2 lie on a line)

(3) $\mu_C([\delta_1]\cap[\delta_2])=F(D_1)+F(D_2)-F(S_1)-F(S_2)$.

In the case of Figure 2 (there are three endpoints of δ_1 and δ_2 on a line)

(3′) $\mu_C([\delta_1]\cap[\delta_2])=F(D)+F(\delta_2)-F(S)$.

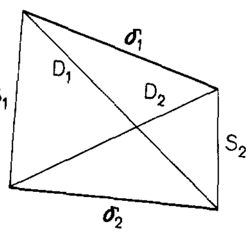

Fig. 1

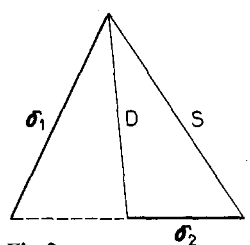

Fig. 2

The following three properties of μ_C are easily verified.

a) $\mu_C\geqq 0$ (follows from (II) and (III)).

b) μ_C is additive on H_C (by the structure of (3) and (3′) which provides cancellation of surplus terms after addition).

c) $A_n\downarrow 0$ in H_C implies $\mu_C(A_n)\to 0$ (follows from (IV)).

a), b) and c) imply, that μ_C can be extended to a measure on the Borel subsets of $[C]$.

It is worth noting that since $[a_i]\cap[a_j]\in H_{ij}$, $\mu_C[C]$ can be obtained by direct summation of (3) and (3′). This yields

$$\mu_C[C]=\sum F(a_i).$$

II. Construction of the Measure $\mu_{C,\alpha}$. Let a segment $\alpha\in\Delta$ be fixed inside C is such a way that, *no vertex of* C *lies on the continuation of* α.

Define the class $H_{C,\alpha}\subset H_C$ as $H_{C,\alpha}=\{[\delta_1]\cap[\delta_2]\in H_C$: no triad of endpoints of the segments δ_2, δ_1 and α which contains at least one endpoint of α lies on a line$\}$.

Clearly

$$H_{C,\alpha}=\bigcup_{i<j} H_{i,j,\alpha}.$$

$H_{i,j,\alpha}$ is the part of $H_{C,\alpha}$ defined by the condition $\delta_1\subset a_i$, $\delta_2\subset a_j$, each $H_{i,j,\alpha}$ is a semi-algebra generating the σ-algebra of Borel subsets of $[a_i]\cap[a_j]$.

The mapping

$$\mu_{C,\alpha}\colon H_{C,\alpha}\to[0,\infty)$$

again is defined separately for the cases presented on Figure 1 and Figure 2.

For δ_1 and δ_2 (Fig. 1)

$$(4)\quad \mu_{C,\alpha}([\delta_1]\cap[\delta_2]) =2F(\alpha)I_{\delta_1,\delta_2}(\alpha)+\sum F(d_i)I_\delta(d_i)-\sum F(s_i)I_\delta(s_i) +F(D_1)I_\alpha(D_1)+F(D_2)I_\alpha(D_2)-F(S_1)I_\alpha(S_1)-F(S_2)I_\alpha(S_2).$$

Here D_1, D_2, S_1, S_2 have been defined on Figure 1, and $I_\alpha(\tau)=1$ if the continuation of $\tau\in\Delta$ intersects α, 0 otherwise.

The segments d_i and s_i have the same significance as in (*) but here each of them is assumed to start from an endpoint of δ. The set $\{s_i\}$ ($\{d_i\}$) consists of those segments of this type which place α and δ_k in the same halfplane (in different halfplanes) with regard to their continuations (see Figs. 3 and 4).

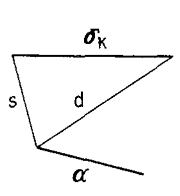

Fig. 3

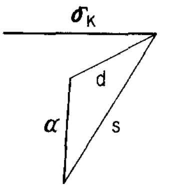

Fig. 4

Furthermore $I_\delta(d)=1$ if the continuation of d intersects δ_l, where $l=1$ if $k=2$, $l=2$ if $k=1$ (in the above notation d passes through the endpoint of δ_k). $I_\delta(s)$ is defined similarly.

Finally $I_{\delta_1,\delta_2}(\alpha)=1$ if the continuation of α intersects both δ_1 and δ_2, 0 otherwise.

For δ_1 and δ_2 situated as shown on Figure 2

$$(4')\quad \mu_{C,\alpha}([\delta_1]\cap[\delta_2])=2F(\alpha)I_{\delta_1,\delta_2}(\alpha)+F(D)I_\alpha(D)-F(S)I_\alpha(S) +\sum F(d_i)I_\delta(d_i)-\sum F(s_i)I_\delta(s_i).$$

Note that the only reason for defining $\mu_{C,\alpha}$ on $H_{C,\alpha}$ rather than on H_C was to reduce the number of cases, which need special formulae for the definition of $\mu_{C,\alpha}$.

Now check the properties of $\mu_{C,\alpha}$.

a) $\mu_{C,\alpha}(A) \geqq 0$, $A \in H_{C,\alpha}$. To check this in an economical way we first note that $\mu_{C,\alpha}(A)$ is linearly additive with respect to α (for fixed A).

Secondly, if $\bar{\alpha}_n \downarrow P$, $P \in D_i$ or $P \in S_i$ ($\bar{\alpha}$ = closure of α) then

(5) $\lim \mu_{C,\alpha_n}(A) = 0 \quad$ for every $A \in H_{C,\alpha}$.

Observing Figure 5 we conclude that

$$F(s_1) + F(s_2) - F(D_1) \to 0$$

and

$$F(d_3) - F(s_3) \to 0.$$

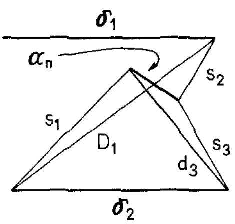

Fig. 5

With $F(\alpha_n) \to 0$ this implies (5) for the case $P \in D$. The case $P \in S$ may be treated similarly.

It follows from these two remarks that the check of a) may be restricted to the cases in which the closure of α does not intersect D_1, D_2, S_1, or S_2.

If α lies outside the convex hull of δ_1 and δ_2 then all the indicators on the right side of (4) (or (4')) vanish yielding:

(6) $\mu_{C,\alpha}([\delta_1] \cap [\delta_2]) = 0 \quad$ if $[\alpha] \subset ([\delta_1] \cap [\delta_2])^c$.

The remaining cases for δ_1 and δ_2 (Fig. 1) are shown on Figure 6. The other cases are treated similarly.

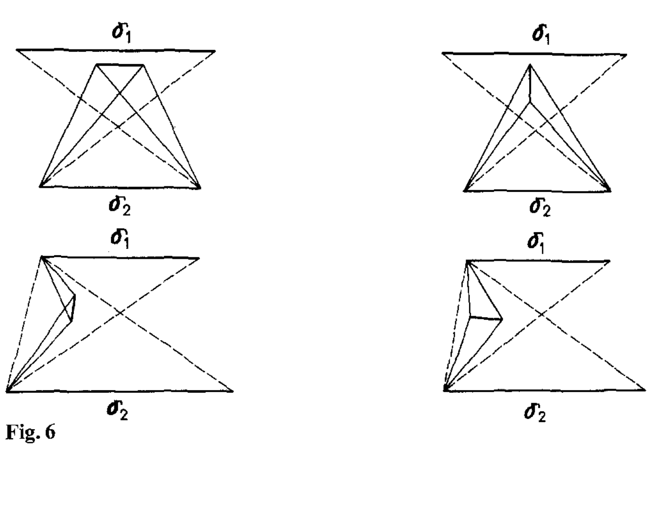

Fig. 6

In each of the cases a) follows by repeated application of the triangle inequality.

b) $\mu_{C,\alpha}$ is additive on $H_{C,\alpha}$ (obvious).

c) $A_n \downarrow \emptyset$ in $H_{C,\alpha}$ implies $\mu_{C,\alpha}(A_n) \to 0$.

To prove c), assume that $A_n = [\delta_1^{(n)}] \cap [\delta_2^{(n)}]$. $A_n \downarrow \emptyset$ implies that $\overline{\delta_1^{(n)}} \downarrow \delta_1$, $\overline{\delta_2^{(n)}} \downarrow \delta_2$ (the bar denotes the closure) and at least one of the δ_1, δ_2 reduces to a point. Suppose $\delta_1 = P$ and Q_1 and Q_2 are the endpoints of δ_2.

We will index segments appearing as the arguments of F in (4) and (4') with an upper (n) to associate them with A_n and let $\tau^{(n)}$ be their generic name. We assume that the upper index is ascribed in a natural way which ensures in each case the existence of the limit:

$$\tau = \lim \tau^{(n)}.$$

Denote by g_τ the line, on which τ lies. In what follows, the sequences $\tau^{(n)}$ are classified according to the type of the set $Z = \{P, Q_1, Q_2, R_1, R_2\} \cap g_\tau$ where R_1 and R_2 are the endpoints of α. First consider the cases when either

$$\operatorname{card} Z = 2$$

or

$$Z = \{P, Q_1, Q_2\}, \{P, R_1, R_2\}, \{Q_i, R_1, R_2\}, \qquad i = 1, 2.$$

In these cases the $\lim I_*(\tau^{(n)})$ exists. I_* is the indicator which corresponds to $\tau^{(n)}$ in (4) or (4'). $\operatorname{Lim} I_*(\tau^{(n)}) = 1$ implies that P is an endpoint of τ. But from this the existence of another sequence follows, $\tau_1^{(n)}$, such that $\lim \tau_1^{(n)} = \lim \tau^{(n)} = \tau$ where the terms corresponding to $\tau^{(k)}$ and $\tau_1^{(n)}$ in (4) (or (4')) have different signs. By the continuity of F, these two terms cancel in the limit. If for the sequence $\tau^{(n)}$ the set Z happens to be of a type different from those mentioned above, then the corresponding indicator needs not possess a limit. Fortunately the sum

$$\varepsilon^{(n)} = \sum_{i \in Y} c(\tau_i^{(n)}) F(\tau_i^{(n)}) I_*(\tau_i^{(n)})$$

where $Y = \{i$: $\lim \tau_i^{(n)}$ lies on the line carrying $Z\}$ and $c(\tau_i) = 1$ if τ_i is of d or D type;

-1 if τ_i is of s or S type;

2 if $\tau_i = \alpha$;

possesses a limit which is equal to zero.

This is an obvious corollary of the following proposition.

Put

$$\mathscr{I}^{(n)} = (I_*(\tau_{i_1}^{(n)}), \ldots, I_*(\tau_{i_k}^{(n)})) \qquad \text{for } \{i_1, \ldots, i_k\} = Y.$$

Proposition 2. *The sum* $\varepsilon^{(n_k)}$ *tends to zero for every subsequence* $\{n_k\}$ *for which* $\mathscr{I}^{(n_k)}$ *remains constant.*

The proof of this statement consists of checking its validity for all possible choices of Z and all possible values of $\mathscr{I}$ which admit infinite repetition in the sequence $\{\mathscr{I}^{(n)}\}$.

The diagrams on Figure 7 make clear that, for every value of $\mathscr{I}$ possessing this property and each point $x \in Z$ from the line carrying the set Z

$$l(x) - m(x) + 2 I_\alpha(x) = 0$$

where

$l(x) = \text{card}\{i\colon\ x \in \lim \tau_i^{(n)},\ \tau_i^{(n)} \text{ is of } d \text{ or } D \text{ type}\},$

$m(x) = \text{card}\{i\colon\ x \in \lim \tau_i^{(n)},\ \tau_i^{(n)} \text{ is of } s \text{ or } S \text{ type}\},$

$I_\alpha(x) = 1 \quad \text{if } x \in \alpha, \quad 0 \text{ otherwise}.$

With this the proof of the proposition and with it the proof of c) is completed, because

$$\lim_{k\to\infty} \varepsilon^{(n_k)} = \int (l(x) - m(x) - 2I_\alpha(x))\, dF_{g_0},$$

where dF_g is the measure on the line g generated by $F(\delta)$, $\delta \subset g$ and $Z \subset g_0$.

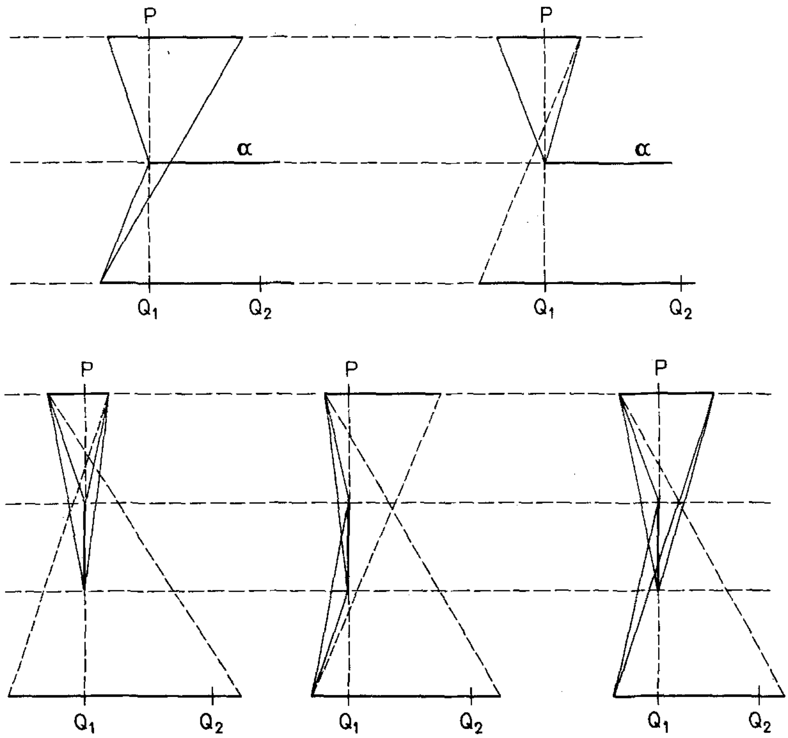

Fig. 7. $\delta_1^{(n)}$, $\delta_2^{(n)}$ and a are drawn in thick lines. Thin continuous lines show those $\tau_i^{(n)}$ for which $I_*(\tau^{(n)}) = 1$. For $Z = \{P, Q, R_1\}$ there are only two (up to numeration) values of $\mathcal{I}$, which admit infinite repetition in the sequence $\mathcal{I}^{(n)}$ (upper row). For $Z = \{P, Q, R_1, R_2\}$ there are three such values if $\mathcal{I}$ (lower row). cases with $Q_1 = Q_2$ may be considered analogously.

It follows from a), b) and c) that $\mu_{C,\alpha}$ may be extended to a measure on the Borel subsets of $[C]$.

III. Construction of the Measure μ. Clearly $[C] \in H_{C,\alpha}$ and a simple summation of (4) and (4′) yields

$$\mu_{C,\alpha}[C] = 2F(\alpha).$$

By virtue of (6) $\mu_{C,\alpha}$ *is concentrated on* $[\alpha]$, that is

$$\mu_{C,\alpha}[\alpha]=\mu_{C,\alpha}[C]=2F(\alpha).$$

At the same time one easily finds that

$$[\delta_1]\cap[\delta_2]\subset\alpha \quad \text{implies } \mu_{C,\alpha}([\delta_1]\cap[\delta_2])=\mu_C(\delta_1]\cap[\delta_2]).$$

This means that $\mu_{C,\alpha}$ *is the restriction of* μ_C *to the set* $[\alpha]$. In particular

$$\mu_{C,\alpha}[\alpha]=\mu_C[\alpha].$$

Hence

(7) $\mu_C[\alpha]=2F(\alpha)$.

Using the continuity of F it is not difficult to show that the restriction on location of α inside C which was adopted at the beginning of II, is not necessary for validity of (7).

Let C_1 be a polygon containing C. For every measure m on the Borel subsets of $[C_1]$ with the property $m[P]=0$ for every $P\in R^2$ for $A\in H_C$ either

$$m(A)=m[D_1]+m[D_2]-m[S_1]-m[S_2]$$

(the case of Fig. 1) or

$$m(A)=m[D]-m[S]-m[\delta_2]$$

(the case of Fig. 2). By the continuity of F and (7) written for μ_{C_1}, $\mu_{C_1}[P]=0$ for every $P\in C_1$.

So we may pose $m=\mu_{C_1}$, and again by (7), (3′) and (3) we get

$$\mu_{C_1}(A)=\mu_C(A) \quad \text{for every } A\in H_C.$$

This means that μ_C *is the restriction of* μ_{C_1} *on the set* $[C]$.

Now take $\{C_n\}$ to be the sequence of squares centered at 0 of side length n.

For every Borel set $A\subset[C_n]$ let

$$\mu(A)=\mu_{C_n}(A).$$

Clearly this defines consistently a measure on $\mathscr{B}$ with the properties.

$$\mu[\alpha]=2F(\alpha), \quad \mu[P]=0 \quad \text{for every } P\in R^2;$$

hence the first assertion of the theorem.

The second assertion is also a corollary of (7) and of the fact that μ_C is completely determined by its values on H_C given by (3) and (3′).

3. Discussion

As has been noted in §1, the right-hand expressions of (3), (3′) and (4), (4′) turn out to be nonnegative because they give the values of $\mu(A)$ for $A=[\delta_1]\cap[\delta_2]$ and $A=[\delta_1]\cap[\delta_2]\cap[\alpha]$ respectively. This naturally poses the problem of whether it is possible to derive other inequalities of the same general type involving a function F satisfying I)–IV) by calculating the measures $\mu(A)$ for properly chosen classes of $A\in\mathscr{B}$.

Such a calculation may be carried out for every subset of G which belongs to the minimal (finite) ring containing the sets $\{[\delta_i]\}_1^n$. This will be done elsewhere. Here we would like to provide the corresponding result only for $\bigcap_{i=1}^{n}[\delta_i]$ and $\bigcup_{i=1}^{n}[\delta_i]$, thereby answering the original Buffon-Sylvester problem posed for general (rather then invariant) measures [3]:

$$\mu(\bigcap[\delta_i])=2\sum F(\delta_i)\,I_{n-1}(\delta_i)+\sum F(d_i)\,I_{n-2}(d_i)-\sum F(s_i)\,I_{n-2}(s_i),$$
$$\mu(\bigcup[\delta_i])=2\sum F(\delta_i)\,I_0(\delta_i)-\sum F(d_i)\,I_0(d_i)+\sum F(s_i)\,I_0(s_i).$$

The notations, which are used here have been explained in §1 where (∗) was introduced; no three terminations of the segments $\{\delta_i\}$ are assumed to lie on a line. Of course, in the case of "invariant" measure μ, the corresponding pseudometric F is the usual Euclidean metric.

Another point which we mention is that the combinatorial nature of the proof of the theorem in §1 implies its validity in a far more general framework. For example, the theorem remains valid when R^2 is replaced by a smooth surface in R^3 which has the property that there is a single geodesic path connecting each pair of its points, and G is replaced by the set of geodesic lines on this surface. Obviously $F(P_1, P_2)$ equal to the geodesic distance between P_1 and P_2 satisfies I)–IV) and hence there exists a unique measure μ on the set of geodesic lines with the property that

$$\mu[\delta]=\text{geodesic length of } \delta.$$

This manner of introducing such μ seems clearer and more thematic then that employing the calculus of variations (as in [4]).

The final remark concerns the possibilities of further investigations connected with the triangle inequality.

The important point is that for any random process of lines (see [5]) the function

$$F(\delta)=\text{prob}\{\delta\in\Delta \text{ is intersected by at least one line of the process}\}$$

satisfies the triangle inequality, but in general fails to be linearly additive. In the case of a random line process with distribution invariant under the Euclidean motions of the plane

$$F(\delta)=F(|\delta|)$$

where $|\delta|$ is the length of δ. The Poisson line process, governed by the invariant measure on G where $F_\lambda=1-\exp(-\lambda|\delta|)$ provides an example of an $F(|\delta|)$ satisfying III) but not II). Any mixture of F_λ (with respect to λ) has the same property. Apparently the intriguing problem of describing classes of $F(|\delta|)$ satisfying the triangle inequality has connections with R. Davidson's problem of describing the class of invariant processes of lines on the plane, which has recently gained much attention [5].

The author is greatly indebted to the referees and to K. Krickeberg for indicating the relation of the present work to the construction of multiparametric Brownian motion [6], [7] as well as for their many remarks towards improving the present manuscript.

References

1. Ambartsumian, R.V.: The solution to the Buffon-Sylvester problem in R^3. Z. Wahrscheinlichkeitstheorie verw. Gebiete **27**, 53–74 (1973)
2. Ambartsumian, R.V.: Combinatorial solution of the Buffon-Sylvester problem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **29**, 25–31 (1974)
3. Sylvester, J.J.: On a funicular solution of Buffon's "Problem of the needle" in its most general form. Acta Math. **14**, 185–205 (1891)
4. Santaló, L.A.: Introduction to Integral geometry. Paris: Hermann 1953.
5. Harding, E. F., Kendall, D. G.: Stochastic geometry. London, New York, Sydney, Toronto: Wiley (1974)
6. Čentsov: Lévy Multiparameter Brownian Motion and Generalized white Noise. Teor. Verojatnost i Primenen. **2**, 281–282 (1957)
7. Lévy, P.: Processus Stochastiques et Mouvement Brownien; éd. 1965, pp. 337–365. Gauthier-Villars Paris

Received February 12, 1976; in revised form May 19, 1976