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Asymptotic Properties of Eulerian Numbers

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1. Introduction

The Eulerian numbers [1, 3] may be defined by means of

$$\frac{1-\lambda}{1-\lambda e^{-x}}=\sum_{n=0}^{\infty}\frac{A_n(\lambda)}{(\lambda-1)^n}\frac{x^n}{n!} \tag{1.1}$$

and

$$A_n(\lambda)=\sum_{k=0}^{n} A_{n,k}\,\lambda^k. \tag{1.2}$$

It follows that

$$A_{0,0}=1, \qquad A_{n,0}=0 \qquad (n>0), \tag{1.3}$$

$$A_{n+1,k}=(n-k+2)\,A_{n,k-1}+k\,A_{n,k} \tag{1.4}$$

and

$$A_{n,k}=A_{n,n-k+1} \qquad (1\leqq k\leqq n). \tag{1.5}$$

The following table of values of $A_{n,k}$, $1\leqq k\leqq n\leqq 6$, is easily computed using (1.4):

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	21	1	
1	57	302	302	57	1

The number $A_{n,k}$ has the following combinatorial interpretation [6, Ch. 8]. A permutation $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$ is said to have a *rise* at a_i if $a_i<a_{i+1}$; also it is customary to count a conventional rise to the left of a_1. Then $A_{n,k}$ is the number of permutations of $(1, 2, \ldots, n)$ with k rises.

Kurtz [5] has proved that

$$A_{n,k}^2>A_{n,k-r}\,A_{n,k+r} \qquad (0<r<k\leqq n). \tag{1.6}$$

It follows that, for odd n, $A_{n,k}$ has a unique maximum at $k=\frac{1}{2}(n+1)$; for even n, the maximum occurs at $k=\frac{1}{2}n$ and $\frac{1}{2}n+1$.

It follows from (1.1) that $A_n(\lambda)$ satisfies the recurrence

$$A_{n+1}(\lambda)=(n+1)\,\lambda\,A_n(\lambda)+(1-\lambda)\,\lambda\,\frac{dA_n(\lambda)}{d\lambda}. \tag{1.7}$$

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Frobenius [3] has noted that the $n-1$ roots of $\lambda^{-1}A_n(\lambda)$ are real, negative and distinct; if λ_0 is a root then λ_0^{-1} is also a root. Moreover the roots of $\lambda^{-1}A_{n+1}(\lambda)$ are separated by those of $\lambda^{-1}A_n(\lambda)$.

It follows from (1.4) that

$$\sum_{k=0}^{n} A_{n,k} = n!. \tag{1.8}$$

Also by repeated differentiation of (1.7) we get

$$A_n'(1) = \tfrac{1}{2}(n+1)! \qquad (n \geqq 1) \tag{1.9}$$

and

$$A_n''(1) = \tfrac{1}{12}(3n-2)(n+1)! \qquad (n>1). \tag{1.10}$$

In the present paper we shall prove the following two theorems.

Theorem 1. *We have*

$$\lim_{n=\infty} \frac{1}{n!} \sum_{k=1}^{[x_n]} A_{n,k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt,$$

where

$$x_n = \sqrt{\tfrac{1}{12}(n+1)}\, x + \tfrac{1}{2}(n+1). \tag{1.11}$$

Theorem 2. *For x_n defined by* (1.11), *we have*

$$\frac{1}{n!} A_{n,[x_n]} = \sqrt{\frac{6}{\pi(n+1)}}\, e^{-\frac{x^2}{2}} + O(n^{-\frac{3}{4}})$$

uniformly for all x.

The proof of Theorem 1 was suggested by Harper's proof [4] of the similar result for Stirling numbers of the second kind. The proof of Theorem 2 makes use of a Berry-Esséen type rate of convergence theorem [2, p. 251] and also the strong logarithmic concavity property (1.6) of the Eulerian numbers.

2. Random Variables and Moments

Following Harper [4], we introduce a triangular array of row independent random variables X_{nk} taking on only the values 0 and 1 by giving their probability densities:

$$P[X'_{nk}=0] = r_{nk}/(1+r_{nk}), \qquad P[X'_{nk}=1] = 1/(1+r_{nk})$$

for $k=1,2,\ldots,n$; and $n=1,2,\ldots$, where $0=r_{n1}<r_{n2}<\cdots<r_{nn}$ are the negatives of the roots of $A_n(\lambda)$. Setting

$$S'_n = \sum_{k=1}^{n} X'_{nk}$$

we have

$$ES'_n = \sum_{k=1}^{n} 1/(1+r_{nk}) \tag{2.1}$$

and

$$\operatorname{Var} S_n' = \sum_{k=1}^{n} \left(1/(1+r_{nk}) - 1/(1+r_{nk})^2\right). \tag{2.2}$$

Now

$$A_n(\lambda) = \prod_{k=1}^{n} (\lambda + r_{nk}), \tag{2.3}$$

and differentiating logarithmically, we get

$$\sum_{k=1}^{n} (\lambda + r_{nk})^{-1} = A_n'(\lambda)/A_n(\lambda). \tag{2.4}$$

For $\lambda = 1$ we have then by (1.7), (1.8), (2.1) and (2.3) that

$$E S_n' = A_n'(1)/n! = (n+1)/2. \tag{2.5}$$

In similar fashion we get from (1.8), (1.9), (1.10) and (2.2)

$$\operatorname{Var} S_n' = \left(A_n''(1) + A_n'(1)\right)/A_n(1) - \left(A_n'(1)/A_n(1)\right)^2 = (n+1)/12. \tag{2.6}$$

Next we compute the distribution function

$$F_n'(x) = P[S_n' \leqq x] \tag{2.7}$$

by noting that

$$\begin{aligned} P[S_n' = p] &= P[\text{exactly } p \text{ of the } X_{nk}' \text{ are } 1] \\ &= \sum \prod_{k_i} (1 + r_{nk_i})^{-1} \prod_{k_j} r_{nk_j}/(1 + r_{nk_j}) \end{aligned} \tag{2.8}$$

where the k_i's are the k's for which $X_{nk} = 1$, and the k_j's are the k's for which $X_{nk} = 0$; the number of k_i's is p and the number of k_j's is $n - p$. Since A_{np} is the elementary symmetric function of degree $n-p$ of $r_{n1}, r_{n2}, \ldots, r_{nn}$, it follows from (2.3 and (2.8) that

$$P[S_n' = p] = A_{n,p}/n!. \tag{2.9}$$

Hence

$$F_n'(x) = \sum_{k=1}^{[x]} A_{n,k}/n!. \tag{2.10}$$

3. The Central Limit Theorem with Error Term

Let

$$X_{n,k} = (X_{n,k}' - E X_{n,k}')$$

and

$$S_n = (S_n' - E S_n')/(\operatorname{Var} S_n')^{\frac{1}{2}}. \tag{3.1}$$

Let $F_{n,k}$ be the distribution function of $X_{n,k}$. Now it is easy to see that the Lindeberg condition

$$\lim_n \sum_k \int_{|x| \geqq \varepsilon} x^2 \, dF_{n,k}(x) = 0$$

holds and the array $X_{n,k}$ satisfies the central limit theorem. The distribution functions F_n of S_n converge pointwise to the normal distribution

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$$

that is

$$F_n(x)=P[s_n \leqq x] \underset{n}{\to} N(x) \quad \text{for all } x. \tag{3.2}$$

But by (2.5), (2.6) and (3.1)

$$P[S_n \leqq x]=P\left[S'_n \leqq x\sqrt{\frac{n+1}{12}}+\frac{n+1}{2}\right]. \tag{3.3}$$

Put

$$x_n = x\sqrt{\frac{n+1}{12}}+\frac{n+1}{2}. \tag{3.4}$$

Then by (2.7) and (2.9)

$$F_n(x)=\sum_{k=1}^{[x_n]} A_{n,k}/n! \underset{n}{\to} N(x) \quad \text{for all } x. \tag{3.5}$$

This proves Theorem 1.

We state now a Berry-Esséen type theorem giving a rate of convergence to the normal distribution for triangular arrays. The result is similar to and proved in exactly the same manner as the Theorem in [2, p. 521]:

If for $k=1,2,\ldots,n$; and $n=1,2,\ldots$

$$(\operatorname{Var} X_{nk})^{-1} E|X_{nk}|^3 < \lambda \tag{3.6}$$

where λ is a constant, and the left side of (3.6) *is* 1 *if* $\operatorname{Var} X_{nk}=0$, *and if*

$$\operatorname{Var} S_n \underset{n}{\to} \infty, \tag{3.7}$$

then for all n and for all x

$$|F_n(x)-N(x)| \leqq 10\,\lambda\,(\operatorname{Var} S_n)^{-\frac{1}{2}}. \tag{3.8}$$

To apply this estimate to our array we compute

$$\begin{aligned} E|X_{nk}|^3 &= E|X'_{nk}-1/(1+r_{nk})|^3 \\ &=(1-1/(1+r_{nk}))^3(1+r_{nk})^{-1}+(1+r_{nk})^{-3}\,r_{nk}(1+r_{nk})^{-1} \\ &= r_{nk}(r_{nk}^2+1)(r_{nk}+1)^{-4}. \end{aligned}$$

Then

$$(\operatorname{Var} X_{nk})^{-1} E|X_{nk}|^3 = (r_{nk}^2+1)(r_{nk}+1)^{-2} \leqq 1$$

since $r_{nk} \geqq 0$, and (3.6) holds with $\lambda=1$. The condition (3.7) is also satisfied, and hence by (3.8)

$$\left|\sum_{k=1}^{[x_n]} \frac{A_{n,k}}{n!}-N(x)\right| < 10\sqrt{12}(n+1)^{-\frac{1}{2}} \tag{3.9}$$

for all x and for all n.

4. Asymptotic Estimates of $A_{n,[x_n]}$

We can use the central limit theorem (3.5) and the logarithmic concavity of the $\{A_{n,k}\}$, (1.6) to establish a local limit theorem giving asymptotic estimates of $A_{n,[x_n]}$.

For $0<x<y$ we have by (3.4) and (3.5)

$$\lim_n \frac{1}{n!} \sum_{[x_n]+1}^{[y_n]} A_{n,k} = \frac{1}{\sqrt{2\pi}} \int_x^y e^{-\frac{t^2}{2}}\, dt, \tag{4.1}$$

where

$$y_n = y\sqrt{\frac{n+1}{12}} + \frac{n+1}{2}.$$

Since $A_{n,k}$ is monotone over the range of summation we get

$$([y_n]-[x_n]) \frac{A_{n,[x_n]}}{n!} \geqq \frac{1}{n!} \sum_{[x_n]+1}^{[y_n]} A_{n,k} \tag{4.2}$$

and

$$([y_n]-[x_n]) \frac{A_{n,[y_n]}}{n!} \leqq \frac{1}{n!} \sum_{[x_n]+1}^{[y_n]} A_{n,k}. \tag{4.3}$$

Next we divide both sides of (4.2) and (4.3) by $y-x$, and noticing that

$$\lim_n \frac{[y_n]-[x_n]}{y-x} \sqrt{\frac{12}{n+1}} = 1,$$

we get from (4.1) that

$$\liminf_n \sqrt{\frac{n+1}{12}} \frac{A_{n,[x_n]}}{n!} \geqq \frac{(y-x)^{-1}}{\sqrt{2\pi}} \int_x^y e^{-\frac{t^2}{2}} dt \tag{4.4}$$

and

$$\limsup_n \sqrt{\frac{n+1}{12}} \frac{A_{n,[y_n]}}{n!} \leqq \frac{(y-x)^{-1}}{\sqrt{2\pi}} \int_x^y e^{-\frac{t^2}{2}} dt. \tag{4.5}$$

If we fix x and let $y \downarrow x$ in (4.4), and if we fix y and let $x \uparrow y$ in (4.5) we get

$$\lim_n \sqrt{\frac{n+1}{12}} \frac{A_{n,[x_n]}}{n!} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{4.6}$$

for all $x>0$. Actually we can take $x=0$ in (4.4) but we cannot take $y=0$ in (4.5).

In similar fashion we get equation (4.6) for all $x<0$, but $x=0$ presents a special problem. If $x=0$, then $x_n=\frac{n+1}{2}$. Let $m=\left[\frac{n+1}{2}\right]$. By symmetry,

$$A_{n,m} = \max_{1\leqq k\leqq n} A_{n,k}.$$

By (4.4) with $x=0$ we have

$$\liminf_n \sqrt{\frac{n+1}{12}} \frac{A_{n,m}}{n!} \geqq \frac{1}{\sqrt{2\pi}}. \tag{4.7}$$

To bound $A_{n,m}$ above use the logarithmic concavity (1.6) applied to $A_{n,m}$, $A_{n,[x_n]}$ and $A_{n,[z_n]}$, where z_n is chosen so that

$$[x_n]-m=[z_n]-[x_n] \tag{4.8}$$

and x is positive. Now

$$A^2_{n,[x_n]} \geqq A_{n,m} A_{n[z_n]} \tag{4.9}$$

and

$$\sqrt{\frac{n+1}{12}}\,\frac{A_{n,m}}{n!} \leqq \frac{\left(\sqrt{\frac{n+1}{12}}\,A_{n,[x_n]}/n!\right)^2}{\sqrt{\frac{n+1}{12}}\,A_{n,[z_n]}/n!}.$$

Simple computations establish

$$\left|z_n - 2x\sqrt{\frac{n+1}{12}} - \frac{n+1}{2}\right| \leqq 3, \tag{4.10}$$

and we get from (4.6)

$$\limsup_n \sqrt{\frac{n+1}{12}}\,\frac{A_{n,m}}{n!} \leqq \frac{\left(e^{-\frac{x^2}{2}}/\sqrt{2\pi}\right)^2}{e^{-\frac{(2x)^2}{2}}/\sqrt{2\pi}} \tag{4.11}$$

for all $x>0$. The left hand side of (4.11) is independent of x, and so letting $x \to 0$ we get

$$\limsup_n \sqrt{\frac{n+1}{12}}\,\frac{A_{n,m}}{n!} \leqq \frac{1}{\sqrt{2\pi}}. \tag{4.12}$$

Now (4.12) combined with (4.7) proves (4.6) for all x.

Finally, we can apply the central limit theorem with Berry-Esséen type error estimates (3.9) to prove Theorem 2. As before, let $0<x<y$. Then by (3.9)

$$([y_n]-[x_n])\,\frac{A_{n,[y_n]}}{n!} \leqq \frac{1}{\sqrt{2\pi}} \int_x^y e^{-\frac{t^2}{2}}\,dt + \frac{20\sqrt{12}}{\sqrt{n+1}}. \tag{4.13}$$

We fix y, let $x=y-n^{-\frac{1}{4}}$ (for n large) and divide both sides of (4.13) by $(y-x)\sqrt{\frac{n+1}{12}}$. Now

$$\left|[y_n]-[x_n]-(y-x)\sqrt{\frac{n+1}{12}}\right| \leqq 2,$$

and we have by (4.13)

$$\frac{A_{n,[y_n]}}{n!} \leqq \sqrt{\frac{12}{n+1}}\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{n^{-\frac{1}{4}}}\left(\int_{y-n^{-\frac{1}{4}}}^{y} e^{-\frac{t^2}{2}}\,dt\right) + O(n^{-\frac{3}{4}}), \tag{4.14}$$

since by (4.6)

$$\sqrt{\frac{12}{n+1}}\,\frac{2}{n^{-\frac{1}{4}}}\,\frac{A_{n,[y_n]}}{n!} = O(n^{-\frac{3}{4}}) \quad \text{as } n \to \infty.$$

By applying the law of the mean we get

$$\frac{1}{n^{-\frac{1}{4}}}\int_{y-n^{-\frac{1}{4}}}^{y} e^{-\frac{t^2}{2}}\,dt = e^{-\frac{y^2}{2}} + O(n^{-\frac{1}{4}}).$$

Hence altogether we get

$$\frac{A_{n,[y_n]}}{n!} \leqq \sqrt{\frac{12}{n+1}}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{y^2}{2}} + O(n^{-\frac{3}{4}}) \tag{4.15}$$

for all $y>0$. The constant in O is independent of y.

In similar fashion we get (as before)

$$\frac{A_{n,[x_n]}}{n!} \geqq \sqrt{\frac{12}{n+1}}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}} + O(n^{-\frac{3}{4}}) \tag{4.16}$$

for all $x \geqq 0$. These arguments carry over to the case $y<x<0$, and so we have

$$\frac{A_{n,[x_n]}}{n!} = \sqrt{\frac{12}{m+1}}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}} + O(n^{-\frac{3}{4}}) \tag{4.17}$$

for all $x \neq 0$. For $x=0$ we have only (4.16). To finish the proof of Theorem 2 we use the logarithmic concavity of $\{A_{n,k}\}$. We choose x_n and z_n as in (4.8). For convenience we work with logarithms. By (4.9)

$$\log\frac{A_{n,m}}{n!} \leqq 2\log\frac{A_{n,[x_n]}}{n!} - \log\frac{A_{n[z_n]}}{n!}, \tag{4.18}$$

and by (4.17)

$$\log\frac{A_{n,[x_n]}}{n!} = \log\left(\sqrt{\frac{12}{n+1}}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\right) - O\left(e^{\frac{x^2}{2}}\,n^{-\frac{1}{4}}\right) \tag{4.19}$$

for all $x \neq 0$. If we bound x by $|x|<M$, then the constant of O is independent of x. Now by (4.10)

$$z_n = (2x+\varepsilon)\sqrt{\frac{n+1}{12}} + \frac{n+1}{2} \quad \text{and} \quad |\varepsilon| \leqq 3\sqrt{\frac{12}{n+1}}.$$

Then (4.19) with x bounded gives

$$\begin{aligned}\log\frac{A_{n,[z_n]}}{n!} &= \log\sqrt{\frac{6}{\pi(n+1)}}\,e^{-\frac{(2x+\varepsilon)^2}{2}} + O(n^{-\frac{1}{4}})\\ &= \log\sqrt{\frac{6}{\pi(n+1)}}\,e^{-\frac{(2x)^2}{2}} + O(n^{-\frac{1}{4}}),\end{aligned} \tag{4.20}$$

and by (4.18), (4.19) and (4.20)

$$\log\frac{A_{n,m}}{n!} \leqq \log\left(\sqrt{\frac{6}{\pi(n+1)}}\,e^{x^2}\right) + O(n^{-\frac{1}{4}}) \quad \text{as} \quad n\to\infty \tag{4.21}$$

uniformly for $0<x<M$. The left hand side of (4.21) is independent of x, as is the constant in the O term, and hence

$$\log \frac{A_{n,m}}{n!} \leqq \log \left(\sqrt{\frac{12}{n+1}} \frac{1}{\sqrt{2\pi}} \right) + O(n^{-\frac{1}{4}}). \tag{4.22}$$

Exponentiating and combining (4.22) and (4.16) we get

$$\frac{A_{n,[x_n]}}{n!} = \sqrt{\frac{6}{\pi(n+1)}}\, e^{-\frac{x^2}{2}} + O(n^{-\frac{3}{4}}) \quad \text{as} \quad n \to \infty$$

uniformly for all x. This proves Theorem 2.

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