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# Metric Spaces in which Prohorov's Theorem Is Not Valid

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## I. Preliminaries

Let $X$ be a Hausdorff topological space. By a measure on $X$ we understand a tight Borel probability measure on $X$. The set of all measures on $X$ is denoted by $P(X)$; this set is a topological space with the usual topology. This topology can be described as follows:

If $\mu_\alpha$ is a net in $P(X)$ and $\mu \in P(X)$ then $\lim \mu_\alpha = \mu$ if and only if $\liminf \mu_\alpha(G) \geqq \mu(G)$ for every open set $G \subset X$.

For convenience a space $X$ is called a Prohorov space if for every compact set $M \subset P(X)$ and every $\varepsilon > 0$ there exists a compact set $A \subset X$ such that $\mu(A) > 1 - \varepsilon$ for each $\mu \in M$.

It is well known that every topologically complete space $X$ (i.e. space which is a $G_\delta$ subspace of some compact space) is a Prohorov space (see Corollary 1 of Theorem 1). Varadarajan [3] claimed to prove that a metric space $X$ is a Prohorov space provided that every Borel measure on $X$ is tight (consequently a separable metric space which is a Borel subset of its completion is a Prohorov space), but his proof is incorrect.

An example of a $K_\sigma$ metric non-Prohorov space (and therefore the proof of non-validity of Varadarajan's theorem) was given by Davies [1]. In this note it is proved that a co-analytic separable metric space is a Prohorov space if and only if it is topologically complete (consequently a separable metric space which is a Borel subset of its completion is a Prohorov space if and only if it is a $G_\delta$ subset of its completion). This theorem gives also a solution of the problem whether the space of rational numbers is a Prohorov space (see e.g. [1]). The reader, who is interesting only in this problem, can find its solution in part III which does not depend on topological results of part II.

We begin with the following trivial lemma which will be used without special mention.

**Lemma 1.** *If $M$ is a compact set of probability measures on a subspace $Y$ of a topological space $X$, then the extensions to $X$ of the measures $\mu \in M$ constitute a compact set of probability measures on $X$.*

The proof of the following Theorem 1 and Corollaries 1, 2 was communicated to me by Dr. Roy O. Davies.

**Theorem 1.** *Every $G_\delta$ subspace of a Prohorov space $X$ is a Prohorov space.*

*Proof.* It is enough to show this for an open subspace $G$ of $X$ (then if $M$ is a compact set of tight probability measures on $G_1 \cap G_2 \cap \ldots$, we can choose for every $n$ a compact set $K_n \subset G_n$ such that $\mu(K_n) > 1 - \frac{1}{2^n}\varepsilon$ for every $\mu \in M$, and then with $K = K_1 \cap K_2 \ldots$ we have $K \subset G_1 \cap G_2 \cap \ldots$ and $\mu(K) > 1 - \varepsilon$ for every $\mu \in M$).

Let $M$ be a compact set of tight probability measures on $G$. Given $\varepsilon > 0$, choose a compact set $K_0 \subset X$ with $\mu(K_0 \cap G) > 1 - \frac{1}{2}\varepsilon$ for every $\mu \in M$.

Since any $\mu \in M$ is tight, there exists for it a compact set $K \subset G$ with $\mu(K) > 1 - \frac{1}{2}\varepsilon$. Since $K$ and $K_0 \setminus G$ are disjoint compact sets, then there exists an open set $G^* \supset K$ with $\overline{G^*} \cap (K_0 \setminus G) = \emptyset$. Thus the sets $\{\mu \in M;\ \mu(G^*) > 1 - \frac{1}{2}\varepsilon\}$, as $G^*$ runs over the open sets satisfying $\overline{G^*} \cap (K_0 \setminus G) = \emptyset$, constitute an open covering of $M$. Since $M$ is compact, it is covered by a finite number of these, say

$$\{\mu \in M;\ \mu(G_m^*) > 1 - \tfrac{1}{2}\varepsilon\}, \qquad m = 1, \ldots, k.$$

The set $K = K_0 \setminus (\overline{G_1^*} \cup \cdots \cup \overline{G_k^*})$ is a compact subset of $G$ and $\mu(K) > 1 - \varepsilon$ for every $\mu \in M$.

**Corollary 1.** *A topologically complete topological space is Prohorov.*

**Corollary 2.** *A locally compact Hausdorff space is Prohorov.*

## II. Some Topological Theorems

In this part some conditions are given under which a metric space contains the space of rational numbers as a $G_\delta$ subspace.

We denote by $\mathcal{M}_k$ $(k = 1, 2, \ldots)$ the set of all sequences of natural numbers with $k$ members.

The union of the sets $\mathcal{M}_k$ will be denoted by $\mathcal{M}$. For $z \in \mathcal{M}_k$, $z = [n_1, \ldots, n_k]$ and natural number $n$ the symbol $[z, n]$ means the sequence $[n_1, \ldots, n_k, n] \in \mathcal{M}_{k+1}$ and for $1 \leqq j \leqq k$ the symbol $z_j$ means the sequence $[n_1, \ldots, n_j] \in \mathcal{M}_j$.

The set of all infinite sequences of natural numbers is denoted by $\mathcal{N}$. For $w \in \mathcal{N}$, $w = [n_1, n_2, \ldots]$ we put $w_k = [n_1, \ldots, n_k] \in \mathcal{M}_k$ and for $z \in \mathcal{M}_k$ we put $\mathcal{N}_z = \{w \in N;\ w_k = z\}$. The sets $\mathcal{N}_z$ $(z \in M)$ constitute a basis of some topology on $\mathcal{N}$; it is well known that the space $\mathcal{N}$ is homeomorphic to the space of irrational numbers with the usual topology (see [2]).

**Lemma 2.** *Let $X_0$ be a metric space and let $X \subset X_0$. Suppose that $G_\delta$ subsets $F_z$ of $X_0$ are given for every $z \in \mathcal{M}$ and put $Y = \bigcup_{w \in \mathcal{N}} \bigcap_{k=1}^{\infty} F_{w_k}$. Let the following conditions hold.*

(i) *$X \cap Y = \emptyset$.*

(ii) *$F_{w_k} \supset F_{w_{k+1}}$ for every $w \in \mathcal{N}$.*

(iii) *$X \cap F_z$ is dense in $F_z$ for every $z \in \mathcal{M}$.*

(iv) *$\bigcup_{n=1}^{\infty} F_{[z,n]}$ is dense in $F_z$ for every $z \in \mathcal{M}$.*

(v) *$F_z$ is dense-in-itself for every $z \in \mathcal{M}$.*

(vi) *There exists $z \in \mathcal{M}_1$ such that $F_z \neq \emptyset$.*

*Then $X$ contains a countable dense-in-itself $G_\delta$ subspace.*

*Proof.* Let $G_z^m$ be open subsets of $X_0$ such that $\bigcap_{m=1}^{\infty} G_z^m = F_z$.

We will construct the points of some countable dense-in-itself $G_\delta$ subspace $S$ of $X$ by induction. We will define the systems of points $x_z$ for $z \in \mathcal{M}$ (the set of all points $x_z$ will be the required set), the system of their neighbourhoods $U_z$ and some subsidiary natural numbers $m_z$.

*I.* We can find $z_0 \in \mathcal{M}_1$, $z_0 = [k_0]$ such that $F_{z_0} \neq \emptyset$. As $X \cap F_{z_0}$ is dense in $F_{z_0}$ and $F_{z_0}$ is nonempty and dense-in-itself we can choose a sequence $x_{[n]} \in X \cap F_{z_0}$ and a sequence $U_{[n]}$ of open neighbourhoods of $x_{[n]}$ such that $U_{[n]} \cap U_{[n']} = \emptyset$ for every $n \neq n'$ (it is sufficient to consider some convergent sequence of different points of $X \cap F_{z_0}$). We put $m_z = k_0$ for every $z \in \mathcal{M}_1$.

*II.* Let $k > 1$ be a natural number and let the points $x_z$, the open neighbourhoods $U_z$ of $x_z$ and the natural numbers, $m_z$ be defined for every $z \in \mathcal{M}_i$, $i < k$. Suppose that $x_z \in F_{[m_{z_1}, \ldots, m_{z_{k-1}}]}$ for every $z \in \mathcal{M}_{k-1}$ and that $U_z \cap U_{z'} = \emptyset$ for every $z, z' \in \mathcal{M}_{k-1}$, $z \neq z'$.

For every $z \in \mathcal{M}_{k-1}$ we can find a sequence $x_{[z,n]}$ and natural numbers $m_{[z,n]}$ such that

a) $x_{[z,n]} \in X \cap U_z \cap F_{[m_{z_1}, \ldots, m_{z_{k-1}}, m_{[z,n]}]}$.

b) $\lim\limits_{n \to \infty} x_{[z,n]} = x_z$.

c) $x_{[z,n]} \neq x_z$ for every $n$ and $x_{[z,n]} \neq x_{[z,n']}$ for every $n \neq n'$.

For every $z \in \mathcal{M}_{k-1}$ and every natural $n$ we choose open neighbourhoods $U_{[z,n]}$ of $x_{[z,n]}$ such that $U_{[z,n]} \subset U_z \cap \bigcap\limits_{i,j<k} G^i_{[m_{z_1}, \ldots, m_{z_j}]}$ and $U_{[z,n]} \cap U_{[z,n']} = \emptyset$ for every $n \neq n'$.

Let $S_k$ be the set of all $x_z$, $z \in \mathcal{M}_k$ and let $S = \bigcup\limits_{k=1}^{\infty} S_k$. The set $S$ is clearly countable and a dense-in-itself subspace of $X$. We only have to prove that $S$ is a $G_\delta$ subset of $X$.

Let $H_{k,q} = \bigcup\limits_{z \in M_k} U_z \cap \left\{x \in X_0;\ \operatorname{dist}(x, S_k) < \frac{1}{q}\right\}$. It is easy to prove that $S_k = \bigcap\limits_{q=1}^{\infty} H_{k,q}$, therefore $S_k$ is a $G_\delta$ subset of $X_0$, it follows that $\bigcup\limits_{j=1}^{p} S_j$ is a $G_\delta$ subset of $X_0$. We choose open subsets $G_{p,q}$ of $X_0$ such that $\bigcup\limits_{j=1}^{p} S_j = \bigcap\limits_{q=1}^{\infty} G_{p,q}$.

Let $Q_{p,q} = \bigcup\limits_{z \in \mathcal{M}_p} U_z \cup G_{p-1,q}$ for natural $p, q$ (we set $G_{0,q} = \emptyset$). Clearly $S \subset Q_{p,q}$ for every $p, q$, therefore $S \subset \bigcap\limits_{p,q} Q_{p,q}$.

Suppose that $x \in \bigcap\limits_{p,q} Q_{p,q}$. If $x \notin \bigcup\limits_{z \in \mathcal{M}_p} U_z$ for some $p$ then $x \in G_{p-1,q}$ for every $q$ and therefore $x \in S$. If $x \in \bigcap\limits_{p,q} Q_{p,q} \setminus S$ then $x \in \bigcup\limits_{z \in \mathcal{M}_p} U_z$ for every $p$. Thus there exists $z \in \mathcal{N}$ such that $x \in U_{z_p}$ for every $p$. For every $i, j$ we choose $p, p > i$, $p > j$. Then $U_{z_p} \subset G^i_{[m_{z_1}, \ldots, m_{z_j}]}$, therefore $x \in \bigcap\limits_{i=1} G^i_{[m_{z_1}, \ldots, m_{z_j}]} = F_{[m_{z_1}, \ldots, m_{z_j}]}$; it follows that $x \in \bigcap\limits_{j=1} F_{[m_{z_1}, \ldots, m_{z_j}]} \subset Y$, therefore $x \notin X$.

Now it can be easily seen that $S = X \cap \bigcap_{p,q} Q_{p,q}$ and therefore $S$ is a $G_\delta$ subset of $X$.

**Theorem 2.** *Let $T$ be a metric space which is of the first category in itself. Then $T$ contains a countable dense-in-itself $G_\delta$ subspace.*

*Proof.* There exist $T_n \subset T$, $T_n$ closed and nowhere dense in $T$, $T = \bigcup_{n=1}^{\infty} T_n$. We set in Lemma 2 $X_0 = X = T$, $F_z = T \setminus \bigcup_{j=1}^{k} T_j$ for $z \in \mathscr{M}_k$.

It is easy to prove that the conditions (i)–(vi) are valid; it follows that Theorem 2 is a consequence of Lemma 2.

Note that a separable metric space $X$ is called a co-analytic space if there exists a complete separable metric space $\tilde{X}$ such that $X \subset \tilde{X}$ and $\tilde{X} \setminus X$ is analytic (see [2]).

**Theorem 3.** *A co-analytic separable metric space is topologically complete if and only if it contains no countable dense-in-itself $G_\delta$ subspace.*

*Proof.* If $T$ is topologically complete metric space and $S \subset T$ a countable $G_\delta$ subspace then $S$ is also topologically complete. Therefore $S$ is not dense-in-itself because in the opposite case $S$ is of the first category in itself and this is a contradiction.

Suppose $T$ is not topologically complete. Let $T_0$ be a separable complete metric space such that $T \subset T_0$ and $T_0 \setminus T$ is analytic. Then $T_0 \setminus T$ is nonempty and consequently there exists a continuous mapping $\varphi$ of $\mathscr{N}$ onto $T_0 \setminus T$. Let $\mathscr{B}$ be a countable basis of $\mathscr{N}$. Let $\mathscr{B}_1$ be the set of all $B \in \mathscr{B}$ for which there exists an $F_\sigma$ subset $F_B$ of $T_0$ such that $\varphi(B) \subset F_B \subset T_0 \setminus T$.

We set $G = \bigcup \mathscr{B}_1$, $\Psi = \mathscr{N} \setminus G$, $F = \bigcup \{F_H;\ H \in \mathscr{B}_1\}$. Then $F$ is an $F_\sigma$ subset of $T_0$, $\varphi(G) \subset F \subset T_0 \setminus T$ and $\Psi$ is nonempty because in the opposite case $F = T_0 \setminus T$ and $T$ is a $G_\delta$ subset of $T_0$.

We set in Lemma 2: $X_0 = T_0$, $X = T$, $F_z = \overline{\varphi(\mathscr{N}_z \cap \Psi)}$. It is easy to prove that $Y = \bigcup_{z \in \mathscr{N}} \bigcap_{k=1}^{\infty} F_{z_k} = \varphi(\Psi)$ therefore the condition (i) of Lemma 2 holds. The condition (ii) is obvious. If we suppose that the condition (iii) is not fulfilled then there exists an open subset $J$ of $T_0$ such that $F_z \cap J \neq \emptyset$ and $F_z \cap J \cap T = \emptyset$. We set $F_0 = F \cup (F_z \cap J)$, $F_0$ is an $F_\sigma$ subset of $T_0$ and $F_0 \subset T_0 \setminus T$. Now $\varphi(\mathscr{N}_z \cap \varphi^{-1}(J)) \subset F_0 \subset T_0 \setminus T$ therefore $\varphi(B) \subset F_0 \subset T_0 \setminus T$ for every set $B \in \mathscr{B}$, $B \subset \mathscr{N}_z \cap \varphi^{-1}(J)$. From this it follows that $\mathscr{N}_z \cap \varphi^{-1}(J) \subset G$ but $\mathscr{N}_z \cap \varphi^{-1}(J) \cap \Psi \neq \emptyset$ and this is a contradiction. Thus the condition (iii) holds. It is clear that

$$\varphi(\mathscr{N}_z \cap \Psi) \subset \bigcup_{n=1}^{\infty} (\Psi \cap \mathscr{N}_{[z,n]}) \subset \bigcup_{n=1}^{\infty} F_{[z,n]}.$$

Thus the condition (iv) is also fulfilled. As both $\varphi(\Psi)$ and $T \cap F_z$ are dense in $F_z$ and $\varphi(\Psi) \cap T = \emptyset$ the set $F_z$ is dense-in-itself, consequently the condition (v) is also valid.

The condition (vi) holds because $Y = \varphi(\Psi) \neq \emptyset$.

According to Lemma 2 there exists a countable dense-in-itself $G_\delta$ subset of $T$.

## III. Some Non-Prohorov Spaces

In this part some conditions on the space $X$ are given in order not to be Prohorov space. These results will be used in the next part to prove the main theorem.

**Theorem 4.** *Let $X_0$ be a compact Hausdorff space and let $\emptyset \neq X \subset X_0$. Let, for every natural n, $\mathscr{G}_n$ be a system of open subsets of $X_0$ and let $H_n$ be the union of all the sets of $\mathscr{G}_n$. Suppose that the following conditions hold.*

(i) *There exists a set $B \subset X_0$, $B \supset X_0 \setminus X$ which is measurable for every measure $\mu \in P(X_0)$ such that for every compact $K \subset B$ and every natural n there exists $G \in \mathscr{G}_n$ such that $K \subset G$.*

(ii) *For every compact $K \subset X$ and every sequence $L_n$ of finite subsets of X there exist a natural number $n_0$ and $x \in X \setminus K$ such that $x \notin \bigcup_{n=n_0}^{\infty} H_n$ and if $x \in G \in \mathscr{G}_m$ then $G \cap L_m = \emptyset$.*

*Then X is not a Prohorov space.*

*If, in addition, $X_0$ is a metric space then the condition (ii) can be replaced by the following condition*

(a) *There exists a sequence of sets $X_k \subset X \setminus \bigcup_{n=k}^{\infty} H_n$ such that*

(aa) *The set $\bigcap_{k=1}^{\infty} T_k \setminus X$ is nonempty for every sequence $T_k$ of open subsets of $X_0$ such that $T_1 \supset X_1$ and $T_k \supset T_{k+1} \supset T_k \cap X_{k+1}$.*

(ab) *For every $x \in X_n$ and for every finite set $L \subset X$ there exists a neighbourhood U of x such that $G \in \mathscr{G}_n$ and $X \cap U \cap G \neq \emptyset$ implies $G \cap L = \emptyset$.*

*Proof.* We set $M = \left\{\mu \in P(X);\ \mu(G \cap X) \leqq \frac{1}{n} \text{ for every } G \in \mathscr{G}_n\right\}$.

The set $M_0 = \left\{\mu \in P(X_0);\ \mu(G) \leqq \frac{1}{n} \text{ for every } G \in \mathscr{G}_n\right\}$ is a compact set of tight probability measures on $X_0$. If $K$ is a compact set, $K \subset B$ and if $\mu \in M_0$ then $\mu(K) \leqq \frac{1}{n}$ for every $n$ and therefore (the set $B$ is $\mu$-measurable) $\mu(B) = 0$. Thus for every $\mu \in M_0$ the measure $T\mu$ which is equal to $\mu$ on Borel subsets of $X$ is a tight probability measure on $X$; moreover $T\mu \in M$. If $\mu_\alpha$ is a net in $M_0$ which converges to $\mu \in M_0$ then $\liminf T\mu_\alpha(G \cap X) = \liminf \mu_\alpha(G) \geqq \mu(G) = T\mu(G \cap X)$. Therefore $T$ is a continuous mapping of $M_0$ onto $M$ and thus $M$ is compact.

Let $K \subset X$ be a compact set. Let $N$ be the set of all functions $\eta: X \to \langle 0, +\infty \rangle$ such that

(a) $\eta(x) = 0$ for $x \in K$,

(b) $\sum_{x \in G \cap X} \eta(x) \leqq \frac{1}{n}$ for every $G \in \mathscr{G}_n$.

According to the Zorn's lemma there exists a maximal element $\eta_0$ of $N$ (as usual, we write $\eta_1 \leqq \eta_2$ if $\eta_1(x) \leqq \eta_2(x)$ for every $x \in X$). We prove that $\sum_{x \in X} \eta_0(x) = +\infty$.

Suppose, on the contrary, that $\sum_{x\in X}\eta_0(x)<+\infty$. Then for every natural $n$ there exists a finite subset $L_n$ of $X$ such that $\sum_{x\in X\setminus L_n}\eta_0(x)<\frac{1}{2n}$. According to the condition (ii) there exist a natural number $n_0$ and $x_0\in X\setminus K$ such that $x_0\notin\bigcup_{n=n_0}^{\infty}H_n$ and if $x_0\in G\in\mathcal{G}_m$ then $G\cap L_m=\emptyset$. We put

$$\eta_1(x)=\eta_0(x)\quad\text{for } x\neq x_0\quad\text{and}\quad \eta_1(x_0)=\eta_0(x_0)+\frac{1}{2n_0}.$$

If $G\in\mathcal{G}_n$, $x_0\notin G$ then

$$\sum_{x\in G\cap X}\eta_1(x)=\sum_{x\in G\cap X}\eta_0(x)\leqq\frac{1}{n}.$$

If $G\in\mathcal{G}_n$, $x_0\in G$ then $n<n_0$ and

$$\sum_{x\in G\cap X}\eta_1(x)=\frac{1}{2n_0}+\sum_{x\in G\cap X}\eta_0(x)\leqq\frac{1}{2n_0}+\sum_{x\in X\setminus L_n}\eta_0(x)\leqq\frac{1}{2n_0}+\frac{1}{2n}\leqq\frac{1}{n}.$$

Therefore $\eta_1\in N$, but $\eta_1>\eta_0$ and this is a contradiction. Thus $\sum_{x\in X}\eta_0(x)=+\infty$.

We can find a function $\alpha\colon X\to R$ such that $0\leqq\alpha(x)\leqq\eta_0(x)$ and $\sum_{x\in X}\alpha(x)=1$. If we put $\mu(A)=\sum_{x\in A}\alpha(x)$ then $\mu\in M$ and $\mu(K)=0$. Thus the set $M$ is not tight and $X$ is not a Prohorov space.

If $X_0$ is a metric space, we prove that (a) implies (ii). Let $L_k$ be a sequence of finite subsets of $X$ and let $K$ be a compact subset of $X$. For every $x\in X_n$ we choose an open neighbourhood $U_n(x)$ such that $\operatorname{diam}(U_n(x))\leqq\frac{1}{n}$ and $G\cap L_n=\emptyset$ for every $G\in\mathcal{G}_n$ such that $X\cap U_n(x)\cap G\neq\emptyset$.

We put $T_k=\bigcap_{n=1}^{k}\ \bigcup_{X_n\cap T_{k-1}}U_n(x)$. Let $y_0\in\bigcap_{k=1}^{\infty}T_k\setminus X$. Then $y_0\notin K$ and therefore there exists a natural number $n_0$ such that $\operatorname{dist}(y_0,K)>\frac{1}{n_0}$. Let $x_0\in T_{n_0}\cap X_{n_0}$ such that $y_0\in U_{n_0}(x_0)$. Then $x_0\notin K$ and $x_0\notin G$ for $G\in\mathcal{G}_m$, $m\geqq n_0$. For every $m<n_0$ there exists $x_m\in X_m$ such that $x_0\in U_m(x_m)$ therefore if $x_0\in G\in\mathcal{G}_m$ then $X\cap G\cap U_m(x_m)\neq\emptyset$; it follows that $G\cap L_m=\emptyset$.

*Remark 1.* If the space $X$ in the preceding theorem is countable, some maximal element of $N$ can be construct by induction in the following way.

Let $x_i$ be a sequence of all elements of $X$ such that $x_i\neq x_j$ for $i\neq j$. We put $\eta_0(x_1)=\inf\left\{\frac{1}{n};\ x_1\in G\in\mathcal{G}_n\right\}$ (here we use the convention $\inf\emptyset=+\infty$) and, if $\eta_0(x_1),\dots,\eta_0(x_k)$ are defined,

$$\eta_0(x_{k+1})=\inf\left\{\frac{1}{n}-\sum_{\substack{x_i\in G\\ i\leqq k}}\eta_0(x_i);\ x_{k+1}\in G\in\mathcal{G}_n\right\}.$$

Thus the proof of the preceding theorem is, in case $X$ is countable, constructive.

**Lemma 3.** *Let $X$ be a metric space of the type $K_\sigma$* (i.e. *$X$ is a countable union of compact spaces*) *and let $X$ be of the first category in itself. Then $X$ is not a Prohorov space.*

*Proof.* We can write $X=\bigcup_{n=1} K_n$ where $K_n$ are nonempty compact sets nowhere dense in $X$. We put $X_1=K_1$ and define the sets $X_n$ for $n>1$ by induction. If $X_{n-1}$ is a compact nowhere dense subset of $X$, we choose, for every natural $m$, finite sets $S_m \subset X$ such that

($\alpha$) $S_m \cap X_{n-1}=\emptyset$ for every $m$,

($\beta$) $\operatorname{dist}(s, X_{n-1}) \leqq \frac{1}{m}$ for every $s \in S_m$,

($\gamma$) $\operatorname{dist}(x, S_m) \leqq \frac{1}{m}$ for every $x \in X_{n-1}$ and every $m$

and put $X_n=X_{n-1} \cup \bigcup_{m=1}^{\infty} S_m \cup K_n$. The set $S_m$ can be constructed in the following way. Let $U_1, \ldots, U_p$ be a finite covering of $X_{n-1}$ consisting of open sets with diameter less then $\frac{1}{m}$. For every $i=1, \ldots, p$ we choose a point $s_i \in U_i \setminus X_{n-1}$ and put $S=\{s_1, s_2, \ldots, s_p\}$. Thus $X_n$ are compact subsets of $X$ such that $X_n \subset X_{n+1}$, every point of $X_n$ is a point of accumulation of $X_{n+1} \setminus X_n$ and $X=\bigcup_{n=1}^{\infty} X_n$.

As $X$ is a separable metric space, there exists a compact metric space $X_0$ such that $X \subset X_0$.

We put $J_{k,n}=\left\{x \in X_0;\ \operatorname{dist}(x, X_n)>\frac{1}{k}\right\}$ for natural $k, n$. For natural $n$ let $\mathscr{G}_n$ be the system of all sets of the form $(X_0 \setminus X_k) \cap J_{k,n}$ ($k$ natural). Then $X_k \subset X \setminus \bigcup_{n=k}^{\infty} H_n$ (where $H_n$ is the union of $\mathscr{G}_n$).

If $K \subset X_0 \setminus X$ is a compact set and $n$ is a natural number then $\operatorname{dist}(K, X_n)>0$ and therefore $K \subset J_{k,n}$ for some natural $k$. As $K \subset X_0 \setminus X \subset X_0 \setminus X_k$ it is $K \subset (X_0 \setminus X_k) \cap J_{k,n} \in \mathscr{G}_n$. Considering that $X_0 \setminus X$ is a Borel subset of $X_0$ we finish the proof of the condition (i) of Theorem 4.

Let $T_n$ be open subsets of $X_0$, $T_1 \supset X_1$ and $T_k \supset T_{k+1} \supset T_k \cap X_{k+1}$ and let $T=\bigcap_{n=1}^{\infty} T_n$. Then $T$ is a $G_\delta$ subset of $X_0$, thus $T$ is topologically complete. Moreover $T \cap X=\bigcup_{k=1}^{\infty} T_k \cap X_k$, every set $T_k \cap X_k$ is closed in $T \cap X$ and nowhere dense in $T \cap X$ (every point of $T_k \cap X_k$ is a point of accumulation of $T_{k+1} \cap (X_{k+1} \setminus X_k)$). Thus $T \cap X$ is a set of the first category in $T$ (and $T$ is nonempty); therefore the condition (aa) of Theorem 4 is proved.

To prove the condition (ab) of Theorem 4 we find for every $x \in X_n$ and finite set $L \subset X$ a natural number $p$ such that $L \subset X_p$ and choose a neighbourhood $U$ of $x$ with diameter less than $\frac{1}{p}$. If $U \cap (X_0 \setminus X_k) \cap J_{k,n} \neq \emptyset$ then $U \cap J_{k,n} \neq \emptyset$ and therefore $k>p$. Thus $\left((X_0 \setminus X_k) \cap J_{k,n}\right) \cap L \subset (X_0 \setminus X_p) \cap X_p=\emptyset$ and the condition (ab) of Theorem 4 holds.

Thus, according to Theorem 4, $X$ is not a Prohorov space.

*Remark 2.* If $X$ is a countable dense-in-itself metric space (e.g. the space of rational numbers) then, according to the preceding Lemma, $X$ is not a Prohorov space. Moreover, according to Remark 1, the proof of this fact is constructive.

## IV. Metric Spaces in which Prohorov's Theorem is not Valid

**Theorem 5.** *Let $X$ be a metric space which is of the first category in itself. Then $X$ is not a Prohorov space.*

*Proof.* Suppose $X$ is a Prohorov space. According to Theorem 2, $X$ contains a countable dense-in-itself $G_\delta$ subspace. According to Lemma 3 this subspace is not a Prohorov space but according to Theorem 1 this subspace is a Prohorov space and this is a contradiction.

**Theorem 6.** *Let $X$ be a co-analytic separable metric space. Then $X$ is a Prohorov space if and only if $X$ is topologically complete.*

*Proof.* 1. If $X$ is topologically complete then, according to Prohorov's theorem (Corollary 1 of Theorem 1) $X$ is a Prohorov space.

2. If $X$ is not topologically complete then it contains a countable dense-in-itself $G_\delta$ subspace (Theorem 3). According to Lemma 3 and Theorem 1 $X$ is not a Prohorov space.

*Remark* 3. There exists a separable metric Prohorov space which is not topologically complete.

*Proof.* Assume the continuum hypothesis.

Let $\Omega$ be the first uncountable ordinal number, $\{G_\alpha;\ \alpha<\Omega\}$ all open subsets of $\langle 0,1\rangle$ which contains the set $Q$ of all rational numbers from $\langle 0,1\rangle$. Choose $y_\alpha\in\bigcap_{\beta\leq\alpha} G_\beta\setminus\{y_\gamma;\ \gamma<\alpha\}$ for $\alpha<\Omega$ and set $Y=\{y_\alpha;\ \alpha<\Omega\}$, $X=\langle 0,1\rangle\setminus Y$.

Let $M$ be a compact subset of $P(X)$, $\varepsilon>0$. According to Prohorov's theorem (Corollary 1 of Theorem 1) applied to $\langle 0,1\rangle-Q$ there exists a compact set $A_0\subset\langle 0,1\rangle-Q$ such that $\mu(A_0)>1-\frac{1}{2}\varepsilon$ for every $\mu\in M$. As $A_0\cap Y$ is countable we can apply Prohorov's theorem to $\langle 0,1\rangle\setminus(A_0\cap Y)$ and obtain a compact set $A_1\subset\langle 0,1\rangle\setminus(A_0\cap Y)$ such that $\mu(A_1)>1-\frac{1}{2}\varepsilon$ for every $\mu\in M$. Therefore $A_0\cap A_1\subset X$, $\mu(A_0\cap A_1)>1-\varepsilon$ for every $\mu\in M$ and $X$ is a Prohorov space. $X$ is not topologically complete because $Y$ is uncountable and contains no uncountable compact set (if $Z\subset Y$ is an uncountable compact set then $Z-Q$ is an uncountable Borel set, therefore there exists an uncountable compact subset $Z_0$ of $Z-Q$. Clearly $Z_0\cap Y$ is countable and $Z_0\subset Y$, this is a contradiction).

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