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Covering the Line with Random Intervals

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To each point (x, y) of the upper half-plane H associate the open interval $(x, x+y)$ on the x-axis R. Let S be the Poisson random subset of H generated by $\lambda \times \mu$ on $R \times Y$, where λ is Lebesgue measure on R and μ is a given measure on the upper y-axis $Y=\{y\colon 0<y<\infty\}$. We give in this note a simple necessary and sufficient condition on μ for the union of the intervals associated with each point of S to cover all of R with probability one.

1. Introduction

Let μ be a nonnegative measure on Y finite on intervals $[\varepsilon, T]$, $0<\varepsilon<T<\infty$ and form $\lambda \times \mu$. Let S be the random subset of the upper half-plane $H=R \times Y$ where the number $N(A)$ of points of S in each Borel subset A of H has a Poisson distribution with mean $\lambda \times \mu(A)$ *and* $N(A_i)$, $i=1, \ldots, n$, are independent if $A_1, \ldots, A_n$ are disjoint. Let U denote the (a.s. countable) union of the intervals associated with points of S. We show that $U=R$ with probability one if (1) holds, and with probability zero if (1) fails, where

$$\int_0^1 dx \exp \int_x^\infty (y-x)\, \mu\{dy\} = \infty. \tag{1}$$

The problem of finding necessary and sufficient conditions for $U=R$ was posed and studied by Mandelbrot [M], who was mainly interested in cases where $U \neq R$ because the complement of U then becomes a random perfect set whose distribution is invariant under translations. An earlier, closely related problem, due to Dvoretzky [D] is to determine the conditions for the union of independently and uniformly distributed arcs of given lengths l_n, $n=1, 2, \ldots$ to cover a circumference C of unit length. Dvoretzky's problem was settled in [S] and we show here that Mandelbrot's problem can be settled by a similar application of the methods of [S]. Mandelbrot states that his version is "more natural for application to both mathematics and physics" and indeed its solution appears much simpler and more natural than that of Dvoretzky's version. Mandelbrot conjectured (personal communication) that the condition (1), which appears in [M] in the equivalent form (40) below, is necessary and sufficient for covering.

The final section gives some examples and remarks.

2. A Lower Bound for the Probability of not Covering an Interval

The method of this section is virtually identical with that of §4 of [S] and is due to Billard and Kahane [K].

Fix ε and T and let $U(\varepsilon, T)$ denote the (a.s. finite) union of intervals associated with those points of S for which $\varepsilon \leqq y \leqq T$. Let $m = m(\varepsilon, T)$ denote the measure of that part of $[0, 1]$ which is left uncovered by $U(\varepsilon, T)$. That is

$$m = \int_0^1 \chi(x)\, dx \tag{2}$$

where

$$\chi(x) = \begin{cases} 1 & \text{if } x \notin U(\varepsilon, T) \\ 0 & \text{if } x \in U(\varepsilon, T). \end{cases} \tag{3}$$

Let φ denote the indicator of the event that $[0, 1] \not\subset U(\varepsilon, T)$:

$$\varphi = \begin{cases} 1 & \text{if } [0, 1] \not\subset U(\varepsilon, T) \\ 0 & \text{if } [0, 1] \subset U(\varepsilon, T). \end{cases} \tag{4}$$

Since $\varphi = 0$ implies $m = 0$, we have

$$m = m\,\varphi. \tag{5}$$

Applying Schwarz's inequality we have

$$(E m)^2 \leqq E m^2\, E \varphi^2. \tag{6}$$

Since $E\varphi^2 = E\varphi = P([0, 1] \not\subset U(\varepsilon, T))$ we have

$$P([0, 1] \not\subset U(\varepsilon, T)) \geqq (E m)^2 / E m^2. \tag{7}$$

The first two moments of m are easy to obtain since from (2) and the invariance of the distribution of S under translations parallel to the x-axis, we get

$$E m = E\chi(0) = P(0 \notin U(\varepsilon, T)), \tag{8}$$

$$\begin{aligned} E m^2 &= \int_0^1 \int_0^1 P(x_1 \notin U(\varepsilon, T), x_2 \notin U(\varepsilon, T))\, dx_1\, dx_2 \\ &= 2 \int_0^1 \int_0^{x_2} P(0 \notin U(\varepsilon, T), x_2 - x_1 \notin U(\varepsilon, T))\, dx_1\, dx_2 \\ &= 2 \int_0^1 (1 - t)\, P(0 \notin U(\varepsilon, T), t \notin U(\varepsilon, T))\, dt \\ &\leqq 2 \int_0^1 P(0 \notin U(\varepsilon, T), t \notin U(\varepsilon, T))\, dt. \end{aligned} \tag{9}$$

From (7), (8) and (9) we obtain

$$P([0, 1] \not\subset U(\varepsilon, T)) \geqq \tfrac{1}{2} P(0 \notin U(\varepsilon, T))^2 \Big/ \int_0^1 P(0 \notin U(\varepsilon, T), t \notin U(\varepsilon, T))\, dt, \tag{10}$$

which is the lower bound we will need for the probability of not covering.

3. An Upper Bound for the Probability of not Covering an Interval

The method of this section is similar to that of §3 of [S] but is considerably simpler because the role of the basic inequality (4) of [S] is here played by an equality ((11) below) which in turn results from the basic property of the Poisson generated set S that the numbers $N(A_i)$ of points in disjoint sets A_i are independent.

Lemma. *If* $t_1 < \cdots < t_j < t < x$ *then*

$$P(x \notin U \mid t \notin U;\ t_i \in U \text{ for all } i) = P(x \notin U \mid t \notin U). \tag{11}$$

Proof. Let $A(t)$, $t \in R$ denote the wedge-shaped region of the upper half plane defined by

$$A(t) = \{(x, y)\colon x < t,\ t - x < y\}. \tag{12}$$

The region $A(x) - A(t)$ is disjoint from $A(t)$ so that

$$\begin{aligned} P(N(A(x)) = 0, N(A(t)) = 0) &= P(N(A(x) - A(t)) = 0;\ N(A(t)) = 0) \\ &= P(N(A(x) - A(t)) = 0)\, P(N(A(t)) = 0). \end{aligned} \tag{13}$$

Similarly the region $A(x) - A(t)$ is disjoint from $A(t) \cup A(t_i)$ for all i, and so

$$\begin{aligned} &P(N(A(x)) = 0;\ N(A(t)) = 0;\ N(A(t_i)) > 0 \text{ for all } i) \\ &\quad = P(N(A(x) - A(t)) = 0;\ N(A(t)) = 0;\ N(A(t_i)) > 0 \text{ for all } i) \\ &\quad = P(N(A(x) - A(t)) = 0)\, P(N(A(t)) = 0;\ N(A(t_i)) > 0 \text{ for all } i). \end{aligned} \tag{14}$$

From (13) and (14) and the fact that $a \notin U$ if and only if $N(A(a)) = 0$ we have, respectively

$$P(x \notin U, t \notin U) = P(N(A(x) - A(t)) = 0)\, P(t \notin U), \tag{15}$$

$$P(x \notin U, t \notin U, t_i \in U \text{ for all } i) = P(N(A(x) - A(t)) =)\, P(t \notin U, t_i \in U \text{ for all } i). \tag{16}$$

Dividing in (15) by $P(t \notin U)$ and in (16) by $P(t \notin U, t_i \in U$ for all $i)$ we obtain (11) immediately. This proves the lemma.

Fix $k > 0$ and let $\xi_k = j/k$ if $0, 1/k, 2/k, \ldots, (j-1)/k$ belong to U but $j/k \notin U$, $j = 0, 1, 2, \ldots$ setting $\xi_k = \infty$ if there is no such j. Thus ξ_k is the first uncovered point of the sequence $0, 1/k, 2/k, \ldots$. Define $m(a, b)$ for $a < b$ to be the measure of the uncovered part of the interval (a, b), so that

$$m(a, b) = \int_a^b \chi(x)\, dx = \lambda((a, b) - U) \tag{17}$$

where χ is given by (3). We have

$$\begin{aligned} E m(0, 2) &= \sum_{0 \leqq j \leqq \infty} E[m(0, 2) \mid \xi_k = j/k]\, P(\xi_k = j/k) \\ &\geqq \sum_{0 \leqq j \leqq k} E[m(0, 2) \mid \xi_k = j/k]\, P(\xi_k = j/k) \\ &\geqq \sum_{0 \leqq j \leqq k} E[m(j/k, (j/k) + 1) \mid \xi_k = j/k]\, P(\xi_k = j/k), \end{aligned} \tag{18}$$

noting that for $j \leqq k$, the interval $(j/k, (j/k) + 1) \subset (0, 2)$. Using (17), we have

$$E[m(j/k, (j/k) + 1) \mid \xi_k = j/k] = \int_{j/k}^{(j/k)+1} P(x \notin U \mid \xi_k = j/k)\, dx. \tag{19}$$

The event $\{\xi_k = j/k\} = \{j/k \notin U;\ (i-1)/k \in U,\ i=1,\ldots,j\}$ and applying (11) with $t=j/k$, $t_i=(i-1)/k$ to the integrand in (19) we obtain, setting $x=(j/k)+t$,

$$E[m(j/k,(j/k)+1)\,|\,\xi_k=j/k] = \int_{j/k}^{(j/k)+1} P(x\notin U\,|\,j/k\notin U)\,dx$$
$$= \int_0^1 P((j/k)+t\notin U\,|\,j/k\notin U)\,dt \tag{20}$$
$$= \int_0^1 P(t\notin U\,|\,0\notin U)\,dt$$

where we have used the translational invariance of the distribution of U in the last equation. From (18) and (20) we get

$$Em(0,2) \geqq \int_0^1 P(t\notin U\,|\,0\notin U)\,dt \sum_{0\leqq j\leqq k} P(\xi_k=j/k). \tag{21}$$

Since the sum on the right in (21) is simply $P(\xi_k \leqq 1)$ and $\xi_k \leqq 1$ if and only if $0, 1/k, \ldots, k/k$ are not all covered we have from (21),

$$P(j/k\notin U \text{ for some } j\leqq k) \leqq Em(0,2) \Big/ \int_0^1 P(t\notin U\,|\,0\notin U)\,dt. \tag{22}$$

Letting $\mu_{\varepsilon,T}(I)=\mu(I\cap[\varepsilon,T])$ and applying (22) to the U set thus obtained we see that (22) is also valid for U replaced by $U(\varepsilon,T)$. From (8) and (22) we obtain then

$$P(j/k\notin U(\varepsilon,T) \text{ for some } j\leqq k) \leqq 2P(0\notin U(\varepsilon,T))^2 \Big/ \int_0^1 P(0\notin U(\varepsilon,T), t\notin U(\varepsilon,T))\,dt. \tag{23}$$

As $k\to\infty$ through powers of 2, the left side of (23) increases to

$$P(t\notin U(\varepsilon,T) \text{ for some } t \text{ a binary rational, } 0\leqq t\leqq 1).$$

Because $U(\varepsilon,T)$ is a finite union of open intervals, the probability that U covers the binary rationals of $[0,1]$ but not all points of $[0,1]$ is zero and so

$$P([0,1]\not\subset U(\varepsilon,T)) \leqq 2P(0\notin U(\varepsilon,T))^2 \Big/ \int_0^1 P(0\notin U(\varepsilon,T), t\notin U(\varepsilon,T))\,dt. \tag{24}$$

But this is (10) with the inequality reversed except for a factor of 4.

4. Proof that (1) Is Necessary and Sufficient for Covering

The method of this section is rather simpler than the corresponding estimates of §§5 and 6 of [S]. The simplicity here results from that of the Poisson formula ((29), (30)) for the probability that there are no points of S in a region.

Since $0\notin U(\varepsilon,T)$ if and only if $N(A)=0$ where

$$A=\{-y<x<0,\ \varepsilon\leqq y\leqq T\} \tag{25}$$

we have

$$P(0\notin U(\varepsilon,T)) = P(N(A)=0). \tag{26}$$

Similarly, $0 \notin U(\varepsilon, T)$, $t \notin U(\varepsilon, T)$ if and only if $N(AUB)=0$ where A is given by (25) and for $t>0$,

$$B=B(t)=\{\max(0, t-y)<x<t, \varepsilon \leqq y \leqq T\} \tag{27}$$

so that

$$P(0 \notin U(\varepsilon, T), t \notin U(\varepsilon, T))=P(N(AUB)=0). \tag{28}$$

Since $N(A)$ and $N(AUB)$ have Poisson distributions with means $\lambda \times \mu(A)$, $\lambda \times \mu(AUB)$ respectively and since A and B are disjoint sets, we have

$$P(N(A)=0)=\exp[-\lambda \times \mu(A)], \tag{29}$$

$$P(N(AUB)=0)=\exp[-\lambda \times \mu(A)-\lambda \times \mu(B)]. \tag{30}$$

From (26), (28), and (29) and (30), for $t>0$,

$$P(0 \notin U(\varepsilon, T), t \notin U(\varepsilon, T))/P(0 \notin U(\varepsilon, T))^2=\exp[\lambda \times \mu(A)-\lambda \times \mu(B(t))]. \tag{31}$$

From (25) and (27) we have

$$\lambda \times \mu(A)=\int_{\varepsilon}^{T} y\, \mu\{dy\}, \tag{32}$$

$$\lambda \times \mu(B(t))=\int_{\varepsilon}^{T} \min(y, t)\, \mu\{dy\}. \tag{33}$$

Subtracting (33) from (32),

$$\lambda \times \mu(A)-\lambda \times \mu(B(t))=\int_{\max(\varepsilon, t)}^{T} (y-t)\, \mu\{dy\}. \tag{34}$$

From (31) and (34) integrating over t, $0<t<1$,

$$\int_0^1 P(0 \notin U(\varepsilon, T), t \notin U(\varepsilon, T))/P(0 \notin U(\varepsilon, T))^2\, dt=\int_0^1 dt \exp \int_{\max(\varepsilon, t)}^{T} (y-t)\, \mu\{dy\}. \tag{35}$$

Denoting the right hand side by $F(\varepsilon, T)$ we have from (10) and (24)

$$\frac{1}{2F(\varepsilon, T)} \leqq P([0, 1] \not\subset U(\varepsilon, T)) \leqq \frac{2}{F(\varepsilon, T)}. \tag{36}$$

As $\varepsilon \to 0$ and $T \to \infty$, $F(\varepsilon, T)$ increases to $F(0, \infty)$, which is the left side of (1). We note that

$$\bigcap_{n=1}^{\infty} \left\{[0, 1] \subset U\left(\frac{1}{n}, n\right)\right\}=\{[0, 1] \subset U\} \tag{37}$$

since the left side is clearly contained in the right while if $[0, 1]$ is covered by a countable union of open intervals of U, some finite subcover exists by the Heine-Borel theorem and so the right side of (37) is contained in the left. From (37) and countable additivity, $P([0, 1] \not\subset U(\varepsilon, T))$ decreases as $\varepsilon \to 0$ and $T \to \infty$ to $P([0, 1] \not\subset U)$. Thus

$$\frac{1}{2F(0, \infty)} \leqq P([0, 1] \not\subset U) \leqq \frac{2}{F(0, \infty)} \tag{38}$$

where $F(0,\infty)$ is the left side of (1). If (1) holds $F(0,\infty)=\infty$ and $P([0,1]\not\subset U)=0$. Since then $P([n,n+1]\not\subset U)=0$ as well, by translational invariance of the distribution of U, $P(R\subset U)=1$ by countable additivity of P. On the other hand if (1) fails so that $F(0,\infty)<\infty$ then $P(R\subset U)\leqq P([0,1]\subset U)<1$ by (38). However we can make the stronger assertion that if (1) fails, $P(R\subset U)=0$. To see this let $p_n=P([n,\infty)\subset U)$. Then $p_n\uparrow p_\infty$ where $p_\infty=P([n,\infty)\subset U$ for some $n)$. The latter event has probability zero or one because it is a tail event, depending only on the behavior of S outside an arbitrarily large rectangle in H. Thus $p_\infty=$zero or one. On the other hand p_n does not depend on n by translational invariance of U. Therefore letting $n\to-\infty$

$$P(R\subset U)=\text{zero or one}. \tag{39}$$

Since $P(R\subset U)<1$ we must have $P(R\subset U)=0$. Thus we have shown that $P(R=U)=$zero or one according as (1) converges or diverges.

5. Remarks

Integration by parts in the exponent of (1) shows that (1) is equivalent to

$$\int_0^1 dx\exp\int_x^\infty \mu([y,\infty))\,dy=\infty. \tag{40}$$

The upper limit 1 on the integral over x in (1) and (40) could equivalently be replaced by any positive number t, since the integrand is continuous in x.

We remark that (1) is not equivalent to

$$\int_0^1 dx\exp\left[\int_x^\infty y\,\mu\{dy\}\right]=\infty, \tag{41}$$

although examples of μ where (1) fails but (41) holds are not simple. To give such an example, consider a sequence $\{l_n\}$, $1=l_1\geqq l_2\ldots\geqq l_n\to 0$ and let μ be the measure assigning unit mass to each l_n. We will show that in this case, $U=R$ a.s. if and only if

$$\sum_{n=1}^\infty \frac{1}{n^2}\exp(l_1+\cdots+l_n)=\infty \tag{42}$$

which is the same condition found in [S] for the union of independent and uniformly distributed arcs of length l_n to cover a unit circumference. Indeed, breaking up the integral in (1) over intervals $l_{n+1}<x<l_n$, (1) becomes

$$\begin{aligned}\infty&=\sum_{n=1}^\infty \int_{l_{n+1}}^{l_n} dx\exp\left(\sum_{j=1}^n (l_j-x)\right)\\ &=\sum_{n=1}^\infty \frac{1}{n}\left(\exp[l_1+\cdots+l_{n+1}-(n+1)\,l_{n+1}]-\exp[l_1+\cdots+l_n-n\,l_n]\right) \qquad (43)\\ &=\sum_{n=2}^\infty \frac{1}{(n-1)\,n}\exp(l_1+\cdots+l_n-n\,l_n)\end{aligned}$$

where we used summation by parts in the last line (noting that $l_1+\cdots+l_n-n\,l_n$ increases in n). In [S], it is shown that the last term in (43) is infinite if and only if (42) holds. Thus (42) holds if and only if $U=R$ a.s. On the other hand (41) becomes in this case by the same method,

$$\begin{aligned}\infty &= \sum_{n=1}^{\infty} \int_{l_{n+1}}^{l_n} dx \exp(l_1+\cdots+l_n)\\ &= \sum_{n=1}^{\infty} (l_n-l_{n+1}) \exp(l_1+\cdots+l_n).\end{aligned} \tag{44}$$

Summing by parts we obtain (41) holds if and only if

$$\begin{aligned}\infty &= \sum_{n=1}^{\infty} l_n[\exp(l_1+\cdots+l_n)-\exp(l_1+\cdots+l_{n-1}))\\ &= \sum_{n=1}^{\infty} l_n(1-\exp(-l_n))\exp(l_1+\cdots+l_n).\end{aligned} \tag{45}$$

Since $l_n\to 0$ and $1-\exp(-u)\sim u$ as $u\to 0$, (41) is equivalent in this case to

$$\sum_{n=1}^{\infty} l_n^2 \exp(l_1+\cdots+l_n)=\infty \tag{46}$$

which is condition (3) of [S]. An example (Example 2) is given in [S] of a sequence $\{l_n\}$ where (46) holds and (42) fails. For the corresponding μ, (41) holds but covering does not take place since (42) and (1) do not hold. Thus (41) is not equivalent to (1).

In [M], μ is said to give a low frequency covering if R is a.s. covered by intervals of length $\geqq 1$ and μ is said to give a high frequency covering if R is a.s. covered by intervals of length <1. Decomposing $\mu=\mu_L+\mu_H$ where

$$\begin{aligned}\mu_L(\{dy\})&=\mu([1,\infty)\cap\{dy\})\\ \mu_H(\{dy\})&=\mu((0,1)\cap\{dy\})\end{aligned} \tag{47}$$

we see that μ gives a low (high) frequency covering if and only if (1) holds with μ replaced by μ_L (μ_H). From (1), μ gives a low frequency covering if and only if

$$\int_1^{\infty} y\,\mu\{dy\}=\infty. \tag{48}$$

High frequency covering is more delicate; to illustrate consider the example

$$\mu^c\{dy\}=c\,y^{-2}\,dy, \qquad 0<y<\infty. \tag{49}$$

μ^c gives a low frequency covering for all $c>0$ since (48) holds for all $c>0$. A high frequency cover exists only for $c\geqq 1$ since μ_H^c satisfies (1) only for $c\geqq 1$.

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