Z. Wahrscheinlichkeitstheorie verw. Geb. 22, 195–204 (1972)

Automorphisms of Baire Measures on Generalized Cubes

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1. Introduction

Let μ be a finite, Lebesgue-Stieltjes measure on $I=[0,1]$, (or more generally a finite Borel measure on a Polish space X). If ϕ is a measure preserving automorphism of the measure algebra of (I,μ), it was shown by von Neumann [7] that ϕ is induced by a (1-1) invertible Borel measurable measure preserving point mapping T on I. The theorem generalizes to any space which is point isomorphic to a Lebesgue-Stieltjes measure on the unit interval, in particular to any finite measure on $S=\prod I_\alpha$, $\alpha\in A$, A countable, $I_\alpha=[0,1]$. Further ϕ need not be measure preserving, in which case, of course, T will only preserve sets of measure zero.

Von Neumann's result was generalized by Maharam [5] to the direct product of uncountably many normalized measures on $[0,1]$, i.e. to the direct product measure on $S=\prod I_\alpha$, $\alpha\in A$, A possibly uncountable. Making heavy use of the ideas and techniques of Maharam we generalize the result to a wide class of finite measures (including most measures encountered in probability theory) on the product σ-algebra (which is the Baire σ-algebra) of $S=\prod I_\alpha$. As in Maharam's paper we do not need ϕ to be measure preserving (and the same proofs work when S is the product of uncountably many Polish spaces).

We do not know if the result is true for an arbitrary finite Baire measure on $S=\prod I_\alpha$. However if it were true, and if further T could be chosen to be Borel measurable on $S=\prod I_\alpha$ (instead of just Baire measurable), then we could show that if ϕ were an automorphism of the measure algebra of a finite regular Borel measure on S, (which is canonically isomorphic to the measure algebra of its Baire restriction) then it could be induced by a (1-1) invertible Borel measurable point mapping of S. By a standard embedding procedure this would generalize to a Radon measure on an arbitrary compact space X. These results are however false, as an example due to Panzone and Segovia ([8], Sec. 5 Example (c)) shows. The example even shows that the result is false for Baire measures on an arbitrary compact space X (as opposed to $S=\prod I_\alpha$).

An early version of this paper contained "proofs" of all the above (hypothetical) statements. The error in the argument was discovered by J.C. Taylor. My thanks are due to him and also to K.N. Gowrisankaran and D.A. Dawson for their comments and suggestions.

We note, however, that on a compact (and even a locally compact) Hausdorff space a set isomorphism of a Radon measure is always induced by a point homomorphism (i.e. a not necessarily invertible map). See [2], Chapter X, Theorem 1.

* This research was supported by a grant from the National Research Council of Canada.

2. Notation

We follow the notation of Maharam's paper [5]. Let S and S' be measure spaces and E and E' their respective measure algebras. A *point isomorphism* T from S to S' is a bijection of S to S' such that T and T^{-1} map measurable sets into measurable sets, and sets of measure zero into sets of measure zero. A *set-isomorphism* ϕ from S to S' is simply an isomorphism of E and E', but not necessarily a measure preserving isomorphism. When $S=S'$ and $E=E'$ we speak of point and set automorphisms. Measurable subsets of S are denoted by X, Y, etc., elements of E by x, y, etc., points of S by p, q, etc. Every T induces a ϕ by the rule $\phi(x)=\{T(X)\}$ where $X\in x$, and $\{Y\}$ denotes the element of E to which Y belongs.

We write $S=S(A)=\prod I_\alpha$, $\alpha\in A$ where each I_α is the closed interval $[0,1]$. For subsets $B\subset A$ we write $S(B)=\prod I_\alpha$, $\alpha\in B$. If $C\subset B$, the projection from $S(B)$ to $S(C)$ is denoted by π_{BC}; π_{AC} is abbreviated to π_C. Each $S(B)$ is a measurable space, the measurable sets being the product σ-algebra of the Borel σ-algebras of the I_α. If B is uncountable this product σ-algebra is the Baire σ-algebra of $S(B)$ and not the Borel σ-algebra. S^B denotes the σ-algebra of cylinders in $S(A)$ of the form $\pi_B^{-1}(X)$, where X is a measurable subset of $S(B)$. Let μ denote a fixed probability measure on $S=S(A)$, (i.e. a Baire measure on S). Then, via S^B and π_B, μ induces measures μ_B on each $S(B)$. The measure algebras of S, S^B and $S(B)$ are denoted respectively by E, E^B and $E(B)$. π_B induces a canonical isomorphism of E^B and $E(B)$. [Note that in Maharam [5], μ is always a direct product measure.] We write $e(B)$ for $\{S(B)\}$, the equivalence class of $S(B)$.

3. Preliminary Lemmas

The following lemma is stated in Maharam [5] for the case when μ is a direct product measure but the proof works equally well for an arbitrary probability measure μ on the product σ-algebra of S.

Lemma 1. (Maharam, Lemma 2.) *Let ϕ be a set automorphism of $S=\prod I_\alpha$, $\alpha\in A$, and let T be a* (1-1) *mapping of S onto itself such that for each finite set $C\subset A$ and for each measurable set K of S^C, $T(K)\in\phi\{K\}$ and $T^{-1}(K)\in\phi^{-1}\{K\}$. Then T is a point automorphism of S, and induces ϕ.*

Definition. (a) Let ϕ be a given automorphism of E. A set $B\subset A$ will be called *invariant under* ϕ if $\phi(E^B)=E^B$. Restricted to E^B, ϕ will be an automorphism of E^B and so will induce an automorphism of $E(B)$.

(b) Let T be a given point automorphism of S. A set $B\subset A$ will be called *invariant under* T if $T(S^B)=S^B$.

Lemma 2. (a) (Maharam, Lemma 3.) *If ϕ is a set automorphism of E, then each countable set $B\subset A$ is contained in some countable subset $\bar{B}$ of A which is invariant under ϕ.*

(b) *If T is a point automorphism of S then each countable set $B\subset A$ is contained in some countable subset $\bar{B}$ of A which is invariant under T.*

Proof. (a) The proof is unchanged from that given in Maharam [5] for the special case when μ is the direct product measure.

(b) The proof of (a) works for (b) with only minor verbal changes.

We also need the following, which is a slight generalization of von Neumann's original theorem.

Lemma 3. (Maharam, Lemma 4.) *Every set automorphism of a Lebesgue-Stieltjes measure on the unit interval can be induced by a point automorphism.*

In addition to these lemmas from [5] we need

Lemma 4. *Let μ be a Lebesgue-Stieltjes probability measure on $I=[0,1]$ and T a point automorphism of (I,μ). Then there exists a point automorphism T' differing from T only on a μ-null set, such that T' and T'^{-1} are of Baire class at most 3 or equivalently of analytically representable class at most Φ_3.*

Proof. By Lusin's Theorem there exists an $\mathscr{F}_\sigma$ set $H=\bigcup_{n=1}^{\infty} F_n$ such that F_n closed, $F_n\subset F_{n+1}$, $\mu(I-H)=0$ and $T|F_n$ is continuous. Since each F_n is compact it follows that $T|F_n$ is a homeomorphism, that TH is also an $\mathscr{F}_\sigma$ set, and that T and T^{-1} are of class Φ_1 from H to TH, and TH to H respectivley.

Since T is (1-1), and $I-H$ and $I-TH=T(I-H)$ have the same cardinal and since these are both $\mathscr{G}_\delta$ sets the cardinal can be finite, countable or c. If $I-H$ is finite, $T|I-H$ is a homeomorphism; if countable T and T^{-1} are of Baire class at most 1 on these sets (the inverse image of any closed set being at worst an $\mathscr{F}_\sigma$).

If $I-H$ is of cardinal c, since it is a $\mathscr{G}_\delta$, it can be metrized to be a complete separable space, and the same is true of $I-TH$. By a Theorem of Kuratowski [3], (p. 451, Remark (ii)), there exists a (1-1) map T_1 of $I-H$ to $I-TH$ such that T_1, T_1^{-1} are both of Baire class 1. In the first two cases we put $T'=T$, in the last case we put

$$T'p=Tp \quad \text{if } p\in H,$$

$$T'p=T_1 p \quad \text{if } p\in I-H.$$

$T'|H$ is of Baire class 1 and H is of multiplicative Baire class 1, $T'|I-H$ is of Baire class at most 1 and $I-H$ is of multiplicative Baire class at most 2. Similar remarks apply to T'^{-1}.

Let F be closed in I, let $F_1=F\cap T'H$, $F_2=F\cap(I-T'H)$. F_1 is closed in $T'H$, $T'|H$ is of Baire class 1, so $T'^{-1}F_1$ is of Baire class 1 in H; since H itself is of Baire class 1 in I, $T'^{-1}F_1$ is of Baire class $1+1=2$ in I. F_2 is closed in $I-T'H$, $T'|I-H$ is of Baire class 1, so $T'^{-1}F_2$ is of Baire class 1 in $I-H$; since $I-H$ is of Baire class 2 in I, $T'^{-1}F_2$ is of Baire class $1+2=3$ in I. So

$$T'^{-1}F=T'^{-1}F_1\cup T'^{-1}F_2$$

is of Baire class at most 3 in I, and so T' is of Baire class at most 3. The same holds for T'^{-1}. Now on I, the functions of Baire class 3 coincide with the analytically representable functions of class Φ_3 so $T', T'^{-1}\in\Phi_3$.

(Remember that $f\in\Phi_3$, if $f(p)=\lim_{m\to\infty}\lim_{n\to\infty}\lim_{r\to\infty} f_{m,n,r}(p)$ where each $f_{m,n,r}$ is continuous.)

Note. For all results on Baire classes, analytically representable classes Φ_α and mappings of $\mathscr{G}_\delta$ sets see Kuratowski [3], §§ 30, 31, 33, 36, 37.

4. Main Lemmas and Theorem

The following lemma is similar to Lemma 5 of Maharam.

Lemma 5. *Let B be any subset of A such that $C=A-B$ is countable. ϕ is a set automorphism of S such that $\phi|E^B$ is the identity mapping. Then there exists a point automorphism T of S inducing ϕ and such that $\pi_B T=\pi_B$.*

Proof. By Lemma 2(a) there exists $C\subset\bar{C}\subset A$ where $\bar{C}$ is countable and invariant under ϕ. So $\phi_1=\pi_{\bar{C}}\phi\pi_{\bar{C}}^{-1}$ is an automorphism of $E(\bar{C})$ induced by ϕ, and as in Maharam, $S(\bar{C})$ is point isomorphic to a Lebesgue-Stieltjes probability measure (or even Lebesgue measure plus a countable number of atoms) on I. So there exists a point isomorphism T_1 of $S(\bar{C})$ which induces ϕ_1. Put $D=\bar{C}-C$. Then $D\subset B$.

Suppose first $D\neq\emptyset$. Then $S(\bar{C})=S(D)\times S(C)$. Now if $x\in E(D)\,\tilde{x}=e(A-\bar{C})\times x\times e(C)$ belongs to both E^B and $E^{\bar{C}}$. Since $\phi|E^B$ is the identity it follows that $\tilde{x}=\phi(\tilde{x})=e(A-\bar{C})\times x\times e(C)$. It follows that

$$\phi_1(x\times e(C))=x\times e(C)$$

for each $x\in E(D)$. Thus, if X is a measurable subset of $S(D)$, $T_1(X\times S(C))$ differs from $X\times S(C)$ by a null set. Apply this to the sets $T_1^i(X_n)$ ($i=0,\pm1,\pm2,\ldots$, $n=1,2,3,\ldots$) where $X_1,X_2,\ldots$ form a separating sequence of generators of the Borel σ-algebra of $S(D)$; we obtain countably many null sets with union N, say. Thus N is a null set with $T_1(N)=N=T_1^{-1}(N)$ and

$$T_1[(X_n\times S(C))-N]=(X_n\times S(C))-N \qquad (n=1,2,3,\ldots).$$

Define a transformation T_2 on $S(\bar{C})$ by

$$T_2(p)=T_1(p) \quad \text{if } p\in S(\bar{C})-N$$
$$T_2(p)=p \quad \text{if } p\in N.$$

Thus T_2 is a point automorphism of $S(\bar{C})$ also inducing ϕ_1 and

$$T_2(X_n\times S(C))=X_n\times S(C) \qquad (n=1,2,3,\ldots).$$

Since $X_1,X_2,\ldots$ is a separating sequence, it follows that for each $p\in S(D)$,

$$T_2(p\times S(C))=p\times S(C)$$

and hence for each $X\subset S(D)$,

$$T_2(X\times S(C))=X\times S(C).$$

Define T on $S=S(A)$ by

$$T(p,q)=(p,T_2(q))$$

where $p\in S(A-\bar{C})$ and $q\in S(\bar{C})$. We certainly have $\pi_B T=\pi_B$. Further $\{TX\}=\phi\{X\}$ if $X\in S^B$ or $S^{\bar{C}}$. Since the set of X such that $\{TX\}=\phi\{X\}$ forms a σ-algebra and S^B and S^C generate the product σ-algebra of $S(A)$, it follows that $\{TX\}=\phi\{X\}$ for all X in the product σ-algebra, i.e. that T induces ϕ.

If $D=\emptyset$, $T_1=T_2$ and T, as defined, obviously has all the required properties.

Note. The above proof differs from the corresponding one in Maharam *only* in the argument showing that T induces ϕ. However as the lemma is vital and nontrivial we include the whole proof.

Lemma 6. *Let B be any subset of A such that $C=A-B$ is countable, ϕ a set automorphism of S such that B is invariant under ϕ and so $\phi|E^B$ induces an automorphism ϕ' of $E(B)$. Suppose there exists a point automorphism T' of $S(B)$ inducing ϕ' and suppose further that there exists some point automorphism P of $S(A)$ extending T',* i.e. *such that $\pi_B P=T'\pi_B$. Then there exists a point automorphism T of $S(A)$ inducing ϕ and such that $\pi_B T=T'\pi_B$.*

Proof. Let ψ be the set automorphism of S induced by P. Then $\phi\psi^{-1}$ is a set automorphism of S and $\phi\psi^{-1}|E^B$ is the identity. So by Lemma 5, there exists a point automorphism Q of S inducing $\phi\psi^{-1}$, and such that $\pi_B Q=\pi_B$. Then QP induces $\phi\psi^{-1}\psi=\phi$ and $\pi_B QP=\pi_B P=T'\pi_B$.

Definition. Let ϕ be a set automorphism of S and let $B\subset A$ be invariant under ϕ and such that the set automorphism ϕ' induced by ϕ on $E(B)$ is induced by a point automorphism T' of $S(B)$. We say that ϕ has the *countable extension property relative to* B if for every set $B_1\supset B$, invariant under ϕ and such that B_1-B is countable, the set automorphism ϕ' induced by ϕ on $E(B_1)$ is induced by a point automorphism T_1' of $S(B_1)$ such that $\pi_{B_1 B}T_1'=T'\pi_{B_1 B}$. If ϕ has the countable extension property relative to B for every B invariant under ϕ such that the set automorphism ϕ' induced by ϕ on $E(B)$ is induced by a point automorphism T' of $S(B)$, then we say that ϕ has the *countable extension property.*

Lemma 7. *Let $S=\prod I_\alpha$, $\alpha\in A$, let μ be a probability measure on the product (Baire) σ-algebra of S. Let ϕ be a set-automorphism of S such that ϕ has the countable extension property. Then there exists a point-automorphism T of S which induces ϕ.*

Proof. Consider the family of ordered pairs (B_λ, T_λ) where (i) B_λ is a subset of A invariant under ϕ, (ii) T_λ is a point automorphism of $S(B_\lambda)$ and (iii) the automorphisms of $E(B_\lambda)$ induced by $\phi|E^{B_\lambda}$ and by T_λ are the same. We say that $(B_\lambda, T_\lambda)<(B_\nu, T_\nu)$ provided that $B_\lambda\subset B_\nu$ and $\pi_{\nu\lambda}T_\nu=T_\lambda\pi_{\nu\lambda}$ on $S(B_\nu)$. (We abbreviate $\pi_{B_\nu B_\lambda}$ to $\pi_{\nu\lambda}$, π_{B_λ} to π_λ). This partial ordering is transitive. Further every linearly ordered sub-family has an upper bound in the family. Let $\{(B_\nu, T_\nu)\colon \nu\in M\}$ be the linearly ordered sub-family. Put $B'=\bigcup B_\nu$, then B' is an invariant subset of A under ϕ. Given $p\in S(B')$ and $\alpha\in B'$, choose any ν such that $\alpha\in B_\nu$ and let $q_\alpha=(T_\nu(\pi_\nu(p)))_\alpha$ which is independent of ν since M is linearly ordered. Define $T'(p)$ by

$$(T'(p))_\alpha=q_\alpha \qquad \text{for all } \alpha\in B'.$$

A straightforward calculation using Lemma 1 shows that (B', T') is a member of our family and that $(B_\nu, T_\nu)<(B', T')$ for each $\nu\in M$.

By Zorn's lemma there exists a maximal member (B, T) of the family (which is clearly not vaccuous since there exist countable subsets of A invariant under ϕ by Lemma 2(a), and these belong to the family by Lemma 3). It is enough to prove that $B=A$ for then by condition (iii) above T induces ϕ. Suppose not, there exists $\alpha\in A-B$, and a countable set $D\subset A$, invariant under ϕ, such that $\alpha\in D$.

Let $B^* = B \cup D$, then B^* is also invariant under ϕ, and $\phi^* = \pi_{B^*} \phi \pi_{B^*}^{-1}$ is a set automorphism of $S^* = S(B^*)$. Since ϕ has the countable extension property, ϕ is induced by a point automorphism T^* extending T. But then (B^*, T^*) is a member of the family, $B \neq B^*$, $(B, T) < (B^*, T^*)$ contradicting the maximality of (B, T). This completes the proof of the lemma.

Note. This proof is virtually identical with that of the theorem in Maharam [5].

Lemmas 6 and 7 give us a good idea of what sort of conditions we need for a set automorphism ϕ of μ on S to be induced by a point automorphism. Lemma 7 tells us that ϕ need only have the countable extension property. Lemma 6 tells us that ϕ has this property if (and clearly only if) every point automorphism T' inducing ϕ on a partial product can be extended to *some* point automorphism (not necessarily inducing ϕ) for every larger partial product with only countably many additional factors. Let us try to analyze heuristically, when this can happen. As in Lemma 6, let B be invariant under ϕ, such that $C = A - B$ is countable, let ϕ' be the automorphism induced by ϕ on $E(B)$ and suppose ϕ' is induced by a point automorphism T'. Suppose now a disintegration (or decomposition) of μ over μ_B exists, i.e. for all (or μ_B-almost all) $p \in S(B)$, there exist probability measures μ_p on $S(A)$, such that $\operatorname{supp}(\mu_p) \subset \pi_{AB}^{-1}(p)$, such that for Y measurable in $S(A)$, $\mu_p(Y)$ is a μ_B-measurable function on $S(B)$ and such that

$$\mu(Y) = \int_{S(B)} \mu_p(Y)\, \mu_B(dp).$$

Then, for the existence of a point automorphism P of $S(A)$ extending T', it is clearly necessary that for almost all p, μ_p and $\mu_{T'p}$ are isomorphic under an isomorphism P_p and that

$$P(p, q) = (T' p, P_p q) \qquad \text{where } p \in S(B),\ q \in S(C).$$

Thus (at least if the required disintegration were true) some sort of homogeneity condition on the μ_p might be sufficient to ensure the extension. Unfortunately the truth of the required disintegration theorem is unknown except when B is countable. Our next lemma is essentially a device to enable us to assume this.

Lemma 8. *Let ϕ be a set automorphism of S. In order that ϕ have the countable extension property, it is sufficient that ϕ have this property for every countable invariant subset of A.*

Proof. Let B be invariant under ϕ, ϕ' the induced automorphism of $E(B)$, and suppose ϕ' is induced by a point automorphism T' of $S(B)$. It is clearly enough to assume that $C = A - B$ is countable and show that T' has an extension to a point automorphism T of $S(A)$ which induces ϕ. By Lemma 6, it is enough to show that T' has an extension to some point automorphism P on $S(A)$.

By Lemma 2(a), there exists, $C \subset \bar{C} \subset A$ where $\bar{C}$ is countable and invariant under ϕ. Let $D = \bar{C} - C = B \cap \bar{C}$. If $D \neq \emptyset$, then by Lemma 2(b), there exists a countable set F, $D \subset F \subset B$, such that F is invariant under T'. Thus for $Y \subset S(F)$, there exists $Y' \subset S(F)$ such that

$$T'\big(Y \times S(B - F)\big) = Y' \times S(B - F).$$

Apply this to a separating sequence of sets Y_n in $S(F)$; we see that for each $p \in S(F)$, there exists $Z \subset S(F)$ such that $T'(p \times S(B-F)) = Z \times S(B-F)$ and so $p \times S(B-F) = T'^{-1}(Z \times S(B-F))$. Since F is also invariant under T'^{-1} and both T' and T'^{-1} are (1-1) it follows that Z cannot contain more than one point p' and so

$$T'(p \times S(B-F)) = p' \times S(B-F).$$

Thus there exists a (1-1) map $\hat{T}$ on $S(F)$, which is obviously a point automorphism, such that

$$\pi_{BF} T' = \hat{T} \pi_{BF}.$$

By hypothesis $\hat{T}$ can be extended to a point automorphism $\hat{P}$ of $S(F \cup C)$. Put $G = F \cup C$. Since $F \supset D$, $G = F \cup \bar{C}$ and so G is invariant under ϕ. (F, being invariant under T' is certainly invariant under ϕ.) Let $\tilde{\phi}$ be the automorphism induced by ϕ on $E(G)$. By Lemma 6, $\tilde{\phi}$ can be induced by a point automorphism $\tilde{T}$ of $S(G)$ such that $\pi_{GF} \tilde{T} = \hat{T} \pi_{GF}$.

Define T on $S(A)$ as follows: Put

$$\begin{aligned} (T_p)_\alpha &= (\tilde{T}_p)_\alpha \qquad \text{if } \alpha \in G \\ &= (T' p)_\alpha \qquad \text{if } \alpha \in B. \end{aligned}$$

This is consistent, since for $\alpha \in F = G \cap B$,

$$(\tilde{T}_p)_\alpha = (T' p)_\alpha = (\hat{T}_p)_\alpha.$$

T is thus a (1-1) map of $S(A)$ onto itself, $\pi_{AB} T = T' \pi_{AB}$. Further $\{TX\} = \phi\{X\}$ if $X \in S^B$ or S^G. So as in the proof of Lemma 5, T induces ϕ.

If $D = \emptyset$, C is invariant under ϕ and so ϕ induces an automorphism ϕ_1 of $E(C)$ which by Lemma 3 is induced by a point automorphism T_1. Define T on $S(A)$ by

$$T(p, q) = (T' p, T_1 q)$$

where $p \in S(B)$, $q \in S(C)$; it is easily seen that T has the required properties.

We now state the disintegration (decomposition) theorem that we need.

Disintegration Theorem. *Suppose F, G countable sets, $F \subset G$, μ a probability measure on $S(G)$, $C = G - F$. Then there exist measures $\mu(p, Y)$ on $S(G)$ for each $p \in S(F)$, such that*

(i) $\operatorname{supp} \mu(p, \cdot) \subset \pi_{GF}^{-1}(p)$;

(ii) *for μ_F-almost all p, $\mu(p, \cdot)$ is a probability measure on $S(G)$;*

(iii) *for each $Y \subset S(G)$, $\mu(p, Y)$ is measurable on $S(F)$; and*

(iv) $\mu(Y) = \int_{S(F)} \mu(p, Y) \, \mu_F(dp)$.

$\mu(p, \cdot)$ may be regarded as a measure on $\pi_{GF}^{-1}(p)$, and since all such fibres are canonically isomorphic to $S(C)$, each $\mu(p, \cdot)$ may be regarded as a measure on $S(C)$. Let $C = \{\alpha_1, \alpha_2, \ldots\}$ be a fixed ordering of C. Let $C_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let q_{α_i} denote points of $S(\{\alpha_i\})$, p as usual points of $S(F)$. Let

$$\mu^{(n)}(p, q_{\alpha_1}, q_{\alpha_2}, \ldots, q_{\alpha_{n-1}}, \cdot)$$

denote the disintegration of $\mu_{F\cup C_n}$ over $\mu_{F\cup C_{n-1}}$, $\mu^{(1)}(p,\cdot)$ that of $\mu_{F\cup C_1}$ over μ_F. Then if $X\subset S(F\cup C_n)$

$$\mu(X\times S(G-(F\cup C_n)))=\underbrace{\int\mu_F(dp)\int\mu^{(1)}(p,dq_{\alpha_1})\int\cdots\int}_{X}\mu^{(n)}(p,q_{\alpha_1},\ldots,q_{\alpha_{n-1}},dq_{\alpha_n});$$

and if $Z\subset S(C_n)$,

$$\mu(p,Z\times S(C-C_n))=\underbrace{\int\mu^{(1)}(p,dq_{\alpha_1})\int\cdots\int}_{Z}\mu^{(n)}(p,q_{\alpha_1},\ldots,q_{\alpha_{n-1}},dq_{\alpha_n}).$$

The *general* disintegration theorem (for spaces which are not product spaces) can be found in [1], Theorem 1 or [2], Chapter IX, Theorem 5 (as well as many other places). The forms with repeated, product space integrals are most conveniently found in Loève [4], p. 360 et seq.

We next show that if $\mu(p,\cdot)$ and $\mu(T'p,\cdot)$ are isomorphic the desired extension always exists.

Lemma 9. *Suppose F, G are countable sets, $F\subset G$, $C=G-F$, μ a probability measure on $S(G)$, ϕ a set automorphism of μ such that F is invariant under ϕ, ϕ' the set automorphism of $S(F)$ induced by ϕ. If ϕ' is induced by a point automorphism T' of $S(F)$ such that, for μ_F-almost all $p\in S(F)$, $\mu(p,\cdot)$ and $\mu(T'p,\cdot)$ considered as measures on $\pi_{GF}^{-1}(p)$ and $\pi_{GF}^{-1}(T'p)$ respectively are point isomorphic, then T' can be extended to a point automorphism of $S(G)$ inducing ϕ.*

Proof. By Lemma 6 it is enough to prove that T' has some extension to a point automorphism of $S(G)$. Let N be the set of p in $S(F)$ for which $\mu(p,\cdot)$ and $\mu(T'p,\cdot)$ are not isomorphic or for which $\mu(p,\cdot)$ is not a probability measure. $\mu_F(N)=0$. Replacing N by $\bigcup_{n=-\infty}^{\infty}T'^nN$ we may assume that $T'N=N$. On N we may change $\mu(p,\cdot)$, putting $\mu(p,Y)=0$ for all $p\in N$ and all Y. $\mu(p,\cdot)$ is then point isomorphic to $\mu(T'p,\cdot)$ for all $p\in S(F)$. Let $\tilde{P}_p$ be a point isomorphism of $(\pi_{GF}^{-1}(p),\mu(p,\cdot))$ to $(\pi_{GF}^{-1}(T'p),\mu(T'p,\cdot))$ (P_p may be even assumed measure preserving, though this is unnecessary).

Since each $\pi_{GF}^{-1}(p)$ is canonically isometric to $S(C)$, each $\tilde{P}_p$ defines a point automorphism $\tilde{Q}_p$ of $S(C)$. If $p\notin N$, by Lemma 4, $\tilde{Q}_p$ may be chosen so that both $\tilde{Q}_p$ and $\tilde{Q}_p^{-1}$ are of Baire class 3 or analytically representable class Φ_3 at most. If $p\in N$, let $\tilde{Q}_pq=q$, $q\in S(C)$, which is a homeomorphism and so of class 0. On $S(G)$ define

$$P(p,q)=(T'p,\tilde{Q}_pq)\qquad\text{where } p\in S(F),\ q\in S(C).$$

P is bijective. If $\tilde{Q}_p$ were continuous for each p, then P would be measurable. [Note that if X, Y, Z are Polish spaces, and $f\colon X\times Y\to Z$ is such that for each $y\in Y$, f is Borel measurable on X, and for each $x\in X$, f is continuous on Y, then f is Borel measurable on $X\times Y$. See [9], Theorem 2, its proof and the remark following the theorem.] Since measurable maps into separable metric spaces are closed under pointwise sequential limits, it is still true that P is measurable if $\tilde{Q}_p$ is of class Φ_3. Clearly, P extends T'. To show that P is a point automorphism, it is enough to show that $\mu(Y)=0$ implies $\mu(PY)=0$. (The reverse implication follows by considering P^{-1} in place of P.) Now if $\mu(Y)=0$, it follows, since $\mu(Y)=\int_{S(F)}\mu(p,Y\cap\pi_{GF}^{-1}(p))\,\mu_F(dp)$

that $\mu(p, Y \cap \pi_{GF}^{-1}(p)) = 0$ for μ_F-almost all p. Now since $\tilde{P}_p$ is a point isomorphism, $\mu(p, Y \cap \pi_{GF}^{-1}(p)) = 0$ implies $\mu(T' p, PY \cap \pi_{GF}^{-1}(T' p)) = 0$; so this holds for μ_F-almost all p. Since T' is a point automorphism, it follows that $\mu(p, PY \cap \pi_{GF}^{-1}(p)) = 0$ for μ_F-almost all p, and so

$$\mu(PY) = \int_{S(F)} \mu(p, PY \cap \pi_{GF}^{-1}(p)) \, \mu_F(dp) = 0.$$

This completes the proof of the lemma.

Lemma 10. *Suppose F, G are countable sets, $F \subset G$, $C = G - F = \{\alpha_1, \alpha_2, \ldots\}$, $C_n = \{\alpha_1, \ldots, \alpha_n\}$. Let μ be a probability measure on $S(G)$, $\mu^{(1)}(p, \cdot)$ the disintegration of $\mu_{F \cup C_1}$ over μ_F, $\mu^{(n)}(p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}, \cdot)$ that of $\mu_{F \cup C_n}$ over $\mu_{F \cup C_{n-1}}$, $\mu(p, \cdot)$ that of $\mu = \mu_G$ over μ_F. If, for all $n \geqq 1$, $\mu^{(n)}(p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}, \cdot)$ and $\mu^{(n)}(p', q'_{\alpha_1}, \ldots, q'_{\alpha_{n-1}}, \cdot)$ are point isomorphic for every pair of points $(p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}})$ $(p', q'_{\alpha_1}, \ldots, q'_{\alpha_{n-1}})$ in $S(F \cup C_{n-1})$ (with $C_0 = \emptyset$), then $\mu(p, \cdot)$ is isomorphic to $\mu(p', \cdot)$ for every pair of points in $S(F)$.*

Proof. Since for each fixed pair $(p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}})$ and $(p', q'_{\alpha_1}, \ldots, q'_{\alpha_{n-1}})$, there is a measure preserving isomorphism of

$$\mu^{(n)}(p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}, \cdot) \quad \text{and} \quad \mu^{(n)}(p', q'_{\alpha_1}, \ldots, q'_{\alpha_{n-1}}, \cdot),$$

and these measures can be canonically regarded as being on $S(\{\alpha_n\})$, this isomorphism induces a measure preserving point isomorphism of

$$(S(\{\alpha_n\}), \mu^{(n)}(p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}, \cdot)) \quad \text{and} \quad (S(\{\alpha_n\}), \mu^{(n)}(p', q'_{\alpha_1}, \ldots, q'_{\alpha_{n-1}}, \cdot)).$$

Now let $p, p' \in S(F)$. There exists a measure preserving point isomorphism $Q_p^{(1)}$ of $\mu^{(1)}(p, \cdot)$ and $\mu^{(1)}(p', \cdot)$ on $S(\{\alpha_1\})$. Put $P_p^{(1)} q_{\alpha_1} = Q_p^{(1)} q_{\alpha_1}$, $T^{(1)}(p, q_{\alpha_1}) = (p', Q_p^{(1)} q_{\alpha_1})$. If $X \subset S(\{\alpha_1\})$, using the repeated integral formula for $\mu(p, \cdot)$ given by the disintegration we see that

$$\mu(p, X \times S(C - C_1)) = \mu(p', P_p^{(1)} X \times S(C - C_1)).$$

Assume that we have defined $P_p^{(n-1)}$ on $S(C_{n-1})$ so that $P_p^{(n-1)}$ is a measurable bijection which is an extension of $P_p^{(n-2)}$ and

$$\mu(p, X \times S(C - C_{n-1})) = \mu(p', P_p^{(n-1)} X \times S(C - C_{n-1}))$$

for $X \subset S(C_{n-1})$; there exists a measure preserving point isomorphism

$$Q^{(n)}_{p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}} \quad \text{of} \quad (S(\{\alpha_n\}), \mu^{(n)}(p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}, \cdot))$$

and

$$(S(\{\alpha_n\}), \mu^{(n)}(p', P_p^{(n-1)}(q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}), \cdot)).$$

Put

$$P_{(p)}^{n}(q_{\alpha_1}, \ldots, q_{\alpha_n}) = (P_p^{(n-1)}(q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}), Q^{(n)}_{p, q_{\alpha_1}, \ldots, q_{\alpha_{n-1}}} q_{\alpha_n}).$$

Then using the repeated integral formula

$$\mu(p, X \times S(C - C_n)) = \mu(p', P_p^{(n)} X \times S(C - C_n)) \qquad \text{for } X \subset S(C_n),$$

and $P_p^{(n)}$ is an extension of $P_p^{(n-1)}$. By this means we obtain a measurable bijection of $S(C)$, which is a measure preserving map of $\mu(p, \cdot)$ and $\mu(p', \cdot)$ for sets which are cylinders with bases in $S(C_n)$ for each n, and hence for all measurable sets in $S(C)$. $T(p,q)=(p', P_p q)$ where $q \in S(C)$, is the required, measure preserving isomorphism of fibres.

We are finally in a position to state and prove our main theorem.

Theorem. *Let* $S=\prod I_\alpha$, $\alpha \in A$. *Let* μ *be a probability measure on the product (Baire)* σ*-algebra of* S, *with the following property: For every countable* $F \subset A$, *and every* $\alpha \in A-F$, *let* $\mu^{(F,\alpha)}(p,\cdot)$ *denote the disintegration of* $\mu_{F \cup \{\alpha\}}$ *on* $S(F \cup \{\alpha\})$ *over* μ_F *on* $S(F)$. *Then for every pair,* $p, p' \in S(F)$, $\mu^{(F,\alpha)}(p,\cdot)$ *and* $\mu^{(F,\alpha)}(p',\cdot)$ *are point isomorphic.*

Then every set automorphism ϕ *of* μ *on* S *is induced by a point automorphism* T *of* S.

Proof. By Lemmas 10 and 9, ϕ has the countable extension property for every countable set F invariant under ϕ. By Lemma 8, ϕ has the countable extension property. The result then follows by Lemma 7.

Corollary. *If the finite dimensional marginal distributions of* μ *are defined by positive densities then every set automorphism* ϕ *of* μ *on* S *is induced by a point automorphism.*

Proof. In this case each $\mu^{(F,\alpha)}(p,\cdot)$ is equivalent to, and so point isomorphic to Lebesgue measure and the condition of the theorem is satisfied.

The corollary covers measures such as Wiener measure. Slight modifications cover many other measures of various stochastic processes.

We do not know whether the theorem is true with no additional hypothesis on the measure μ. Possibly the homogeneity condition on fibres must always be satisfied. It might be possible to show this using the much harder measure algebraic disintegration theorem of Maharam [6].

Added in Proof. The theorem *is* true without additional hypothesis on the measure. The proof will appear in a subsequent paper in this journal.

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(Received February 15, 1971)