Z. Wahrscheinlichkeitstheorie verw. Gebiete 31, 235–253 (1975)


# Multivariate Point Processes: Predictable Projection, Radon-Nikodym Derivatives, Representation of Martingales

Jean Jacod*

## Introduction

1. A point process over the half-line $]0, \infty[$ is a strictly increasing sequence $(T_n)_{n \geqq 1}$ of positive random variables defined on a measurable space $(\Omega, \mathscr{F})$. A *multivariate point process* (also called marked point process) is a point process $(T_n)$ for which a random variable $X_n$ is associated to each $T_n$. The variables $X_n$ take their values in a measurable space $(E, \mathscr{E})$ (the set of "marks"). With our definition, we may have a finite accumulation point $T_\infty = \lim T_n$, but no point after $T_\infty$. The reason for allowing $T_\infty < \infty$ will become apparent later.

A multivariate point process $(T_n, X_n)$ is completely characterized by the following discrete *random measure* on $]0, \infty[\times E$:

$$\mu(\omega; dt, dx) = \sum_{n \geqq 1} \varepsilon_{(T_n(\omega), X_n(\omega))}(dt, dx)\, 1_{\{T_n(\omega) < \infty\}}, \tag{$*$}$$

where $\varepsilon_a$ denotes the Dirac measure located at point $a$.

When $E$ consists of one point, the multivariate point process reduces to a point process. Our formulation also includes the "jump processes", as introduced by Boël, Varaiya and Wong [2]: if $(Z_t)$ is a jump process, $T_n$ is the $n$-th jump and $X_n = Z_{T_n}$.

2. Let $(\mathscr{F}_t)_{t \geqq 0}$ be an increasing and right-continuous family of sub-$\sigma$-algebras of $\mathscr{F}$ such that each $T_n$ is a stopping time and each $X_n$ is $\mathscr{F}_{T_n}$-measurable. Let $P$ be a probability measure on $(\Omega, \mathscr{F})$. In Theorem (2.1) we introduce our main tool: we prove the existence and uniqueness of a positive random measure $\nu(\omega; dt, dx)$ on $]0, \infty[\times E$ called the *predictable projection* of $\mu$ and such that (i) for each $B \in \mathscr{E}$ the process $(\nu(]0, t] \times B))_{t \geqq 0}$ is predictable (or natural: we recall the definition of these terms below), and (ii) for each $B \in \mathscr{E}$ and $n \geqq 1$,

$$\big(\nu(]0, t \wedge T_n] \times B) - \mu(]0, t \wedge T_n] \times B)\big)_{t \geqq 0}$$

is a martingale. Equivalently, these two conditions mean that for each $B$, $(\nu(]0, t] \times B))$ is the so-called "dual predictable projection" of $(\mu(]0, t] \times B))$ [7]. One can easily see that, in addition, $\nu(\{t\} \times E) \leqq 1$ for each $t$ and $\nu([T_\infty, \infty[\times E) = 0$.

When the multivariate point process is only a point process, its predictable projection is also known as its "stochastic intensity" (Brémaud [3, 4], Jacod [10]).

3. The original aim of this research was to consider the converse of the previous result. Namely, let a multivariate point process to which the random measure $\mu$

* This research was conducted in the Statistics Department of Princeton University, while the author held a U.S.-France exchange award from the National Science Foundation.

is associated by (*), and a predictable random measure $\nu$ satisfying $\nu(\{t\} \times E) \leqq 1$ and $\nu([T_\infty, \infty[ \times E) = 0$, be defined on $(\Omega, (\mathscr{F}_t), \mathscr{F})$. *Is it possible to construct one (and only one?) probability measure on* $(\Omega, \mathscr{F})$ *such that* $\nu$ *is the predictable projection of* $\mu$?

Clearly enough, without further assumption the answer is a negative one. However let the "minimal" family of $\sigma$-algebras be defined by

$$\mathscr{G}_t = \sigma\big(\mu(]0, s] \times B) : s \leqq t, B \in \mathscr{E}\big),$$

and let us assume the following:

(A.1) we have $\mathscr{F}_t = \mathscr{F}_0 \vee \mathscr{G}_t$ [1] for each $t$.

In view of the above problem, we have three main results:

a) *Uniqueness.* Two probability measures having the same restriction to $\mathscr{F}_0$ and leading to the same predictable projection for $\mu$ must coincide on $\mathscr{F}_\infty$.

b) *First Result of Existence.* Let us assume that $\Omega$ is the set of all possible multivariate point processes, that $\mu$ is the random measure associated by (*) to the canonical multivariate point process defined on $\Omega$, and that $\mathscr{F}_t = \mathscr{G}_t$. For any predictable random measure $\nu$ satisfying $\nu(\{t\} \times E) \leqq 1$ and $\nu([T_\infty, \infty[ \times E) = 0$ there exists one and only one probability measure on $(\Omega, \mathscr{F}_\infty)$ for which $\nu$ is the predictable projection of $\mu$. The same kind of result also holds under slightly more general assumptions, which are explicitely stated in Theorem (3.6).

c) *Second Result of Existence.* If $P$ is a probability for which the predictable projection $\nu$ is known, we give in Theorem (5.2) a necessary and sufficient condition for another predictable random measure $\nu'$ which satisfies $\nu' \ll \nu$ to be the predictable projection for another probability measure $P'$ satisfying $P' \ll P$.

4. In the course of proving (c) we get two results which are interesting by themselves.

d) *Radon-Nikodym Derivatives.* Let $P$ and $P'$ be two probability measures with $P' \ll P$. Then one can find a positive function $Y$ on $\Omega \times [0, \infty[ \times E$ such that if $\nu$ is the predictable projection of $\mu$ for $P$, then $\nu'(\omega; dt, dx) = \nu(\omega; dt, dx)\, Y(\omega, t, x)$ is the predictable projection of $\mu$ for $P'$ (this holds without (A.1)). Moreover one gives an explicit version for the Radon-Nikodym derivative $E\left(\frac{dP'}{dP}\middle|\mathscr{F}_t\right)$ in terms of $Y$ (Eq. (14) and Theorem (5.1)). These results are similar to the well-known results (of Girsanov's type) relative to the Wiener process. For an extensive discussion of possible uses of results in this direction, we refer to Kailath [11].

e) *Representation of Martingales.* Each martingale (or local martingale) $(Z_t)$ can be written as an integral

$$Z_t = Z_0 + \int_E \int_0^t X(s, x)[\nu(ds, dx) - \mu(ds, dx)],$$

where $X$ is a "predictable" process defined on $\Omega \times [0, \infty[ \times E$ (a more rigourous statement is given in Theorem (5.4)). This result is similar to the known representation of martingales as the stochastic integral of a predictable process with respect to a fundamental martingale, for Poisson and Wiener processes.

[1] $\mathscr{F}_0 \vee \mathscr{G}_t$ is the $\sigma$-algebra generated by $\mathscr{F}_0 \cup \mathscr{G}_t$.

5. Results c), d) and e) were already partially known, at least when $\mu$ is a Poisson random measure under $P$ (cf. Sections 4 and 5 for precise references), and e) was known for arbitrary point processes (Chou and Meyer [5]). But except in [5], the previous proofs were based upon the theory of stochastic integrals, which is rather sophisticated, and implies some unnecessary assumptions.

On the other hand, this paper is "elementary" in the sense that it does not ask for a difficult theory as a prerequisite. In particular each apparently stochastic integral is to be interpreted as an ordinary integral for each $\omega \in \Omega$. In addition our proofs are often shorter and more general than the previous proofs (except once more [5] for e)). However the simplicity of our approach depends strongly on the particular structure of $\mu$, and the stochastic integrals approach probably gives more insight. For example we refer to Van Schuppen and Wong [21] for the study of similar problems in a more general setting, and to Orey [14] for a general review of these problems.

## 1. Notations and Preliminaries

1. Let $(\Omega, \mathscr{F})$ be a measurable space on which is defined an increasing and right-continuous family $(\mathscr{F}_t)_{t \geq 0}$ of $\sigma$-algebras included in $\mathscr{F}$.

We make a frequent use of the books of Meyer [12] and Dellacherie [7] concerning predictable stopping times and processes. However these notions are usually defined on a probability space $(\Omega, \mathscr{F}, P)$, the family $(\mathscr{F}_t)$ being complete with respect to $P$. Here we are interested in constructing $P$. Therefore we need a definition for these notions which is independent of $P$, and which is as follows:

*Definition. A real-valued process* $(X_t)_{t \geq 0}$ is called *predictable* if the application $(\omega, t) \rightsquigarrow X_t(\omega)$ is measurable with respect to the $\sigma$-algebra $\mathscr{P}$ of $\Omega \times [0, \infty[$ generated by the applications $(\omega, t) \rightsquigarrow Y_t(\omega)$ which are $\mathscr{F}_t$-measurable in $\omega$ and left-continuous in $t$.

*A stopping time* $T$ is called *predictable* if the process $X_t = 1_{\{T \leq t\}}$ is predictable.

The following facts are well known: let $P$ be a probability measure on $(\Omega, \mathscr{F})$ and $(\bar{\mathscr{F}}_t)$ be the usual completion of $(\mathscr{F}_t)$ for $P$. Then a process is predictable in the sense of [7] (one could say $(\bar{\mathscr{F}}_t)$-predictable) if and only if it is equal to a predictable process in the above sense, except on a $P$-null set. A similar statement holds for predictable stopping times. Using these facts, the reader can verify that each result of Dellacherie [7] we use in the sequel applies for the above notion of predictability.

2. If $(A, \mathscr{A})$ is any measurable space, we denote by $b\mathscr{A}$ (resp. $\mathscr{A}^+$) the set of all real-valued $\mathscr{A}$-measurable functions on $A$, which are bounded (resp. non-negative). If $T$ is a non-negative random variable defined on $\Omega$, its *graph* is the set $[T] = \{(\omega, T(\omega)) : T(\omega) < \infty\}$. By *increasing process* we mean a process $(B_t)_{t \geq 0}$ whose paths are increasing, right-continuous, and satisfy $B_0 = 0$. For such a process we put $\Delta B_t = B_t - B_{t-}$ and $B_\infty = \lim_{t \uparrow \infty} B_t$. We recall that an increasing process is predictable if and only if it is *natural* in the sense of Meyer [12].

We consider a space $E$ which is Lusin (i.e.: $E$ is a Borel subset of a compact metric space), and an extra point $\Delta$. We put $E_\Delta = E \cup \{\Delta\}$, $\tilde{E} = ]0, \infty[ \times E$, $\tilde{E}_\Delta = E \cup \{(\infty, \Delta)\}$ and $\tilde{\Omega} = \Omega \times [0, \infty[ \times E$. Let $\mathscr{E}$ (resp. $\mathscr{E}_\Delta$, $\tilde{\mathscr{E}}$, $\tilde{\mathscr{E}}_\Delta$) denote the Borel $\sigma$-algebra of $E$ (resp. $E_\Delta$, $\tilde{E}$, $\tilde{E}_\Delta$), and $\tilde{\mathscr{P}} = \mathscr{P} \otimes \mathscr{E}$.

In this paper, a *random measure* always means a positive transition measure $\eta(\omega; dt, dx)$ from $(\Omega, \mathscr{F})$ over $(E, \mathscr{E})$.

*Definition.* A *random measure* $\eta$ is called *predictable* if, for each $X \in \tilde{\mathscr{P}}^+$, the process $(\eta X)_t$ defined by

$$(\eta X)_t(\omega) = \int_E \int_0^t X(\omega, s, x)\, \eta(\omega; ds, dx)$$

is predictable.

3. We suppose that a *multivariate point process* is given on $\Omega$. In other words we have a sequence $(T_n, X_n)_{n \geqq 1}$ of random variables with values in $(\tilde{E}_\Delta, \tilde{\mathscr{E}}_\Delta)$, such that:

(i) each $T_n$ is a stopping time, and $T_n \leqq T_{n+1}$,

(ii) each $X_n$ is $\mathscr{F}_{T_n}$-measurable,

(iii) if $T_n < \infty$, then $T_n < T_{n+1}$.

We put $T_0 = 0$, $T_\infty = \lim T_n$, $S_n = T_n - T_{n-1}$ on $\{T_{n-1} < \infty\}$ and $S_n = \infty$ on $\{T_{n-1} = \infty\}$. We define a random measure $\mu$ by

$$\mu(\omega; dt, dx) = \sum_{n \geqq 1} 1_{\{T_n(\omega) < \infty\}}\, \varepsilon_{(T_n(\omega), X_n(\omega))}(dt, dx).$$

Let $\mathscr{G}_t = \sigma(\mu(]0, s] \times B) : s \leqq t, B \in \mathscr{E})$. Clearly $\mathscr{G}_t \subset \mathscr{F}_t$, and the family $(\mathscr{G}_t)_{t \geqq 0}$ is increasing and right-continuous (the last property depends upon the fact that the sequence $(T_n)$ is strictly increasing). At last we shall need the assumption (A.1) as written in the introduction.

We shall also consider the counting process $N_t = \mu(]0, t] \times E)$. When $E$ reduces to one point, that is when the multivariate point process reduces to an ordinary point process on $]0, \infty[$, it is completely described by $N = (N_t)_{t \geqq 0}$ and we need not consider $\mu$ at all.

## 2. Existence of a Predictable Projection

In this section we assume that a probability measure $P$ on $(\Omega, \mathscr{F})$ is given. The next result (existence of a predictable projection) is stated in the most convenient form for our purposes. Other equivalent, and more usual, statements are given below.

(2.1) **Theorem.** *There exists one and only one (up to a modification on a P-null set) predictable random measure* $\nu$ *such that for each* $X \in \tilde{\mathscr{P}}^+$ *we have:*

$$E\Big(\int_{\tilde{E}} X(t, x)\, \mu(dt, dx)\Big) = E\Big(\int_{\tilde{E}} X(t, x)\, \nu(dt, dx)\Big). \tag{1}$$

This result is a corollary of the following very general lemma, of independent interest. For each random measure $\eta$ we define a positive measure $M_\eta$ on $(\tilde{\Omega}, \tilde{\mathscr{P}})$ by

$$M_\eta(X) = E\Big(\int_{\tilde{E}} X(t, x)\, \eta(dt, dx)\Big), \tag{2}$$

where $X \in \tilde{\mathscr{P}}^+$. Then we have:

(2.2) **Lemma.** *Let* $\eta$ *be a random measure such that the measure* $M_\eta$ *is* $\sigma$*-finite. There exists one and only one (up to a modification on a P-null set) predictable* *random measure $\eta'$ such that for each $X \in \tilde{\mathscr{P}}^+$ we have:*

$$E\Big(\int_{\tilde{E}} X(t,x)\,\eta(dt,dx)\Big) = E\Big(\int_{\tilde{E}} X(t,x)\,\eta'(dt,dx)\Big). \tag{3}$$

In other words, the predictable measure $\eta'$ is characterized by $M_\eta = M_{\eta'}$. The measure $\eta'$ is called the *predictable projection* of $\eta$ (one should say "dual predictable projection", but no confusion can arise here).

*Proof of* (2.2). a) Let us suppose that $M_\eta(\tilde{\Omega}) < \infty$. The formula $m(.) = M_\eta(., E)$ defines a positive finite measure on $(\Omega \times [0,\infty[, \mathscr{P})$ and there exists a transition probability $B(\omega, t; dx)$ from this space over $(E, \mathscr{E})$ such that

$$M_\eta(d\omega, dt, dx) = m(d\omega, dt)\, B(\omega, t; dx)$$

(the fact that $E$ is Lusin intervenes here).

According to Dellacherie [7] there exists a predictable increasing process $(A_t)$ such that for each $X \in \mathscr{P}^+$, $m(X) = E(\int X_t\, dA_t)$. If we put

$$\eta'(\omega; dt, dx) = A(\omega, dt)\, B(\omega, t; dx),$$

it is clear that $\eta'$ is a predictable random measure satisfying (3) (by a monotone class argument). But (3) also implies that for each $C \in \mathscr{E}$, the increasing process $\eta'(]0,t] \times C)$ is the "dual predictable projection" of the increasing process $\eta(]0,t] \times C)$, which is integrable. Therefore $\eta'(]0,t] \times C)$ is uniquely defined, up to a modification on a $P$-null set. From the separability of $\mathscr{E}$, the same sort of uniqueness holds for the random measure $\eta'$.

b) Let us deal now with the general case. We can find a $\tilde{\mathscr{P}}$-measurable partition $(\tilde{\Omega}_n)$ of $\tilde{\Omega}$ satisfying $M_\eta(\tilde{\Omega}_n) < \infty$ for each $n$. Set

$$\eta_n(\omega; dt, dx) = \eta(\omega; dt, dx)\, 1_{\tilde{\Omega}_n}(\omega, t, x).$$

From the first part, each $\eta_n$ admits a unique predictable projection $\eta'_n$, and we let the reader verify by himself that $\eta' = \sum_{(n)} \eta'_n$ is the unique predictable solution of (3).

*Proof of* (2.1). Each of the following subsets of $\tilde{\Omega}$ is in $\tilde{\mathscr{P}}$ and has a $M_\mu$-measure smaller or equal to 1: $[0] \times E$, $]T_n, T_{n+1}] \times E$, and $[T_\infty, \infty[ \times E$. Therefore (2.2) applies.

(2.3) **Proposition.** *One can choose a version of $\nu$ satisfying identically*

$$\begin{aligned} \nu(\{t\} \times E) &\leq 1 \\ \nu([T_\infty, \infty[ \times E) &= 0. \end{aligned} \tag{4}$$

*Proof.* Let $\nu$ be any version of the predictable projection of $\mu$. If $S$ is a predictable stopping time and if $B \in \mathscr{F}_{S-}$, we can apply (1) to $X(\omega, t, x) = 1_B(\omega)\, 1_{[S]}(\omega, t)$, which leads to

$$\nu(\{S\} \times E) = E(\Delta N_S | \mathscr{F}_{S-}) \quad \text{on } \{S < \infty\}. \tag{5}$$

Therefore if $B = \{(\omega, t): \nu(\omega; \{t\} \times E) > 1\}$ and if $S$ is any predictable stopping time with $[S] \subset B$, we have $P(S < \infty) = 0$. But $B \in \mathscr{P}$, so the section theorem for predictable sets [7] shows that $C = \{\omega: \exists t \text{ with } \nu(\omega; \{t\} \times E) > 1\}$ satisfies $P(C) = 0$.

Now (1) applied to $X(\omega, t, x)=1_{[T_\infty, \infty[}(\omega, t)$ implies that

$$D=\{\omega: \nu(\omega; [T_\infty, \infty[\times E)>0\}$$

satisfies $P(D)=0$. Then $\nu'(\omega; dt, dx)=1_{B^c \cap [0, T_\infty[}(\omega, t)\, \nu(\omega; dt, dx)$ is another version of the predictable projection of $\mu$, which satisfies (4).

*Equivalent Formulations to* (2.1). Using the definition of $\tilde{\mathscr{P}}$, the fact that $\mathscr{P}$ is generated by the stochastic intervals $[0, T]$ where $T$ is any stopping time, and the characterization of the dual predictable projection of an increasing process by Dellacherie [7], one may easily prove that:

(2.4) *The random measure* $\nu$ *is characterized by* (4) *and the fact that* $(\nu(]0, t] \times B))_{t \geqq 0}$ *is the dual predictable projection of* $(\mu(]0, t] \times B))_{t \geqq 0}$ *for each* $B \in \mathscr{E}$.

(2.5) *The random measure* $\nu$ *is characterized by* (4) *and*

(i) $(\nu(]0, t] \times B))_{t \geqq 0}$ *is predictable for each* $B \in \mathscr{E}$,

(ii) $E(\nu(]0, T] \times B))=E(\mu(]0, T] \times B))$ *for each* $B \in \mathscr{E}$ *and each stopping time* $T$.

(2.6) *The random measure* $\nu$ *is characterized by* (4) *and*

(i) $(\nu(]0, t] \times B))_{t \geqq 0}$ *is predictable for each* $B \in \mathscr{E}$,

(ii) $(\nu(]0, t \wedge T_n] \times B)-\mu(]0, t \wedge T_n] \times B))_{t \geqq 0}$ *is a uniformly integrable martingale for each* $n$ *and each* $B \in \mathscr{E}$.

When $P(T_\infty < \infty)=0$ one may also replace (ii) in (2.6) by:

$$(\nu(]0, t] \times B)-\mu(]0, t] \times B))_{t \geqq 0}$$

is a local martingale.

*Comments.* 1. One may give a slightly shorter proof of Theorem (2.1), without using Lemma (2.2). We have chosen this method because (2.2) is interesting on its own.

2. When $E$ reduces to one point, (2.2) as well as (2.1) and its equivalent formulations are in Dellacherie [7], or are trivial extensions of his results. $\nu$ is then completely characterized by the increasing process $A_t=\nu(]0, t] \times E)$, and (4) yields $\Delta A_t \leqq 1$ and $A_\infty = A_{(T_\infty)-}$.

3. For a general $E$, (2.6) has been shown by Boël, Varaiya and Wong [2] when each $T_n$ is totally inaccessible and $P(T_\infty < \infty)=0$. (2.2) was also already known in some cases. For example if $\eta$ is an integer-valued random measure such that $M_\eta$ is $\sigma$-finite, then $\eta$ is a Poisson measure on $E$ if and only if one can choose $\eta'(\omega; dt, dx) = dt\, F(dx)$ for some $\sigma$-finite measure $F$ on $E$ (Meyer [13]).

Also let $(Z_t)$ be a Hunt process, and $\eta$ the random measure associated to its jumps by

$$\eta(dt, dx)=\sum_{s>0} 1_{\{Z_{s-} \neq Z_s\}}\, \varepsilon_{(s, Z_s)}(dt, dx).$$

Then the existence of $\eta'$ is known under the name of "Lévy system" (S. Watanabe [22], Benveniste and Jacod [1]).

4. The next example shows why the results of the following paragraph are to be expected. Suppose $E$ is countable, and $(Z_t)$ is a "minimal" right-continuous Markov chain with values in $E_\Delta$. $(Z_t)$ is a pure jump process and we denote by $T_n$

its $n$-th jump, and $X_0=Z_0$, $X_n=Z_{T_n}$. The minimality of the chain means that $Z_t=\Delta$ if and only if $t\geqq T_\infty$. One knows that the probability law of this chain is entirely defined by a transition probability $Q(i,j)$ on $E$, and a positive function $\lambda(i)$ on $E$: $(X_n)_{n\geqq 0}$ is a Markov chain admitting $Q$ for its transition, and conditionally with respect to $\mathscr{F}_{T_n}$, $S_{n+1}$ is an exponential variable with mean $1/\lambda(X_n)$. Then the predictable projection of $\mu=\sum_{(n)}\varepsilon_{(T_n,X_n)}$ is

$$v(dt,i)=\sum_{n\geqq 0} 1_{\{T_n<t\leqq T_{n+1}\}}\,\lambda(X_n)\,dt\,Q(X_n,i).$$

## 3. An Explicit Form for the Predictable Projection – Two Applications

1. Let $P$ be a probability measure on $(\Omega,\mathscr{F})$ and $G_n(\omega;dt,dx)$ a regular version (which always exists) of the conditional law of $(S_{n+1},X_{n+1})$ with respect to $\mathscr{F}_{T_n}$. Let $H_n(\omega,dt)=G_n(\omega;dt,E_\Delta)$, which is the conditional law of $S_{n+1}$. For each $\omega$, $G_n(\omega;.)$ (resp. $H_n(\omega,.)$) is a probability measure on $\tilde{E}_\Delta$ (resp. $]0,\infty]$).

The following result has been proven for a point process by Papangelou [15] in the stationary case and by Jacod [10] in the general case (cf. also Dellacherie [6] when there is only one point $T_1$).

(3.1) **Proposition.** *Under* (A.1) *the following formula defines a version of the predictable projection of* $\mu$ (*which satisfies* (4)).

$$v(dt,dx)=\sum_{n\geqq 0}\frac{G_n(dt-T_n,dx)}{H_n([t-T_n,\infty])}\,1_{\{T_n<t\leqq T_{n+1}\}}. \tag{6}$$

We need parts of the following lemmas, in which we have put all the technical results we shall use in this paper, concerning the structure of stopping times and predictable processes.

(3.2) **Lemma.** *Assume* (A.1). *If $T$ is a stopping time, for each $n<\infty$* (resp. $n=\infty$) *there exists $R_n\in(\mathscr{F}_{T_n})^+$ such that $T\wedge T_{n+1}=(T_n+R_n)\wedge T_{n+1}$* (resp. $T=T_\infty+R_\infty$) *on* $\{T_n\leqq T\}$.

*Proof.* The proof is the same for $n$ finite or infinite, provided we read $T_{n+1}=\infty$ when $n=\infty$. For each $s>0$ we put $F_s=\{T_n+s<T_{n+1}\}$. The $\sigma$-algebra $\mathscr{F}_{(T_n+s)-}$ is generated by $\mathscr{F}_0$ and the sets $D=\{\mu(]0,r]\times B)=p\}\cap\{r<T_n+s\}$. Set

$$D'=\{\mu(]0,r]\times B)=p\}\cap\{r<T_n\}+\{\mu(]0,T_n]\times B)=p\}\cap\{T_n\leqq r<T_n+s\}.$$

We have $D'\in\mathscr{F}_{T_n}$ and $D\cap F_s=D'\cap F_s$. It follows easily that $\mathscr{F}_{(T_n+s)-}\cap F_s=\mathscr{F}_{T_n}\cap F_s$. But when $s\downarrow t$, then $\mathscr{F}_{(T_n+s)-}\downarrow\mathscr{F}_{T_n+t}$ and $F_s\uparrow F_t$, which implies that $\mathscr{F}_{T_n+t}\cap F_t=\mathscr{F}_{T_n}\cap F_t$. Therefore one can find $G_t\in\mathscr{F}_{T_n}$ such that $\{T<T_n+t\}\cap F_t=G_t\cap F_t$. One easily checks that the random variable $R_n$ defined by $\{R_n<t\}=\bigcup_{r\in\mathbb{Q},r<t}G_r$ answers the question.

(3.3) **Lemma.** *Under* (A.1) *a process $(X_t)$ is predictable if and only if $X_0$ is $\mathscr{F}_0$-measurable and if for each $n$ there exists a $\mathscr{F}_{T_n}$-measurable*[2] *process $(Y_t^n)$ which satisfies $Y_t^n=X_{T_n+t}$ on $\{0\leqq t\leqq S_{n+1}\}$* ($t\geqq 0$ *if* $n=\infty$).

[2] That is, such that $(\omega,t)\rightsquigarrow Y_t^n(\omega)$ is $\mathscr{F}_{T_n}\otimes\mathscr{R}_+$-measurable, where $\mathscr{R}_+$ is the Borel $\sigma$-algebra of $[0,\infty[$.

*Proof. Necessary Condition.* As $\mathscr{P}$ is generated by the sets $A \times \{0\}$ (with $A \in \mathscr{F}_0$) and the stochastic intervals $[0, T]$ (with any stopping time $T$), we need only to show that each process $X_t = 1_{\{t \leqq T\}}$ (where $T$ is a stopping time) satisfies the condition. $X_0$ is clearly $\mathscr{F}_0$-measurable. If $(R_n)$ is the family of variables associated to $T$ by (3.2), it is straightforward to check that the condition is met by the processes $Y_t^n = 1_{\{T_n \leqq T\} \cap \{t \leqq R_n\}}$.

*Sufficient Condition.* We recall the following fact (Dellacherie [7]): each set $A \times \{0\}$ (with $A \in \mathscr{F}_0$), $A \times ]T_n, T_{n+1}]$ (with $A \in \mathscr{F}_{T_n}$), $A \times [T_\infty, \infty[$ (with $A \in \mathscr{F}_{(T_\infty)-}$) is in $\mathscr{P}$, the last one because $T_\infty$ is predictable. As $\mathscr{F}_{(T_\infty)-} = \bigvee_{(n)} \mathscr{F}_{T_n} = \mathscr{F}_\infty$, for a process $(X_t)$ satisfying the above condition the result follows easily from:

$$\begin{aligned}\{(\omega, t)\colon X_t(\omega) > a\} = \{(\omega, 0)\colon X_0(\omega) > a\} \\ + \sum_{n \geqq 0} \{(\omega, t)\colon T_n(\omega) < t \leqq T_{n+1}(\omega), Y^n_{t - T_n(\omega)}(\omega) > a\} \\ + \{(\omega, t)\colon T_\infty(\omega) \leqq t, Y^\infty_{t - T_\infty(\omega)}(\omega) > a\}.\end{aligned}$$

*Proof of* (3.1). Let us define $\nu$ by (6). Clearly enough $\nu$ satisfies (4). From

$$\nu(]0, t] \times B) = \sum_{n \geqq 0} 1_{\{T_n < t \leqq T_{n+1}\}} \left( \sum_{p=0}^{n-1} \int_0^{S_{p+1}} \frac{G_p(ds, B)}{H_p([s, \infty])} + \int_0^{t - T_n} \frac{G_n(ds, B)}{H_n([s, \infty])} \right),$$

we see that for each $B \in \mathscr{E}$, $(\nu(]0, t] \times B))_{t \geqq 0}$ satisfies the condition of Lemma (3.3). Therefore $\nu$ is predictable. Using (2.5), we need only to prove that $E(\nu(]0, T] \times B)) = E(\mu(]0, T] \times B))$ for each stopping time $T$. Let $(R_n)$ be the family of variables associated to $T$ by (3.2). We have

$$\begin{aligned} &E(1_{\{T_n \leqq T\}} \nu(]T_n, T_{n+1} \wedge T] \times B)) \\ &= E\left(1_{\{T_n \leqq T\}} \int H_n(ds) \left(1_{\{R_n < s\}} \int \frac{G_n(du, B)}{H_n([u, \infty])} 1_{\{u \leqq R_n\}} \right.\right. \\ &\quad \left.\left. + 1_{\{s \leqq R_n\}} \int \frac{G_n(du, B)}{H_n([u, \infty])} 1_{\{u \leqq s\} \cap \{u < \infty\}} \right)\right) \\ &= E\left(1_{\{T_n \leqq T\}} \int \frac{G_n(du, B)}{H_n([u, \infty])} 1_{\{u \leqq R_n\} \cap \{u < \infty\}} (H_n(]R_n, \infty]) + H_n([u, R_n]))\right) \\ &= E(1_{\{T_n \leqq T\}} \int G_n(du, B) 1_{\{u \leqq R_n\} \cap \{u < \infty\}}) = E(1_{\{T_n \leqq T\}} \mu(]T_n, T_{n+1} \wedge T] \times B)). \end{aligned}$$

If we add these relationships for all $n \geqq 0$, we get the result.

2. *Uniqueness Theorem.* As a first application of (3.1) we give the following uniqueness result, which so far seems unknown, even for point processes (however, see Orey [14] for results in this direction).

(3.4) **Theorem.** *Assume* (A.1), *and let $P$ and $P'$ be two probability measures on* $(\Omega, \mathscr{F})$ *such that*

(i) *their restriction $P_0$ and $P'_0$ to $(\Omega, \mathscr{F}_0)$ are identical,*

(ii) *$\mu$ admits the same predictable projection for $P$ and $P'$. Then $P$ and $P'$ coincide on* $(\Omega, \mathscr{F}_\infty)$.

Here again we need a lemma, whose proof is given for example in [10]. We denote by $\mathscr{A}$ the class of all (deterministic) right-continuous increasing functions $F$ such that $F(0)=0$, $\Delta F_t = F(t)-F(t-)\leqq 1$ identically, and $F(s)=F(t)$ for each $s\geqq t$ when $\Delta F_t = 1$.

(3.5) **Lemma.** *The formulas*

$$H(]0,t])=1-e^{-F(t)}\prod_{s\leqq t}((1-\Delta F_s)\,e^{\Delta F_s}), \tag{7}$$

$$F(t)=\int_0^t \frac{H(ds)}{H([s,\infty])}, \tag{8}$$

*define a bijective correspondance between the class $\mathscr{A}$, and the class of all probability measures on* $]0,\infty]$.

*Proof of* (3.4). If $P$ and $P'$ do not coincide on $\mathscr{F}_\infty$ there exists $n\geqq 0$ such that $P$ and $P'$ coincide on $\mathscr{F}_{T_n}$, and not on $\mathscr{F}_{T_{n+1}}$. Let $G_n$ (resp. $G'_n$) be the conditional law of $(S_{n+1}, X_{n+1})$ with respect to $\mathscr{F}_{T_n}$, for $P$ (resp. $P'$) and $H_n(.)=G_n(.,E_\Delta)$ (resp. $H'_n(.)=G'_n(.,E_\Delta)$). Let $\nu$ be the common predictable projection of $\mu$. Using Lemma (3.3) we see that for each $B\in\mathscr{E}$ there exists a $\mathscr{F}_{T_n}$-measurable increasing process $(C_t(B))$ such that $\nu(]T_n, T_n+t]\times B)=C_t(B)$ for $t\leqq S_{n+1}$. Set

$$D_t(B)=\int_0^t \frac{G_n(ds,B)}{H_n([s,\infty])}, \qquad D'_t(B)=\int_0^t \frac{G'_n(ds,B)}{H'_n([s,\infty])}. \tag{9}$$

If $\mathscr{E}_1$ is a countable algebra which generate $\mathscr{E}$, we put $R=\inf(t: D_t(B)\neq C_t(B)$ for some $B\in\mathscr{E}_1)$, $S=\inf(t: H_n([t,\infty])=0)$, and we define $R'$ and $S'$ in the same way, using $(D'_t(B))$ and $H'_n$.

It follows from Proposition (3.1) that on $\{T_n<\infty, t\leqq S_{n+1}\}$ we have a.s. $D_t(B)=\nu(]T_n, T_n+t]\times B)$. Therefore $P(T_n<\infty, R<S_{n+1})=0$. But $R$ is $\mathscr{F}_{T_n}$-measurable, thus by definition of $H_n$,

$$E(1_{\{T_n<\infty\}}\,H_n(]R,\infty]))=P(T_n<\infty, R<S_{n+1})=0,$$

which yields $P(R<S, T_n<\infty)=0$. In the same way, $P'(R'<S', T_n<\infty)=0$. But $R$, $R'$, $S$, $S'$ and $T_n$ are $\mathscr{F}_{T_n}$-measurable, and $P$ and $P'$ coincide on $\mathscr{F}_{T_n}$. Therefore if $\Omega_n=\{T_n<\infty, S\leqq R, S'\leqq R'\}$ and $\Omega'_n=\{T_n<\infty\}$, we have $P(\Omega'_n-\Omega_n)=P'(\Omega'_n-\Omega_n)=0$.

Let $\omega\in\Omega_n$. We have $D_t(E_\Delta)=D'_t(E_\Delta)$ for each $t<S\wedge S'$ (because $S\leqq R$ *and* $S'\leqq R'$). But if $S<\infty$, either $D_{S-}(E_\Delta)=\infty$, or $\Delta D_S(E_\Delta)=1$ and $D_t(E_\Delta)=D_S(E_\Delta)$ for $t\geqq S$. The same holds for $D'_t(E_\Delta)$, and it follows easily that the two processes $(D_t(E_\Delta))$ and $(D'_t(E_\Delta))$ are identical, which implies $H_n(\omega,.)=H'_n(\omega,.)$ from Lemma (3.5). The same reasonning proves that $(D_t(B))$ and $(D'_t(B))$ are identical for each $B\in\mathscr{E}_1$, and from (9) we deduce that $G_n(\omega;.,B)=G'_n(\omega;.,B)$. In other words, $G_n$ and $G'_n$ coincide on $\Omega_n$. As $\mathscr{F}_{T_{n+1}}=\mathscr{F}_{T_n}\vee\sigma(S_{n+1}, X_{n+1})$, $P$ and $P'$ coincide on $\mathscr{F}_{T_{n+1}}\cap\Omega_n$. By hypothesis they also coincide on $\mathscr{F}_{T_{n+1}}\cap\{T_n=\infty\}=\mathscr{F}_{T_n}\cap\{T_n=\infty\}$, and they do not charge $\Omega'_n-\Omega_n$. Therefore they are identical on $\mathscr{F}_{T_{n+1}}$, which brings a contradiction.

3. *Construction of P when ν is known.* Here we come to the main concern of this paper: the converse to Theorem (2.1). Actually, a glance to (2.3) and (3.4)

leads us to ask the following question: let $P_0$ be a probability measure on $(\Omega, \mathscr{F}_0)$, and $\nu$ be a predictable random measure satisfying (4). Can we find a (unique) measure $P$ on $(\Omega, \mathscr{F}_\infty)$ whose restriction to $\mathscr{F}_0$ is $P_0$, and for which $\nu$ is the predictable projection of $\mu$?

Undoubtedly we cannot answer this question in the general case, because $\nu$ brings some information on $\mu$ only, that is the $\sigma$-algebras $\mathscr{G}_t$. Under (A.1) we already know that if $P$ exists, it is unique, but $P$ does not need to exist (for example suppose that $\mathscr{G}_t \subset \mathscr{F}_0$ for each $t$. Then for any probability measure, the predictable projection of $\mu$ is $\mu$ itself, and any other random measure $\nu$ will bring a negative answer to the above question).

However we can answer this question under a stronger assumption, which is as follows. Let $\Omega'$ be the set of all possible multivariate point processes, that is of all sequences $(T'_n, X'_n)_{n \geq 1}$ satisfying conditions (i)–(iii) of Section 1–3.

(A.2). $(\Omega'', \mathscr{F}'')$ is an arbitrary measurable space, $\Omega = \Omega' \times \Omega''$, the multivariate point process is $(T_n, X_n)(\omega', \omega'') = (T'_n, X'_n)(\omega')$ where $\omega = (\omega', \omega'') \in \Omega$, and the $\sigma$-algebras are $\mathscr{F}_0 = \{\emptyset, \Omega'\} \otimes \mathscr{F}''$ and $\mathscr{F}_t = \mathscr{F}_0 \vee \mathscr{G}_t$.

In particular this includes the case of "self-exciting" multivariate point processes, when $\Omega''$ reduces to one point (in other words, it suffices to consider $\Omega = \Omega'$). The space $(\Omega'', \mathscr{F}'')$ may also have the signification of the "past" before $t = 0$.

(3.6) **Theorem.** *Assume* (A.2). *Let $P_0$ be a probability measure on $(\Omega, \mathscr{F}_0)$ and $\nu$ a predictable random measure satisfying* (4). *Then there exists a unique probability measure $P$ on $(\Omega, \mathscr{F}_\infty)$ whose restriction to $\mathscr{F}_0$ is $P_0$, and for which $\nu$ is the predictable projection of $\mu$.*

*Proof.* Using Lemma (3.3) we see that for each $n$ and $B \in \mathscr{E}$ there exists a $\mathscr{F}_{T_n}$-measurable increasing process $(C_t^{B,n})$ such that $C_t^{B,n} = \nu(]T_n, T_n + t] \times B)$ on $\{T_n + t \leq T_{n+1}\}$. Due to the particular structure of $\Omega$ we see that $C_t^{B,n}$ is uniquely determined for all $t \geq 0$, provided we put $C_t^{B,n} = 0$ if $T_n = \infty$. We see also that each process $(C_t^{B,n})$ is "$\sigma$-additive" in $B$. Therefore if $C_t^n = C_t^{E,n}$, there exists a transition probability $\hat{C}^n(\omega, t; dx)$ from $(\Omega \times [0, \infty[, \mathscr{F}_{T_n} \otimes \mathscr{R}_+)$ over $(E, \mathscr{E})$ for which we have

$$C_t^{B,n}(\omega) = \int_0^t \hat{C}^n(\omega, s; B)\, dC_s^n(\omega).$$

Put $\alpha_n = \inf(t: \Delta C_t^n = 1)$ and $F_t^n = C_{t \wedge \alpha_n}^n$. For each $n$ and $\omega$, $(F_t^n(\omega))_{t \geq 0}$ belongs to the class $\mathscr{A}$ and from Lemma (3.5) there exists a probability measure $H_n(\omega, .)$ on $]0, \infty]$ satisfying (7) and (8). Actually (7) shows that $H_n$ is a transition from $(\Omega, \mathscr{F}_{T_n})$ over $]0, \infty]$ (because $(F_t^n)$ is $\mathscr{F}_{T_n}$-measurable). Let $G_n$ be the following transition probability from $(\Omega, \mathscr{F}_{T_n})$ over $(E_\Delta, \mathscr{E}_\Delta)$:

$$G_n(\omega; dt, dx) = H_n(\omega, dt)\, \hat{C}^n(\omega, t; dx)\, 1_{\{t < \infty\}} + H_n(\omega, \{\infty\})\, \varepsilon_{(\infty, \Delta)}(dt, dx).$$

Next we construct $P$. Let $P_n$ be a probability on $(\Omega, \mathscr{F}_{T_n})$. Using once more the structure of $\Omega$, we see that the next formula (where $B \in \mathscr{F}_{T_n}$ and $C \in \mathscr{E}_\Delta$) defines a probability $P_{n+1}$ on $(\Omega, \mathscr{F}_{T_{n+1}})$:

$$P_{n+1}(B \cap \{(S_{n+1}, X_{n+1}) \in C\}) = \int P_n(d\omega)\, 1_B(\omega)\, G_n(\omega; C). \tag{10}$$

Starting from $P_0$, which is given, we define $P_n$ by recurrence, using (10). Clearly the restriction of $P_{n+1}$ to $\mathscr{F}_{T_n}$ is $P_n$. But $(\Omega, \mathscr{F}_\infty)$ (resp. $(\Omega, \mathscr{F}_{T_n})$) is isomorphic to

$(E_\Delta^{\mathbb{N}} \times \Omega'', \mathscr{E}_\Delta^{\mathbb{N}\otimes} \otimes \mathscr{F}'')$ (resp. $(E_\Delta^{\{1,\dots,n\}} \times \Omega'', \mathscr{E}_\Delta^{\{1,\dots,n\}\otimes} \otimes \mathscr{F}'')$). So the family $(P_n)$ can be extended in one and only one way to give a probability $P$ on $(\Omega, \mathscr{F}_\infty)$.

We have now to prove that $P$ solves our problem. By construction the restriction of $P$ to $\mathscr{F}_0$ is $P_0$, and $G_n$ is the conditional law of $(S_{n+1}, X_{n+1})$ with respect to $\mathscr{F}_{T_n}$. Using (6), (8) and the definitions of $G_n$, $\alpha_n$ and $(C_t^{B,n})$, we see that a version $\nu'$ of the predictable projection of $\mu$ is given by:

$$\begin{aligned}\nu'(dt, dx) &= \sum_{n \geq 0} F^n(dt - T_n)\, \hat{C}^n(t - T_n, dx)\, 1_{\{T_n < t \leqq T_{n+1}\}}.\\ &= \sum_{n \geq 0} \nu(dt, dx)\, 1_{\{T_n < t \leqq T_{n+1} \wedge (T_n + \alpha_n)\}}.\end{aligned}$$

But from (7) we have $H_n(]\alpha_n, \infty])=0$ if $\alpha_n < \infty$, which yields $P(\alpha_n < S_{n+1})=0$. In other words $\nu$ and $\nu'$ are identical except on a $P$-null set, and $\nu$ is also a version of the predictable projection of $\mu$ for $P$. At last uniqueness for $P$ has already been shown.

*Remark.* If we impose $T_\infty = \infty$ a-priori, we have nothing like the previous result. For example let us suppose that $E$ reduces to one point. If $T_\infty = \infty$ everywhere, for any measure $P$ the predictable projection of $N$ is an increasing process $A = (A_t)$ with $\Delta A_t \leqq 1$ and $A_t < \infty$ a.s. But let us define the following process by recurrence: $A_0 = 0$, $A_{T_n + t} = A_{T_n} + n^2 t$ on $\{t \leqq S_{n+1}\}$. On the space $\Omega_1'$ of all point processes with $T_\infty = \infty$, $A$ is a continuous increasing process with $A_t < \infty$ everywhere. However it is impossible to find a probability $P_1$ on $\Omega_1'$ for which $A$ is the predictable projection of $N$. The previous result shows the existence of a unique probability $P$ on $\Omega'$ which answers the question, but we have $P(\Omega_1') = 0$ (in fact for $P$ we have $E(T_\infty) = \sum 1/n^2 < \infty$).

## 4. Absolute Continuity of Predictable Projections

In this section we suppose that $P$ is a probability measure on $(\Omega, \mathscr{F})$ and that $\nu$ is a version of the predictable projection of $\mu$ which satisfies (4).

The next theorem asserts that if $P'$ is another probability measure on $(\Omega, \mathscr{F})$ which is absolutely continuous with respect to $P$ (we write $P' \ll P$), then one can choose a version $\nu'$ of the predictable projection of $\mu$ for $P'$ such that $\nu' \ll \nu$ (that is, $\nu'(\omega; .) \ll \nu(\omega; .)$ for $P$-almost every $\omega$). However let us remark that we may change $\nu'$ on a $P'$-null set, and thus find another version of $\nu'$ which does not necessarily satisfies $\nu' \ll \nu$, in case $P \ll P'$ does not hold.

(4.1) **Theorem.** *Let $P'$ be a probability measure on $(\Omega, \mathscr{F})$ satisfying $P' \ll P$. There exists a finite $Y \in \tilde{\mathscr{P}}^+$ such that*

(i) *$\nu' = Y \cdot \nu$ [3] is a version of the predictable projection of $\mu$ for $P'$, and*

(ii) *$\nu'(\{t\} \times E) \leqq 1$ and $\nu'(\{t\} \times E) = 1$ whenever $\nu(\{t\} \times E) = 1$.*

As for (2.1), this theorem is a particular case of a more general result, which is interesting in itself. For each random measure $\eta$, (2) defines two measures $M_\eta$ and $M'_\eta$ on $(\tilde{\Omega}, \tilde{\mathscr{P}})$, corresponding to $P$ and $P'$.

[3] $Y \cdot \nu$ stands for the random measure $Y(\omega, t, x)\, \nu(\omega; dt, dx)$.

(4.2) **Lemma.** *Let us suppose that $P' \ll P$. Let $\eta$ be a random measure such that $M_\eta$ is $\sigma$-finite. If $\rho$ is the predictable projection of $\eta$ for $P$, let us assume that $M'_\rho$ and $M'_\eta$ are $\sigma$-finite. Then there exists a finite $Y \in \tilde{\mathscr{P}}^+$ such that $Y \cdot \rho$ is a version of the predictable projection of $\eta$ for $P'$* (in other words, $M'_\eta \ll M'_\rho$ and $Y$ is a version of the Radon-Nikodym derivative $dM'_\eta/dM'_\rho$).

*Proof.* Let $Z$ be a version of the Radon-Nikodym derivative $dP'/dP$, and $(Z_t)$ be a version of the martingale $E(Z|\mathscr{F}_t)$ which is right-continuous and has left-hand limits. As $M'_\eta$ is $\sigma$-finite, $\eta$ admits a predictable projection $\rho'$ for $P'$. Let $X \in \tilde{\mathscr{P}}^+$ be such that $M_\eta(X)$ and $M'_\eta(X)$ are finite. Then the three processes $(\eta(X 1_{\{Z_- > 0\}}))_t$, $(\rho'(X 1_{\{Z_- = 0\}}))_t$ and $(\rho X)_t$ are integrable increasing processes, the two last ones being predictable. Therefore we can apply the following result [7, IV-T-47 and V-T-27] to these processes: if $(B_t)$ is an increasing process, $E(ZB_\infty) = E(\int Z_t \, dB_t)$, and $E(ZB_\infty) = E(\int Z_{t-} \, dB_t)$ when $(B_t)$ is predictable. Using $M'_\eta = M'_{\rho'}$ and $M_\eta = M_\rho$, we obtain:

$$\begin{aligned} M'_\eta(X 1_{\{Z_- = 0\}}) &= E'\Big(\int_E X(t,x)\, 1_{\{Z_{t-} = 0\}}\, \rho'(dt,dx)\Big) \\ &= E\Big(\int_E X(t,x)\, 1_{\{Z_{t-} = 0\}}\, Z_{t-}\, \rho'(dt,dx)\Big) = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} M'_\eta(X) &= E'\Big(\int_E X(t,x)\, 1_{\{Z_{t-} > 0\}}\, \eta(dt,dx)\Big) \\ &= E\Big(\int_E X(t,x)\, 1_{\{Z_{t-} > 0\}}\, Z_t\, \eta(dt,dx)\Big), \end{aligned} \tag{12}$$

$$M'_\rho(X) = M_\rho(X Z_-) = E\Big(\int_E X(t,x)\, Z_{t-}\, \eta(dt,dx)\Big), \tag{13}$$

(11) being used to get (12). From (12) and (13) we deduce that $M'_\eta \ll M'_\rho$, and any finite $Y \in \tilde{\mathscr{P}}^+$ such that $M'_\eta = Y \cdot M'_\rho$ answers the question.

*Proof of* (4.1). (i) We have already seen that $M_\mu$ and $M'_\mu$ are $\sigma$-finite. Let $S_n = \inf(t: \nu(]0,t] \times E) \geqq n)$. We have $\nu(]0, S_n] \times E) \leqq n+1$, and $\lim S_n \geqq T_\infty$ a.s. Therefore $M'_\nu$ is finite on each of the following $\tilde{\mathscr{P}}$-measurable sets: $[0] \times E$, $]S_n, S_{n+1}] \times E$ and $[S_\infty, \infty[ \times E$ (with $S_\infty = \lim S_n$). Then (i) follows from Lemma (4.2), and we put $\nu' = Y \cdot \nu$.

(ii) To simplify the notations, we put $\alpha_t = \nu(\{t\} \times E)$ and $\alpha'_t = \nu'(\{t\} \times E)$. Let $Y'(\omega,t,x) = Y(\omega,t,x)$ if $\alpha_t < 1$ and $\alpha'_t \leqq 1$, or if $\alpha_t = \alpha'_t = 1$ and $Y'(\omega,t,x) = 1$ if not. If we show that $Y'$ differs from $Y$ only on a $P'$-null set, then $Y'$ will satisfy (i) as well as (ii). For this it is sufficient to prove that, if $S = \inf(t: \alpha_t = 1 \neq \alpha'_t)$ and $S' = \inf(t: \alpha'_t > 1)$, then $P'(S \wedge S' < \infty) = 0$. But (2.3) applied to $P'$ implies $P'(S' < \infty) = 0$. The stopping time $S$ is predictable, and from (5) we get $\alpha_S = E(\Delta N_S | \mathscr{F}_{S-}) = 1$ on $\{S < \infty\}$, thus $\Delta N_S = 1$ $P$-a.s. on $\{S < \infty\}$. From (5) again, $\alpha'_S = E'(\Delta N_S | \mathscr{F}_{S-}) = 1$ $P'$-a.s. on $\{S < \infty\}$, which implies $P'(S < \infty) = 0$.

*Remark.* One could give a slightly more explicit form for $Y$. Namely one can show (a) that there exists a finite $Z' \in \tilde{\mathscr{P}}^+$ such that $Z' \cdot \nu$ is the predictable projection of the random measure $\eta(dt,dx) = Z_t\, \mu(dt,dx)$, and (b) that $Y = 1_{\{Z_- > 0\}} \dfrac{Z'}{Z_-}$ satisfies (4.1)(i).

We turn now to the converse of (4.1), namely the possibility of finding a $P' \ll P$ for which the predictable projection of $\mu$ is $Y \cdot \nu$, where $Y \in \tilde{\mathscr{P}}^+$ is given. If we can solve this problem, we shall have another method for constructing a probability when the predictable projection of $\mu$ is known, under some conditions hopefully more general than (A.2).

For notational simplicity, we put $\alpha_t = \nu(\{t\} \times E)$ as above. Let also

$$S = \inf\big(t: t > T_\infty, \text{ or } \nu(]0, t] \times E) = \infty\big).$$

If $Y \in \tilde{\mathscr{P}}^+$, we define $\hat{Y}$ by $\hat{Y}_t = \int_E Y(t, x)\, \nu(\{t\}, dx)$. At last the "continuous part" of $\nu$ is defined by $\nu^c(dt, dx) = 1_{\{\alpha_t = 0\}}\, \nu(dt, dx)$: in fact, $\nu^c(]0, t] \times E)$ is exactly the continuous part of the increasing process $\nu(]0, t] \times E)$.

The next result is crucial. It is well known when $\mu$ is a Poisson measure for $P$, and the proof given here partly borrows from Brémaud [3]. There is also some similitude with results of Ito and Watanabe [8].

(4.3) **Proposition.** *Let $Z_0 \in \mathscr{F}_0^+$ be finite and such that $E(Z_0) = 1$. Let $Y \in \tilde{\mathscr{P}}^+$ be finite and such that $\hat{Y} \leqq 1$, and $\hat{Y}_t = 1$ wherever $\alpha_t = 1$. Then*

$$Z_t = \begin{cases} Z_0\Big(\prod_{T_n \leqq t} Y(T_n, X_n)\Big)\Big(\prod_{s \leqq t,\, s \neq T_n} \Big(1 + \dfrac{\alpha_s - \hat{Y}_s}{1 - \alpha_s}\Big)\Big) & \\ \qquad \cdot \exp \int_0^t \int_E (1 - Y(s, x))\, \nu^c(ds, dx) & \text{if } t < S, \\ \liminf_{s \uparrow S} Z_s & \text{if } S \leqq t \end{cases} \tag{14}$$

*is a right-continuous supermartingale $(Z_t)_{0 \leqq t \leqq \infty}$.*

In (14) we make the usual convention $0/0 = 0$. We shall see below that this formula makes sense. The assumptions made on $Y$ are the conditions (4.1)(ii).

(4.4) **Lemma.** *If $(B_t)$ is a predictable increasing process and if $S_\infty = \inf(t: B_t = \infty)$, there exists a sequence $(S_n)$ of stopping times increasing* a.s. *towards $S_\infty$ and such that $E(B_{S_n}) < \infty$ for each $n$.*

*Proof.* Put $R_n = \inf(t: B_t \geqq n)$. The set $\{(\omega, t): n \leqq B_t(\omega), t \leqq R_n(\omega)\}$ is predictable, and it is the graph of a random variable $V_n$. Thus $V_n$ is a predictable stopping time [6, IV-T-15]. There exists a stopping time $V'_n$ such that $V'_n < V_n$ and that $P(V_n - V'_n \leqq 2^{-n}) \leqq 2^{-n}$. Put $S'_n = R_n \wedge V'_n$ and $S_n = S'_1 \vee \cdots \vee S'_n$. We have $S'_n \leqq S_n \leqq R_n$ and $P(R_n - S'_n \leqq 2^{-n}) \leqq 2^{-n}$, therefore $S_n$ increases a.s. towards $S_\infty$. Also $B_{S'_n} \leqq n$, thus $B_{S_n} \leqq n$ and the result follows.

*Proof of* (4.3). a) Let $\nu' = Y \cdot \nu$, $X_s = \dfrac{1}{1 - \alpha_s} 1_{\{\alpha_s < 1\}}$, $T'_n = \inf(t: t \geqq T_n$, or $\nu(]0, t] \times E) \geqq n$, or $\nu'(]0, t] \times E) \geqq n)$, and $T'_\infty = \lim T'_n$, which satisfies $T'_\infty \leqq S$. Using the assumption that $\hat{Y}_t = 1$ if $\alpha_t = 1$, for each $t < S$ we have clearly

$$Z_t \leqq Z_0\Big(\prod_{T_n \leqq t} Y(T_n, X_n)\Big)\Big(\prod_{s \leqq t} X_s\Big) \exp\big(\nu^c(]0, t] \times E)\big).$$

But $\prod_{s \leqq t} X_s$ is increasing in $t$ and finite for $t < S$ (this comes from an easy computation). Then it follows that:

(i) $Z_t$ is well defined (there is no undetermined form in the product (14)). Clearly $(Z_t)$ is right-continuous, and each $Z_t$ is $\mathscr{F}_t$-measurable.

(ii) If $R_n = \inf(t: Z_t \geqq n)$, then $\lim R_n \geqq S$.

On the other hand let $T'_\infty \leqq t < S$. Then either $\nu'^c(]0, t] \times E) = \infty$ and

$$\exp \int_0^t \int_E (1 - Y(s, x))\, \nu^c(ds, dx) = 0,$$

or $\sum_{s \leqq t} \hat{Y}_s = \infty$, which implies that

$$\prod_{s \leqq t, s \neq T_n} \left(1 + \frac{\alpha_s - \hat{Y}_s}{1 - \alpha_s}\right) = 0.$$

Thus in both cases we have $Z_{t-} = Z_t = 0$. Therefore for each $t \geqq T'_\infty$, we have $Z_t = \liminf_{s \uparrow T'_\infty} Z_s$. Now, using the fact that $Z_t$ is non-negative, Fatou's lemma, and the above fact, we see that the proposition will be proven if we exhibit a sequence $(S_n)$ of stopping times increasing a.s. towards $T'_\infty$, and such that $(Z_{t \wedge S_n})_{t \geqq 0}$ is a uniformly integrable martingale for each $n$.

b) Let us recall an elementary fact. If $a_t$ (resp. $b_t$) is a continuous (resp. purely discontinuous) function from $[0, s]$ into $\mathbb{R}$, with bounded variations, and if $c_t = c_0 (\prod_{r \leqq t} (1 + \Delta b_r)) \exp(a_t - a_0)$ on $[0, s]$, then

$$c_t = c_0 + \int_0^t c_{r-}(da_r + db_r).$$

Let us fix $\omega \in \Omega$ and $s < T'_\infty(\omega)$. We can use this result with $c_t = Z_t$ and

$$a_t = \int_0^t \int_E (1 - Y(r, x))\, \nu^c(dr, dx),$$

$$b_t = \sum_{T_n \leqq t} (Y(T_n, X_n) - 1) + \sum_{r \leqq t, r \neq T_n} \frac{\alpha_r - \hat{Y}_r}{1 - \alpha_r}.$$

(We have $s < T_\infty$, $\nu(]0, s] \times E) < \infty$ and $\nu'(]0, s] \times E) < \infty$, which imply that $a_t$ and $b_t$ have bounded variations on $[0, s]$.) As everything is bounded for $t \leqq s$, we can write:

$$\begin{aligned} a_t + b_t &= \int_0^t \int_E (1 - Y(r, x))(\nu(dr, dx) - \mu(dr, dx)) + \sum_{r \leqq t} \frac{\alpha_r - \hat{Y}_r}{1 - \alpha_r} \\ &\quad - \sum_{T_n \leqq t} \frac{\alpha_{T_n} - \hat{Y}_{T_n}}{1 - \alpha_{T_n}} - \sum_{r \leqq t} (\alpha_r - \hat{Y}_r) \\ &= \int_0^t \int_E \left(1 - Y(r, x) + \frac{\alpha_r - \hat{Y}_r}{1 - \alpha_r}\right)(\nu(dr, dx) - \mu(dr, dx)). \end{aligned}$$

Finally if $V(t, x) = Z_{t-}\left(1 - Y(t, x) + \frac{\alpha_t - \hat{Y}_t}{1 - \alpha_t}\right)$, we get:

$$Z_t = Z_0 + \int_0^t \int_E V(r, x)(\nu(dr, dx) - \mu(dr, dx)) \tag{15}$$

for $t \leqq s$ (we recall that $s < T'_\infty(\omega)$). In addition if we use the statement (ii) above, and once again the bounded variation of $a_t$ and $b_t$, we see that $(v(|V|))_s < \infty$.

On the other hand $V$ is $\tilde{\mathscr{P}}$-measurable. If we apply (4.4) to the increasing process $(v(|V|))_t$, we conclude to the existence of a sequence $(S_n)$ of stopping times which increases a.s. towards $T'_\infty$ and such that $E((v(|V|))_{S_n}) < \infty$ for each $n$. From (1) we see that $E((\mu(|V|))_{S_n}) < \infty$ also holds. Let $Z^n_t = Z_{t \wedge S_n}$. From (15),

$$Z^n_t \leqq Z_0 + ((\mu + v)(|V|))_{S_n \wedge t},$$

which proves that the process $(Z^n_t)$ is uniformly integrable. At last if $B \in \mathscr{F}_t$, then $X(r, x) = V(r, x)\, 1_B\, 1_{\{t \wedge S_n < r \leqq (t+s) \wedge S_n\}}$ is $\tilde{\mathscr{P}}$-measurable, and from (1) we obtain (everything is finite):

$$E(1_B(Z^n_{t+s} - Z^n_t)) = E\big(\int_{\tilde{E}} X(r, x)(v(dr, dx) - \mu(dr, dx))\big) = 0.$$

Therefore $(Z^n_t)$ is a martingale and (4.3) is proven.

We can now state a partial converse to (4.1):

(4.5) **Theorem.** *Let $Z_0$ and $Y$ be like in* (4.3), *and $(Z_t)$ be defined by* (14). *If $E(Z_\infty) = 1$, then $Y \cdot v$ is a version of the predictable projection of $\mu$ for the probability measure $P'$ defined on $(\Omega, \mathscr{F})$ by $P'(d\omega) = P(d\omega)\, Z_\infty(\omega)$.*

This result partially solves the problem stated before Proposition (4.3). More generally one could show that if $T$ is a stopping time for which $E(Z_T) = 1$, then the multivariate point process truncated at $T$ (i.e.: $\mu_T = 1_{]0,T]} \cdot \mu$) admits the random measure $Y \cdot v$ truncated at $T$ (i.e.: $v'_T = 1_{]0,T]}\, Y \cdot v$) as its predictable projection for the probability measure $P'$ defined on $(\Omega, \mathscr{F})$ by $P' = Z_T \cdot P$. The proof would be similar.

Theorem (4.5) is the counterpart of Girsanov's theorem, for pure jump processes. When the increasing process $(v(]0, t] \times E))_{t \geqq 0}$ is absolutely continuous with respect to the Lebesgue measure, it was already known: cf. for example Grigelionis [8] (even when $\mu$ is not a multivariate point process, but a more general integer-valued random measure).

*Proof.* Let $X \in \tilde{\mathscr{P}}^+$ be such that $M_\mu(XY) < \infty$. From (14) we have $Z_{T_n} = Z_{(T_n)-} \cdot Y(T_n, X_n)$, and clearly $(Z_t)$ has left-hand limits except possibly at $S$, and a.s. a left-hand limit at $S$. Then using the method which allowed us to get (11), (12) and (13), we obtain:

$$\begin{aligned} M'_\mu(X) &= E\big(\int_{\tilde{E}} Z_t\, X(t, x)\, \mu(dt, dx)\big) = \sum_{n \geqq 0} E(X(T_n, X_n)\, Z_{T_n}) \\ &= \sum_{n \geqq 0} E(X(T_n, X_n)\, Y(T_n, X_n)\, Z_{(T_n)-}) \\ &= E\big(\int_{\tilde{E}} X(t, x)\, Y(t, x)\, Z_{t-}\, \mu(dt, dx)\big) = M'_v(XY), \end{aligned}$$

and the result follows.

## 5. Radon-Nikodym Derivatives and Representation of Martingales Under (A.1)

1. We start with the same assumptions that in Section 4. Theorem (4.5) tells us that if $Y$ and $(Z_t)$ are like in (4.3) one can find a probability measure $P' \ll P$ for

which $Y \cdot \nu$ is the predictable projection of $\mu$, under the additional assumption that $E(Z_\infty)=1$. It turns out that, under (A.1), this additional assumption is necessary as well as sufficient:

(5.1) **Theorem.** *Assume* (A.1). *If* $P' \ll P$, *let* $Y$ *be the process introduced in* (4.1), $Z$ *a version of the Radon-Nikodym derivative* $dP'/dP$ *and* $Z_0$ *a non-negative finite version of* $E(Z|\mathscr{F}_0)$. *Then* (14) *gives a version* $(Z_t)$ *of the martingale* $E(Z|\mathscr{F}_t)$.

In the same way that for (4.5) one would have similar results for the restriction to any stochastic interval $[0, T]$, where $T$ is a stopping time, under the only assumption that $P'_T \ll P_T$ where $P'_T$ and $P_T$ are the restrictions of $P'$ and $P$ to $(\Omega, \mathscr{F}_T)$.

When $\mathscr{F}_t = \mathscr{G}_t$, when $\mu$ is a Poisson random measure for $P$, and under the (unnecessary because of (4.1)) assumption that the predictable projection $\nu'$ for $P'$ satisfies $\nu' \ll \nu$, this result has been shown by various authors: Skorokhod [19] when $Y$ is deterministic, Snyder [20], Rubin [16] and Brémaud [3] for point processes (with strong assumptions in [20] and [16]), Segall-Kailath [17] and Boël-Varaiya-Wong [2] for a general space $E$.

*Proof.* Let $\nu' = Y \cdot \nu$, $(Z_t)$ defined by (14), $T'_\infty$ and $(S_n)$ like in the proof of (4.3). Let $\mu_n = 1_{]0,S_n]} \cdot \mu$, $\nu_n = 1_{]0,S_n]} \cdot \nu$, $\nu'_n = 1_{]0,S_n]} \cdot \nu'$, $Z^n_t = Z_{t \wedge S_n}$, $\mathscr{F}^n_t = \mathscr{F}_{t \wedge S_n}$. We denote by $P^n$ (resp. $P'^n$) the restriction of $P$ (resp. $P'$) to $(\Omega, \mathscr{F}^n_\infty)$.

On the space $(\Omega, \mathscr{F}^n_\infty)$ equipped with the family $(\mathscr{F}^n_t)$, $\mu_n$ admits $\nu_n$ (resp. $\nu'_n$) as its predictable projection for $P^n$ (resp. $P'^n$). By definition of $S_n$, $(Z^n_t)$ is a uniformly integrable martingale. Thus $E^n(Z^n_\infty)=1$ and $P''^n = Z^n_\infty \cdot P^n$ defines a probability on $(\Omega, \mathscr{F}^n_\infty)$. But one checks easily that when we start with $Y$, $Z_0$, $\mu_n$ and $\nu_n$, (14) gives the process $(Z^n_t)$. Therefore (4.5) implies that $\nu'_n$ is also the predictable projection of $\mu_n$ for $P''^n$. Now the restrictions of $P'^n$ and $P''^n$ to $\mathscr{F}_0$ are identical (they admit the same density $Z_0$ with respect to $P$), and $\mathscr{F}^n_t = \mathscr{F}^n_0 \vee \mathscr{G}^n_t$ where $(\mathscr{G}^n_t)$ is the family of $\sigma$-algebras generated by $\mu_n$. So (3.4) implies that $P'^n = P''^n$ on $(\Omega, \mathscr{F}^n_\infty)$. In other words $Z^n_\infty = E(Z|\mathscr{F}_{S_n})$ (because $\mathscr{F}_{S_n} = \mathscr{F}^n_\infty$), which implies $Z_{S_n \wedge t} = E(Z|\mathscr{F}_{S_n \wedge t})$ for each $t \geqq 0$.

With the notations of (4.3), we have clearly $P(S < T_\infty) = P'(T'_\infty < S) = 0$ and thus $Z=0$ $P$-a.s. on $\{T'_\infty < T_\infty\}$. We have seen that $Z_t = 0$ if $T'_\infty \leqq t < S$, and thus except on a $P$-null set, we have $Z_t = 0$ if $T'_\infty \leqq t < T_\infty$. We obtain:

$$\begin{aligned} E(Z|\mathscr{F}_{T_n}) &= \lim_{(p)} E(Z\, 1_{\{T_n < S_p\}} | \mathscr{F}_{T_n}) + E(Z\, 1_{\{T'_\infty \leqq T_n\}} | \mathscr{F}_{T_n}) \\ &= \lim_{(p)} Z_{T_n} 1_{\{T_n < S_p\}} = 1_{\{T_n < T'_\infty\}} Z_{T_n} = Z_{T_n} \qquad P\text{-a.s.} \end{aligned}$$

One easily deduces that $Z_{T_n \wedge t} = E(Z|\mathscr{F}_{T_n \wedge t})$ for each $t \geqq 0$. We have also seen that $Z_t = \lim\inf_{s \uparrow T_\infty} Z_s$ if $t \geqq T_\infty$. Then

$$\begin{aligned} E(Z|\mathscr{F}_t) &= \lim E(Z\, 1_{\{t \leqq T_n\}} | \mathscr{F}_t) + E(Z\, 1_{\{T_\infty \leqq t\}} | \mathscr{F}_t) \\ &= \lim Z_t 1_{\{t \leqq T_n\}} + 1_{\{T_\infty \leqq t\}} E(Z | \bigvee_{(n)} \mathscr{F}_{T_n}) \\ &= Z_t 1_{\{t < T_\infty\}} + 1_{\{T_\infty \leqq t\}} \lim Z_{T_n} = Z_t. \end{aligned}$$

The result (c) announced in the introduction is then a corollary of the previous theorem:

(5.2) **Theorem.** *Assume* (A.1). *Let* $Z_0$ *and* $Y$ *be like in* (4.3), *and* $(Z_t)$ *be defined by* (14). *Then there exists a probability* $P'$ *on* $(\Omega, \mathscr{F})$ *such that* $P' \ll P$, *that* $Y \cdot \nu$ *is a version of the predictable projection of* $\mu$ *for* $P'$, *and that the restriction of* $P'$ *to* $\mathscr{F}_0$ *admits* $Z_0$ *as its density with respect to* $P$, *if and only if* $E(Z_\infty)=1$. *In this case the restriction of* $P'$ *to* $\mathscr{F}_\infty$ *is unique.*

2. *Representation of Martingales.* As a corollary of the preceding results, we obtain a representation of each martingale (or local martingale) as the integral of a $\tilde{\mathscr{P}}$-measurable process defined on $\tilde{\Omega}$, with respect to the random measure $\nu-\mu$. More precisely we have the two results:

(5.3) **Proposition.** *Let* $X$ *be a finite* $\tilde{\mathscr{P}}$*-measurable function on* $\tilde{\Omega}$, *satisfying:*

$$\int_0^t \int_E |X(s,x)|\, \nu(ds,dx) < \infty \qquad \text{a.s. } on\ \{t < T_\infty\}. \tag{16}$$

*Let* $(Z_t)$ *be a right-continuous process adapted to* $(\mathscr{F}_t)$, *such that*

$$Z_t = Z_0 + \int_0^t \int_E X(s,x)\big(\nu(ds,dx) - \mu(ds,dx)\big) \qquad \text{a.s. } on\ \{t < T_\infty\}. \tag{17}$$

*Then there exists a sequence* $(S_n)$ *of stopping times increasing a.s. towards* $T_\infty$, *for which* $(Z_{t\wedge S_n})_{t\geqq 0}$ *is a uniformly integrable martingale for each n.*

This statement could be expressed as follows: if $X$ satisfies (16), then (17) defines a "local martingale on $[0, T_\infty[$". However (17) makes sense only where the inequality in (16) holds, so (17) actually defines $Z_t$ only on a $P$-full set. If we extend arbitrarily $Z_t$ outside this set, $Z_t$ may happen to be measurable with respect to the completed $\sigma$-algebra $\bar{\mathscr{F}}_t$ of $\mathscr{F}_t$, but not with respect to $\mathscr{F}_t$ itself.

(5.4) **Theorem.** *Assume* (A.1). *Let* $(Z_t)$ *be a right-continuous process. Then there exists a sequence* $(S_n)$ *of stopping times increasing a.s. towards* $T_\infty$, *for which* $(Z_{t\wedge S_n})_{t\geqq 0}$ *is a uniformly integrable martingale for each n, if and only if there exists a finite* $\tilde{\mathscr{P}}$*-measurable function* $X$ *on* $\tilde{\Omega}$ *satisfying* (16) *and* (17).

We emphazise the fact (17) is an ordinary integral (which can evidently be also considered as a "stochastic integral", although there is no need for that). Theorem (5.4) generalizes well-known results concerning the representation of martingales as stochastic integrals of a predictable process with respect to a fundamental martingale (for Poisson and Wiener processes).

Theorem (5.4) has been shown (when $P(T_\infty=\infty)=1$) for point processes by Chou and Meyer [5] by a direct and simpler method (which can undoubtedly be extended to our situation). One can also show (5.4) by using the theory of stochastic integrals with respect to square-integrable martingales, and then deduce (5.1) from (5.4) (Boël-Varaiya-Wong [2], Segall-Kailath [20]), but besides the fact that this method requires a difficult theory, its application is rather tedious and involves some unnecessary assumptions.

*Proof of* (5.3). From (16) and (4.4), there exists a sequence $(S_n)$ of stopping times increasing a.s. towards $T_\infty$ and such that $E(\nu(|X|)_{S_n})$ is finite for each $n$. One can then duplicate the end of the proof of (4.3) (with $V=X$).

*Proof of* (5.4). The direct part is another formulation of (5.3). For the converse, let us start with a positive uniformly integrable martingale $(Z_t)$ with $E(Z_0)=1$. Applying (5.1) to $P'=Z_\infty \cdot P$, we see that $(Z_t)$ is a.s. of the form (14), for a suitable $Y \in \tilde{\mathscr{P}}^+$. After throwing away a $P$-null set, we can suppose that $(Z_t)$ satisfies (14) identically, and also $S=T_\infty$. Using the notations of (4.3), we see that (15) holds for $t<T'_\infty$. But $Z_t=Z_{(T'_\infty)-}=0$ if $T'_\infty \leqq t < T_\infty$, thus $V=V 1_{]0,T'_\infty]}$ and (15) holds for $t<T_\infty$.

In other words, if $X=V$, $Z_t$ equals the right-hand side of (17) for each $t<T_\infty$. From (4.3) we also know that (16) holds for $t<T'_\infty$, and it remains to prove (16) when $t=T'_\infty<T$. As $t<S$, we have $\beta=\sup\left(\frac{\alpha_s}{1-\alpha_s} 1_{\{\alpha_s<1\}}, s \leqq t\right)<\infty$ and from (14) there exists $\gamma<\infty$ such that

$$Z_{s-} \leqq \gamma \exp\Big(-\sum_{r \leqq s} \hat{Y}_r 1_{\{\alpha_r<1\}} - \nu'^c(]0,s] \times E)\Big) \tag{18}$$

for $s \leqq t$. The left-hand side of (16) is smaller than

$$\begin{aligned} &\int_0^t Z_{s-}\left(1+\frac{\alpha_s}{1-\alpha_s} 1_{\{\alpha_s<1\}}\right) \nu(ds,E) \\ &\quad + \int_0^t Z_{s-}\left(Y(s,x)+\frac{\hat{Y}_s}{1-\alpha_s} 1_{\{\alpha_s<1\}}\right) \nu(ds,dx) \end{aligned} \tag{19}$$

(by using the form of $V$). From the finiteness of $\beta$, $\gamma$ and $\nu(]0,t] \times E)$, the first part of (19) is finite. From (18), the second part of (19) is smaller than

$$\int_0^t \exp(-\nu'^c(]0,s] \times E))\, \nu'^c(ds,E) + \gamma(1+\beta) \sum_{s \leqq t} \hat{Y}_s \exp\Big(-\sum_{r \leqq s} \hat{Y}_r\Big),$$

which is finite. Thus (16) holds for each $t<T_\infty$.

Now let $(Z_t)$ be any uniformly integrable martingale. If $Z^+_\infty$ (resp. $Z^-_\infty$) is the positive (resp. negative) part of $Z_\infty$, one can apply the preceding result to the martingales

$$Z^+_t = \frac{E(Z^+_\infty | \mathscr{F}_t)}{E(Z^+_\infty)}, \qquad Z^-_t = \frac{E(Z^-_\infty | \mathscr{F}_t)}{E(Z^-_\infty)}.$$

At last let $(Z_t)$ be a process satisfying the condition stated in (5.4). Put $Z^n_t = E(Z_{S_n \wedge t} - Z_{S_{n-1} \wedge t} | \mathscr{F}_{S_n \wedge t})$: each $(Z^n_t)$ is a uniformly integrable martingale, to which corresponds a $\tilde{\mathscr{P}}$-measurable $X_n$ satisfying (16) and (17). We have $Z^n_t=0$ if $t \leqq S_{n-1}$ and $Z^n_t = Z^n_{S_n}$ if $t \geqq S_n$, thus $X'_n = X_n 1_{]S_{n-1}, S_n]}$ still satisfies (16) and (17). But $Z_t = \sum_{(n)} Z^n_t$ if $t<T_\infty$, therefore $X=\sum_{(n)} X'_n$ solves the problem.

## References

1. Benveniste, A., Jacod, J.: Systèmes de Lévy des processus de Markov. Invent. Math., **21**, 183–198 (1973)
2. Boel, R., Varaiya, P., Wong, E.: Martingales on jump processes I, II. El. Res. Lab. Berkeley. M-407 and M-409 (1973)
3. Brémaud, P.: A martingale approach to point processes. PhD thesis. El. Res. Lab. Berkeley. M-345 (1972)

4. Brémaud, P.: The martingale theory of point processes over the real half-line. IRIA Int. Colloquium on Control Theory (1974). Lect. Notes in Economics and Math. Systems. Berlin, Heidelberg, New York: Springer (To appear)
5. Chou, C.S., Meyer, P.A.: La représentation des martingales relatives à un processus ponctuel discret. C. R. Acad. Sci. Paris, A, **278**, 1561–1563 (1974)
6. Dellacherie, C.: Un exemple de la théorie génerale des processus, Sém. Proba. IV. Lecture Notes **124**, Berlin, Heidelberg, New York: Springer 1970
7. Dellacherie, C.: Capacités et processus stochastiques. Berlin, Heidelberg, New York: Springer 1972
8. Grigelionis, B.: On nonlinear filtering theory and absolute continuity of measures, corresponding to stochastic processes, Lecture Notes 330, Berlin, Heidelberg, New York: Springer 1973
9. Ito, K., Watanabe, S.: Transformation of Markov processes by multiplicative functionals. Ann. Inst. Fourier XV, 13–30 (1965)
10. Jacod, J.: On the stochastic intensity of a random point process over the half-line. Tech. Report **51**, Dept. Stat. Princeton University (1973)
11. Kailath, T.: The structure of Radon-Nikodym derivatives with respect to Wiener and related measures. Ann. Math. Statist. **42**, 1054–1067 (1971)
12. Meyer, P.A.: Probability and potential. Waltham: Blaisdell 1966
13. Meyer, P.A.: Processus ponctuels selon K. Ito, Sém. Proba. V, Lecture Notes **191**, Berlin, Heidelberg, New York: Springer 1971
13. Meyer, P.A.: Processus ponctuels selon K. Ito, Sém. Proba. V, Lecture Notes **191**, Berlin: Springer 1971
14. Orey, S.: Radon-Nikodym derivative of probability measures: martingale methods. Dpt. Found. Math. Sc., Tokyo Un. of Education (1974)
15. Papangelou, F.: Integrability of expected increments of point processes and a related change of scale. Trans. Amer. Math. Soc. **165**, 483–506 (1972)
16. Rubin, I.: Regular point processes and their detection. IEEE, Trans. Information Theory **18**, 5 (1972)
17. Segall, A., Kailath, T.: Radon-Nikodym derivatives with respect to measures induced by discontinuous independent-increments processes. To appear
18. Segall, A., Kailath, T.: On the representation of martingales as stochastic integrals. To appear
19. Skorohod, A.V.: Random processes with independent increments, Rep. AD 645769, Foreign Technology Division
20. Snyder, D.: Filtering and detection for doubly stochastic Poisson processes. IEEE, Trans. Information Theory **18**, 1 (1972)
21. van Schuppen, J.H., Wong, E.: Transformations of local martingales under a change of law. El. Res. Lab. Berkeley, M-385 (1973)
22. Watanabe, S.: On discontinuous additive functionals and Lévy measures of a Markov process. Japan. J. Math. **34**, 53–79 (1964)

Jean Jacod
Laboratoire de Probabilités
Université de Rennes
Avenue du Général Leclerc
F-35000 Rennes
France

*(Received October 10, 1974; in revised form December 18, 1974)*