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Duality of Lévy Systems

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1. Introduction

This paper is a continuation of [2]. In [2] we set up a duality theory for multiplicative functionals (MF's) of dual processes. In the present paper we apply this duality theory to the study of Lévy systems and to the closely related subject of quasi-left-continuous (q.l.c.) pure jump additive functionals (AF's) of such processes. Since this paper is a sequel to [2] and depends heavily on the results of that paper, we follow scrupulously the notation and terminology of [1] and [2]. In order to save space we do not include a lengthy resumé of the results of [2]; rather we give specific references as they are needed.

Throughout this paper X and $\hat{X}$ are two standard processes in duality relative to a Radon measure $\xi(dx)=dx$. More specifically X and $\hat{X}$ satisfy the conditions on p. 259 of [1]. However, we make no regularity assumptions on the resolvents (U^α) and $(\hat{U}^\alpha)$ of X and $\hat{X}$.

We now outline the contents of this paper. In Section 2 we give an important example of dual terminal times (see (4.11) of [2]) which is the key to later results. A corollary to our main result is the following. Let d be a metric for E and let K and L be Borel subsets of E such that $d(K,L)>0$. Then

$$(1.1)\qquad \begin{aligned} T_{K,L}&=\inf\{t:\ X_{t-}\in K,\ X_t\in L\}\\ \hat{T}_{L,K}&=\inf\{t:\ \hat{X}_{t-}\in L,\ \hat{X}_t\in K\}\end{aligned}$$

are dual exact terminal times.

In Section 3 we show that the Lévy systems $(N(x,dy),H_t)$ and $(\hat{N}(dy,x),\hat{H}_t)$ for X and $\hat{X}$ may be chosen so that H and $\hat{H}$ are dual CAF's in $\mathbf{A}_c(X)$ and $\mathbf{A}_c(\hat{X})$ and if μ is the measure associated with H and $\hat{H}$ (Section 9 of [2]), then N and $\hat{N}$ are dual kernels relative to μ, that is, $\int f(Ng)\,d\mu=\int(f\hat{N})g\,d\mu$ for nonnegative f and g. Here $\mathbf{A}_c(X)$ denotes the class of CAF's of X which are finite on $[0,\zeta)$. Finally in Section 4 these results are applied to the study of q.l.c. pure jump AF's. Combining the results of Section 4 with those of Revuz [7] and [8] yields a complete description of all AF's of X and an explicit duality theory for finite additive functionals.

Throughout this paper we often omit the qualifying phrase "almost surely" when writing equalities between random variables. Also the equality of MF's or AF's always means equivalence. Finally we often (but not always) omit the hat "^" in those places where it is redundant; for example, in the notation of (1.1) we

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often write $\hat{E}^x\{f(X_t);\, t<T_{L,K}\}$ in place of $\hat{E}^x\{f(\hat{X}_t);\, t<\hat{T}_{L,K}\}$. Our guiding principle is clarity rather than consistency!

2. Some Dual Terminal Times

Let d be a fixed metric for E and d_2 be the metric on $E\times E$ defined by

$$d_2[(x_1,x_2),(y_1,y_2)]=d(x_1,y_1)+d(x_2,y_2).$$

Let D be the diagonal of $E\times E$ and let $F=E\times E-D=\{(x,y)\in E\times E\colon x\neq y\}$. Since D is closed, F is open in $E\times E$ and is a locally compact separable metric space in the relative topology. Let $i\colon E\times E\to E\times E$ be defined by $i(x,y)=(y,x)$. If B is a Borel subset of $E\times E$ define $\hat{B}=iB$. Clearly $\hat{B}$ is again a Borel subset of $E\times E$ and $I_{\hat{B}}(x,y)=I_B(y,x)$. In particular if $B\subset F$ then $\hat{B}\subset F$.

If B is a Borel subset of F which, of course, is the same as requiring B to be a Borel subset of $E\times E$ contained in F, define

$$(2.1)\qquad \begin{aligned} T(B)&=\inf\{t\colon (X_{t-},X_t)\in B\}\\ \hat{T}(B)&=\inf\{t\colon (\hat{X}_{t-},\hat{X}_t)\in B\}. \end{aligned}$$

Here and in the sequel $X_{0-}=X_0$ and a similar convention holds for $\hat{X}$. We can now state the main result of this section.

(2.2) **Theorem.** *If B is a Borel subset of F, then $T(B)$ and $\hat{T}(\hat{B})$ are dual terminal times.*

The proof of Theorem 2.2 is rather long and so we will break it up into several steps. Recall that if T and $\hat{T}$ are terminal times for X and $\hat{X}$, then T and $\hat{T}$ are dual provided

$$E^f\{g(X_t);\ t<T\}=\hat{E}^g\{f(X_t);\ t<T\}$$

for all $t\geqq 0$ and $f,g\in \mathbf{C}_K^+$. Here $E^f\{\cdot\}=\int f(x)\,E^x\{\cdot\}\,dx$. We write $T\leftrightarrow\hat{T}$ if T and $\hat{T}$ are dual terminal times.

Since the process $t\to(X_{t-},X_t)$ on $E\times E$ is progressively measurable, it is an immediate consequence of (IV; T.47) and (IV; T.52) of [3] that $T(B)$ is a stopping time. Obviously $T(B)$ is then a terminal time. Of course, if B is an arbitrary Borel subset of F, then $T(B)$ may be identically zero. However, if $d_2(B,D)>0$ then $T(B)>0$ almost surely, and hence is a thin exact terminal time. By duality these remarks apply to $\hat{T}(B)$ as well.

We will need the following elementary lemma in our discussion.

(2.3) **Lemma.** *Let $F(x_0,x_1,\ldots,x_n)$ be a bounded Borel function on E^{n+1} and let f and g be bounded nonnegative ξ-integrable functions. If $0=t_0<t_1<\cdots<t_n$ with $t_j-t_{j-1}=s$ for $j=1,2,\ldots,n$, then*

$$E^f\{F(X_{t_0},X_{t_1},\ldots,X_{t_n})\,g(X_{t_n})\}=\hat{E}^g\{F(X_{t_n},X_{t_{n-1}},\ldots,X_{t_0})\,f(X_{t_n})\}.$$

Proof. It suffices to consider the case in which $F(x_0,x_1,\ldots,x_n)=\prod_{j=0}^{n} f_j(x_j)$ with each $f_j\in b\,\mathscr{E}_+$. In this case the proof proceeds by a straightforward induction on n. We leave the details to the reader.

(2.4) **Corollary.** *Let $G(x, y)$ be a bounded Borel function on $E\times E$. If f, g, and $\{t_j\}$ are as in Lemma* 2.3, *then*

$$E^f\left\{\prod_{j=1}^{n} G(X_{t_{j-1}}, X_{t_j})\, g(X_{t_n})\right\}=\hat{E}^g\left\{\prod_{j=1}^{n} G(X_{t_j}, X_{t_{j-1}})\, f(X_{t_n})\right\}.$$

Proof. Apply (2.1) with $F(x_0, x_1, \ldots, x_n)=\prod_{j=1}^{n} G(x_{j-1}, x_j)$ and observe that $F(x_n, x_{n-1}, \ldots, x_0)=\prod_{j=1}^{n} G(x_j, x_{j-1})$.

With these trivialitities out of the way we come to the main step in our proof.

(2.5) **Proposition.** *Let G be a nonvoid open subset of F such that $d_2(G, D)>0$. Then $T(G)$ and $\hat{T}(\hat{G})$ are dual (exact) terminal times.*

Proof. Let $\{G_n\}$ be an increasing sequence of open subsets of G with compact closures $\bar{G}_n$ such that $\bar{G}_n\subset G_{n+1}$ for each n and $\bigcup G_n=G$. Let $\delta_n=1/2\, d_2(\bar{G}_n, G_{n+1}^c)$. Since $\bar{G}_n$ is compact $\delta_n>0$ for each n. To simplify the notation let $T=T(G)$, $T_n=T(G_n)$, $\hat{T}=\hat{T}(\hat{G})$, and $\hat{T}_n=\hat{T}(\hat{G}_n)$. Now fix $t>0$. In what follows we omit the phrase "almost surely" in those places where it is obviously required. Define $g_n=1-I_{G_n}$ and $\hat{g}_n=1-I_{\hat{G}_n}$. Thus $\hat{g}_n(x, y)=g_n(y, x)$. For each $k\geqq 1$ let $t_{k,j}=j\,t\,2^{-k}$, $0\leqq j\leqq 2^k$, be the dyadic points of subdivision of the interval $[0, t]$. For each $n\geqq 1$ and $k\geqq 1$ define

$$M(n, k)=\prod_{j=1}^{2^k} g_n(X_{t_{k,j-1}}, X_{t_{k,j}}) \tag{2.6}$$

and define $\hat{M}(n, k)$ similarly using $\hat{g}_n$ and the process $\hat{X}$. In particular $M(n, k)=1$ if and only if $(X_{t_{k,j-1}}, X_{t_{k,j}})\notin G_n$ for *every* j, $1\leqq j\leqq 2^k$. We now discuss T but the results apply to $\hat{T}$ as well by duality.

Note first of all that g_n and $M(n, k)$ are decreasing functions of n. Since $s\to X_s$ is right continuous and has left limits, given $\varepsilon>0$ we can choose $0=\tau_0<\tau_1<\cdots<\tau_q=t$ so that the oscillation of $s\to X_s(\omega)$ on $[\tau_{j-1}, \tau_j)$ is less than ε for $1\leqq j\leqq q$. Of course, $\{\tau_j\}$ depends on ω in general.

Now suppose that $T>t$. Then $I_G(X_{s-}, X_s)=0$ for all $s\leqq t$. Fix $n\geqq 1$ and choose $\{\tau_j\}$ as above with $\varepsilon=\delta_n$. Choose m so that $t\,2^{-m}<\min_{1\leqq j\leqq q}[\tau_j-\tau_{j-1}]$. Thus if $k\geqq m$ for each j, $1\leqq j\leqq 2^k$, the closure of $I_j^k=(t_{k,j-1}, t_{k,j}]$ contains at most one τ_i. If there is no τ_i in I_j^k, then

$$d(X_{t_{k,j-1}}, X_{t_{k,j}})<\delta_n=1/2\, d_2(\bar{G}_n, G_{n+1}^c),$$

and since $D\subset G_{n+1}^c$ this implies that $g_n(X_{t_{k,j-1}}, X_{t_{k,j}})=1$. If on the other hand $\tau_i\in I_j^k$, then

$$d_2[(X_{\tau_i-}, X_{\tau_i}), (X_{t_{k,j-1}}, X_{t_{k,j}})]<2\delta_n,$$

and since $(X_{\tau_i-}, X_{\tau_i})\in G^c\subset G_{n+1}^c$ it follows that $g_n(X_{t_{k,j-1}}, X_{t_{k,j}})=1$ in this case too. Therefore we have proved that

$$\{T>t\}\subset\bigcap_n\bigcup_m\bigcap_{k\geqq m}\{M(n, k)=1\}. \tag{2.7}$$

Next let $\Gamma_n = \bigcap_m \bigcup_{k \geqq m} \{M(n,k)=1\}$. We claim that for $n \geqq 2$

$$(2.8) \qquad \Gamma_n \subset \{T_{n-1} > t\}.$$

To this end fix $n \geqq 2$ and let $\omega \in \Gamma_n$. Choose $\{\tau_i\}$ as above but with $\varepsilon = \delta_{n-1}$. Let s be a discontinuity of $u \to X_u(\omega)$ with $s \leqq t$. Of course, $s > 0$. Suppose firstly that $s = \tau_i$ for some i. Choose m so that $t 2^{-m} < \min_{1 \leqq j \leqq q} [\tau_j - \tau_{j-1}]$ and then choose $k \geqq m$ so that $\omega \in \{M(n,k)=1\}$. Then there exists a j so that $s = \tau_i \in I_j^k = (t_{k,j-1}, t_{k,j}]$, and since $M(n,k)=1$ we must have $(X_{t_{k,j-1}}, X_{t_{k,j}}) \in G_n^c$. Also

$$d_2 [(X_{s-}, X_s), (X_{t_{k,j-1}}, X_{t_{k,j}})] < 2\delta_{n-1}$$

and so (X_{s-}, X_s) is not in G_{n-1}. Next suppose $s \neq \tau_i$ for all i. Then we can choose a k so that $\omega \in \{M(n,k)=1\}$ and $t 2^{-k} < \min_{1 \leqq i \leqq q} |s - \tau_i|$. Again there exists a j so $s \in I_j^k$, and by the choice of k the closure of I_j^k contains no τ_i. Therefore, $d(X_{s-}, X_s) < \delta_{n-1}$ and so (X_{s-}, X_s) is not in G_{n-1} in this case also. Consequently $T_{n-1}(\omega) > t$ establishing (2.8).

It is immediate since $d_2(G, D) > 0$ that $\{T > t\} = \bigcap_{n \geqq 2} \{T_{n-1} > t\}$, and so it follows from (2.7) and (2.8) that

$$(2.9) \qquad \{T > t\} \subset \bigcap_n \liminf_k \{M(n,k)=1\} \subset \bigcap_n \limsup_k \{M(n,k)=1\} \subset \{T > t\}.$$

Finally using (2.4), (2.9), and the dual of (2.9) we see that for $f, g \in \mathbf{C}_K^+$

$$E^f\{g(X_t); t < T\} = \lim_n \liminf_k E^f\{M(n,k)\, g(X_t)\}$$
$$= \lim_n \liminf_k \hat{E}^g\{M(n,k) f(X_t)\} = \hat{E}^g\{f(X_t); t < T\}.$$

But this states that $T = T(G)$ and $\hat{T} = \hat{T}(\hat{G})$ are dual terminal times, proving Proposition 2.5.

(2.10) **Proposition.** (i) *If $\{B_n\}$ is an increasing sequence of Borel subset of F with union $B \subset F$, then $T(B_n) \downarrow T(B)$.*

(ii) *If $\{B_n\}$ is a decreasing sequence of Borel subsets of F with intersection B and if $d_2(B_1, D) > 0$, then $T(B_n) \uparrow T(B)$ and $T(B_n) = T(B)$ for all sufficiently large n on $\{T(B) < \infty\}$.*

Proof. We restrict our attention to (ii) since (i) is clear. Obviously $\{T(B_n)\}$ increases and $T(B_n) \leqq T(B)$ for all n. Suppose $T(B) = t < \infty$ and $T(B_n) < t$ for all n. Then for each n there exists s_n with $0 < s_n < t$ such that $(X_{s_n-}, X_{s_n}) \in B_n \subset B_1$. But $r = d_2(B_1, D) > 0$ and since there are only a finite number of discontinuities of $u \to X_u$ on $[0, t]$ with a saltus exceeding r, there are only finitely many distinct s_n. Thus by passing to a subsequence we may assume that $s_n = s < t$ for all n. Consequently $(X_{s-}, X_s) \in \cap B_n = B$ and so $T(B) \leqq s$ contradicting the fact that $T(B) = t > s$. Hence $T(B_n) = T(B)$ for all sufficiently large n if $T(B) < \infty$. If $T(B) = \infty$ the same argument shows that $T(B_n) \uparrow T(B)$, completing the proof of (2.10).

Let $\{B_n\}$ be a sequence of Borel subsets of F which satisfies the conditions in (2.10i) or in (2.10ii). Then it is an easy consequence of (2.10) and its dual that if $T(B_n) \leftrightarrow \hat{T}(\hat{B}_n)$ for each n, then $T(B) \leftrightarrow \hat{T}(\hat{B})$. See the proof of (6.5) in [2]. In particular if B is a compact subset of F or an arbitrary open subset of F, then $T(B)$ and $\hat{T}(\hat{B})$ are dual terminal times.

We can now complete the proof of Theorem 2.2 by a standard capacity argument. Fix f, $g \in \mathbf{C}_K^+$ and $t>0$. If B is a Borel subset of F define

$$\begin{aligned} \varphi(B) &= E^f\{g(X_t);\ T(B) \leqq t\} \\ \psi(B) &= \hat{E}^g\{f(X_t);\ T(B) \leqq t\}. \end{aligned} \tag{2.11}$$

It follows from the remarks in the preceding paragraph that $\varphi(B)=\psi(\hat{B})$ if B is either compact or open. A standard argument shows that φ and ψ restricted to the compact subsets of F are Choquet capacities in the sense of [1, I-10.5] and that $\varphi^*(B)=\varphi(B)$ and $\psi^*(B)=\psi(B)$ for all Borel sets B contained in F. See the proof of (I-10.12) and (I-10.15) of [1]. Of course, the right continuity of φ and ψ comes from Proposition 2.10. Now let B be a Borel subset of F. Then there exists an increasing sequence $\{K_n\}$ of compact subsets of B and a decreasing sequence $\{G_n\}$ of open supersets of B such that

$$\lim_n \varphi(K_n) = \varphi(B) = \lim_n \varphi(G_n)$$

$$\lim_n \psi(\hat{K}_n) = \psi(\hat{B}) = \lim_n \psi(\hat{G}_n).$$

Consequently $\varphi(B)=\psi(\hat{B})$ for each Borel subset B of F, establishing Theorem 2.2.

The following corollaries are immediate consequences of Theorem 2.2.

(2.12) **Corollary.** *Let K and L be disjoint Borel subsets of E. Then $T_{K,L}=T(K\times L)$ and $\hat{T}_{L,K}=\hat{T}(L\times K)$ are dual terminal times. If $d(K,L)>0$, then $T_{K,L}$ and $\hat{T}_{L,K}$ are exact.*

(2.13) **Corollary.** *Let Γ be a Borel subset of $(0,\infty)$. Then $T=\inf\{t\colon d(X_{t-},X_t)\in\Gamma\}$ and $\hat{T}=\inf\{t\colon d(\hat{X}_{t-},\hat{X}_t)\in\Gamma\}$ are dual terminal times. If Γ is at a positive distance from the origin, then T and $\hat{T}$ are exact.*

3. Lévy Systems

We begin with a lemma that will be needed in our discussion of Lévy systems.

(3.1) **Lemma.** *Let T and $\hat{T}$ be exact terminal times of X and $\hat{X}$ with all points of E permanent for both T and $\hat{T}$, that is, $P^x(T>0)=1$ and $\hat{P}^x(\hat{T}>0)=1$ for all x. Let T_n be the iterates of T, that is, $T_0=0$, $T_{n+1}=T_n+T\circ\theta_{T_n}$ for $n\geqq 0$, and let $\hat{T}_n$ be the iterates of $\hat{T}$. Fix a constant a with $0<a<1$ and let $a_0=0$ and $a_n=a$ for $n\geqq 1$. Define*

$$M_t = \prod_{T_n \leqq t} (1-a_n); \qquad \hat{M}_t = \prod_{\hat{T}_n \leqq t} (1-a_n).$$

Then M and $\hat{M}$ are exact MF's of X and $\hat{X}$, and M and $\hat{M}$ are dual if and only if T and $\hat{T}$ are dual.

Proof. It is routine that M and $\hat{M}$ are MF's of X and $\hat{X}$. Clearly all points are permanent for both M and $\hat{M}$ and so they are exact. Observe that if $S=\inf\{t\colon M_t=0\}$, then $S=\lim T_n$, and a similar statement holds for $\hat{M}$. Suppose M and $\hat{M}$ are dual. Then it follows from [2; 4.12] that T and $\hat{T}$ are dual because $I_{[0,T)}(t)=\lim_{\lambda\to\infty}(M_t)^\lambda$ and $I_{[0,\hat{T})}(t)=\lim_{\lambda\to\infty}(\hat{M}_t)^\lambda$.

Conversely suppose that T and $\hat{T}$ are dual terminal times. Let (V^α), $(\hat{V}^\alpha)$, (W^α), and $(\hat{W}^\alpha)$ be the resolvents corresponding to T, $\hat{T}$, M, and $\hat{M}$ respectively. By assumption (V^α) and $(\hat{V}^\alpha)$ are dual resolvents. Fix $g \in \mathbf{C}_K^+$. Then for $\alpha > 0$

$$\begin{aligned} W^\alpha g(x) &= E^x \int_0^\infty e^{-\alpha t} g(X_t) M_t \, dt \\ &= \sum_{n=0}^\infty E^x \int_{T_n}^{T_{n+1}} e^{-\alpha t} g(X_t)(1-a)^n \, dt \\ &= \sum_{n=0}^\infty (1-a)^n E^x \{e^{-\alpha T_n} V^\alpha g(X_{T_n})\} \\ &= \sum_{n=0}^\infty (1-a)^n P_{T_n}^\alpha V^\alpha g(x). \end{aligned}$$

But if $h \in b\mathscr{E}_+^*$,

$$P_{T_{n+1}}^\alpha h(x) = E^x \{e^{-\alpha T_n} E^{X(T_n)} [e^{-\alpha T} h(X_T)]\} = P_{T_n}^\alpha P_T^\alpha h(x),$$

and combining these computations we obtain

$$W^\alpha g(x) = \sum_{n=0}^\infty (1-a)^n (P_T^\alpha)^n V^\alpha g(x), \tag{3.2}$$

where, of course, $(P_T^\alpha)^0$ is the identity. Also if $u^\alpha(x,y) < \infty$ we have $v^\alpha(x,y) = u^\alpha(x,y) - P_T^\alpha u^\alpha(x,y) = u^\alpha(x,y) - u^\alpha \hat{P}_T^\alpha(x,y)$, and so

$$P_T^\alpha v^\alpha(x,y) = P_T^\alpha u^\alpha(x,y) - (P_T^\alpha)^2 u^\alpha(x,y) = u^\alpha \hat{P}_T^\alpha(x,y) - u^\alpha (\hat{P}_T^\alpha)^2(x,y) = v^\alpha \hat{P}_T^\alpha(x,y).$$

Now if $f \in C_K^+$ by using (3.2) and its dual and the above, we obtain

$$(f, W^\alpha g) = \sum_{n=0}^\infty (1-a)^n (f, (P_T^\alpha)^n V^\alpha g) = \sum_{n=0}^\infty (1-a)^n (f \hat{V}^\alpha (\hat{P}_T^\alpha)^n, g) = (f \hat{W}^\alpha, g),$$

and so (W^α) and $(\hat{W}^\alpha)$ are dual resolvents. This completes the proof of Lemma 3.1.

We turn now to the discussion of Lévy systems. Let $\mathbf{A}(\hat{\mathbf{A}})$ denote the class of all AF's of $X(\hat{X})$ that are finite on the interval $[0, \zeta)$ $([0, \hat{\zeta}))$. Let $\mathbf{A}_c$ and $\hat{\mathbf{A}}_c$ denote the set of continuous elements in $\mathbf{A}$ and $\hat{\mathbf{A}}$ respectively. A pair (N, H) where $N = (N(x, B))$ is a kernel on $(E, \mathscr{E})$ satisfying $N(x, \{x\}) = 0$ for all x and which is *proper* in the sense that E is a countable disjoint union of Borel sets B_n such that $N(\cdot, B_n)$ is finite for each n, and where $H \in \mathbf{A}_c$ is called a *Lévy system* for X provided that for each nonnegative Borel function Φ on $E \times E$ *vanishing on the diagonal* one has for each $x \in E$ and $t \geqq 0$

$$E^x \{\sum_{s \leqq t} \Phi(X_{s-}, X_s)\} = E^x \int_0^t \int \Phi(X_s, y) N(X_s, dy) \, dH_s. \tag{3.3}$$

Moreover one may assume that for a fixed $\alpha > 0$, H has a bounded α-potential. It is well-known ([5] or [10]) that if X is special standard, then X has a Lévy system. Also it is proved in [9] that if X and $\hat{X}$ are dual processes then X has a Lévy system without any additional assumption. Thus we may assume that both X and $\hat{X}$ have Lévy systems (N, H) and $(\hat{N}, \hat{H})$ respectively, and the main purpose of this section is to investigate the relationship between these systems. Of course, we write the action of $\hat{N}$ on $f \in \mathscr{E}_+$ as $f\hat{N}(x) = \int f(y)\hat{N}(dy, x)$ in keeping with our standard notational scheme. Finally it is known (see [2] or [7]) that there exists a unique σ-finite measure μ on $(E, \mathscr{E})$ which doesn't charge semipolar sets and such that

$$U_H^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dH_t = \int u^\alpha(x, y) f(y)\, \mu(dy)$$

for all $\alpha \geqq 0$ and $f \in \mathscr{E}_+$. The corresponding measure for $\hat{H}$ is written $\hat{\mu}$. In particular if the left (or right) side of (3.3) is finite for all t, then letting $A_t(\Phi) = \sum_{s \leqq t} \Phi(X_{s-}, X_s)$ it follows from (VII-T.17) of [3] that

$$E^x \int_0^\infty e^{-\alpha t}\, dA_t(\Phi) = \int u^\alpha(x, z) \int \Phi(z, y) N(z, dy)\, \mu(dz) \tag{3.4}$$

for each $\alpha \geqq 0$. Finally if A and $\hat{A}$ are in $\mathbf{A}_c$ and $\hat{\mathbf{A}}_c$ respectively we say that A and $\hat{A}$ are *dual* (additive functionals) provided $M_t = \exp(-A_t)$ and $\hat{M}_t = \exp(-\hat{A}_t)$ are dual MF's. It is known that A and $\hat{A}$ are dual if and only if $\mu = \hat{\mu}$ where μ and $\hat{\mu}$ are the measures corresponding to A and $\hat{A}$ respectively. See section 9 of [2] or [7].

We are now in a position to state the main result of this section.

(3.5) **Theorem.** *The Lévy systems (N, H) and $(\hat{N}, \hat{H})$ for X and $\hat{X}$ respectively may be chosen so that H and $\hat{H}$ are dual CAF's and such that if μ is the measure corresponding to H and $\hat{H}$, then N and $\hat{N}$ are dual kernels relative to μ, that is*

$$\int f(Ng)\, d\mu = \int (f\hat{N})\, g\, d\mu \qquad \text{for all } f, g \in \mathscr{E}_+.$$

Proof. Fix $\alpha > 0$. Let $(K(x, dy), J_t)$ and $(\hat{K}(dy, x), \hat{J}_t)$ by Lévy systems for X and $\hat{X}$ such that J and $\hat{J}$ have bounded α-potentials. Let ν and $\hat{\nu}$ denote the measures corresponding to J and $\hat{J}$ respectively.

Let B be a compact subset of $E \times E - D$ and let $T = T(B)$ be the exact terminal time defined in (2.1). Let $\varphi(x) = E^x\{e^{-\alpha T}\}$. Then $0 \leqq \varphi < 1$ and so $K_n = \{\varphi \leqq 1 - 1/n\} \uparrow E$ as $n \to \infty$. Let $C_n = B \cap (E \times K_n)$. Fix n and let $R = T(C_n)$. If $R_0, R_1, \ldots$ are the iterates of R, $X(R_k) \in K_n$ for $k \geqq 1$ and so $\sum_{k \geqq 1} E^x(e^{-\alpha R_k}) \leqq \sum_{k \geqq 1} (1 - 1/n)^{k-1} < \infty$. Therefore the additive functional $A_t = \sum_{R_k \leqq t} 1$ has a bounded α-potential. Similarly there exists an increasing sequence $\{L_n\}$ of Borel subsets of E whose union is E and such that if $F_n = \hat{B} \cap (E \times L_n)$ and $\hat{R} = \hat{T}(F_n)$ and if $(\hat{R}_k)$ and $\hat{A}$ are defined analogously to (R_k) and A, then $\hat{A}$ has a bounded α-potential. Let $\Gamma_n = B \cap (L_n \times K_n)$. Then $\Gamma_n \uparrow B$, $\hat{\Gamma}_n = \hat{B} \cap (K_n \times L_n)$, and the additive functionals $A_t(\Gamma_n) = \sum_{s \leqq t} I_{\Gamma_n}(X_{s-}, X_s)$ and $\hat{A}_t(\hat{\Gamma}_n) = \sum_{s \leqq t} I_{\Gamma_n}(\hat{X}_s, \hat{X}_{s-})$ have bounded α-potentials. Finally $E \times E - D$ is the union of an increasing sequence of compacts and so there exists an increasing sequence $\{\Gamma_n\}$ of Borel subsets of $E \times E - D$ whose union is $E \times E - D$ such that for each n, Γ_n is at a positive distance from D and such that $A(\Gamma_n)$ and $\hat{A}(\hat{\Gamma}_n)$ have bounded α-potentials.

We now fix n and let $\Gamma = \Gamma_n$, $T = T(\Gamma)$, and $\hat{T} = \hat{T}(\hat{\Gamma})$. By Theorem 2.2, T and $\hat{T}$ are dual exact terminal times. Let M and $\hat{M}$ be defined with respect to T and $\hat{T}$ as in Lemma 3.1 so that M and $\hat{M}$ are exact MF's. Also if $A_t = aA_t(\Gamma)$ and

$\hat{A}_t = a\hat{A}_t(\hat{\Gamma})$, then A and $\hat{A}$ have bounded α-potentials and

$$dA_t = -(M_{t-})^{-1}\, dM_t; \quad d\hat{A}_t = -(\hat{M}_{t-})^{-1}\, d\hat{M}_t. \tag{3.6}$$

Here a is the constant in Lemma 3.1.

Now by the definition of a Lévy system, or more exactly (3.4)

$$U_A^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dA_t = a E^x \sum_t e^{-\alpha t} I_\Gamma(X_{t-}, X_t) f(X_t)$$
$$= a \int u^\alpha(x, z) \int I_\Gamma(z, y) f(y) K(z, dy)\, \nu(dz),$$

for $f \in \mathscr{E}_+$. Let $L(x, dy) = a I_\Gamma(x, y) K(x, dy)$ for notational simplicity. Then the above becomes

$$U_A^\alpha f = U^\alpha(Lf \cdot \nu). \tag{3.7}$$

If we write $\hat{U}_A^\alpha$ for $\hat{U}_{\hat{A}}^\alpha$ and let $\hat{L}(dy, x) = a\hat{K}(dy, x) I_\Gamma(y, x)$, then the dual of (3.7) is

$$f\hat{U}_A^\alpha = (\hat{\nu} \cdot f\hat{L})\, \hat{U}^\alpha. \tag{3.8}$$

In (3.7) and (3.8) and in what follows we adopt the notation $h \cdot \nu$ for the measure $\gamma(dx) = h(x)\, \nu(dx)$ whenever $h \in \mathscr{E}_+$. Similarly we write $\hat{\nu} \cdot h$ for the measure $\hat{\gamma}(dx) = \hat{\nu}(dx)\, h(x)$.

Let (V^α) and $(\hat{V}^\alpha)$ be the resolvents corresponding to M and $\hat{M}$. Then a standard calculation (see [2] or [4]) using (3.6) and the duality of M and $\hat{M}$ yields

$$U_A^\alpha v^\alpha = P_M^\alpha u^\alpha = u^\alpha \hat{P}_M^\alpha = v^\alpha \hat{U}_A^\alpha.$$

Combining this and (3.7) we see that

$$\int u^\alpha(x, y)\, L V^\alpha f(y)\, \nu(dy) = U_A^\alpha V^\alpha f(x) = \int u^\alpha(x, y) \int \hat{P}_M^\alpha(dy, z) f(z)\, dz. \tag{3.9}$$

Using the uniqueness theorem for potentials of measures [1; VI-1.15] this yields for $f \in b\mathscr{E}_+$

$$LV^\alpha f \cdot \nu = \int \hat{P}_M^\alpha(\cdot, z) f(z)\, dz, \tag{3.10}$$

and dually

$$\hat{\nu} \cdot (f\hat{V}^\alpha \hat{L}) = \int f(y)\, dy\, P_M^\alpha(y, \cdot). \tag{3.11}$$

Moreover (3.9) can be written as

$$\int u^\alpha(x, y) \int \hat{P}_M^\alpha(dy, z) f(z)\, dz = \iint u^\alpha(x, y)\, L v^\alpha(y, z) f(z)\, \nu(dy)\, dz$$

and so for each fixed x

$$u^\alpha \hat{P}_M^\alpha(x, z) = \int u^\alpha(x, y)\, L v^\alpha(y, z)\, \nu(dy) \tag{3.12}$$

almost everywhere in z. But the left side of (3.12) is α-coexcessive as a function of z and hence cofinely continuous while the right side is $\alpha - (\hat{X}, \hat{M})$ excessive, as a function of z and hence cofinely continuous ($\hat{E}_M = E$). Therefore (3.12) holds identically. Similarly

$$P_M^\alpha u^\alpha(x, z) = \int v^\alpha \hat{L}(x, y)\, u^\alpha(y, z)\, \hat{\nu}(dy). \tag{3.13}$$

Now fix $f, g \in \mathbf{C}_K^+$. Then

$$(f, P_M^\alpha U^\alpha g) = (f \hat{U}^\alpha \hat{P}_M^\alpha, g),$$

and using (3.10) and (3.11) this equality can be written

$$\int (f \hat{V}^\alpha \hat{L}) U^\alpha g \, d\hat{\nu} = \int f \hat{U}^\alpha (L V^\alpha g) d\nu. \tag{3.14}$$

Now from (3.12)

$$f \hat{V}^\alpha(z) = f \hat{U}^\alpha(z) - f \hat{U}^\alpha \hat{P}_M^\alpha(z) = f \hat{U}^\alpha(z) - \int f \hat{U}^\alpha(y) L v^\alpha(y, z) \nu(dy).$$

But $|f \hat{U}^\alpha| \leqq 1/\alpha \|f\|$ and so by (3.8)

$$\int (f \hat{U}^\alpha \hat{L}) U^\alpha g \, d\hat{\nu} \leqq \alpha^{-1} \|f\| ((\hat{\nu} \cdot 1\hat{L}) \hat{U}^\alpha, g) = \alpha^{-1} \|f\| (1 \hat{U}_A^\alpha, g) < \infty,$$

because $1 \hat{U}_A^\alpha$ is bounded and $g \in \mathbf{C}_K^+$. Therefore we can substitute the above expression for $f \hat{V}^\alpha$ into the left side of (3.14) to obtain

$$\int (f \hat{V}^\alpha \hat{L}) U^\alpha g \, d\hat{\nu} = \int (f \hat{U}^\alpha \hat{L}) U^\alpha g \, d\hat{\nu} - \iint f \hat{U}^\alpha(x) L v^\alpha \hat{L}(x, y) U^\alpha g(y) \nu(dx) \hat{\nu}(dy),$$

where of course,

$$L v^\alpha \hat{L}(x, y) = \iint L(x, dz) v^\alpha(z, w) \hat{L}(dw, y).$$

Similarly the right side of (3.14) becomes

$$\int f \hat{U}^\alpha (L V^\alpha g) d\nu = \int f \hat{U}^\alpha (L U^\alpha g) d\nu - \iint f \hat{U}^\alpha(x) L v^\alpha \hat{L}(x, y) U^\alpha g(y) \nu(dx) \hat{\nu}(dy).$$

Consequently

$$\int (f \hat{U}^\alpha \hat{L}) U^\alpha g \, d\hat{\nu} = \int f \hat{U}^\alpha (L U^\alpha g) \, d\nu,$$

and using the uniqueness theorem for potentials twice we obtain

$$\hat{L}(dx, y) \hat{\nu}(dy) = \nu(dx) L(x, dy) \tag{3.15}$$

as measures on $E \times E$. Perhaps we should point out that the measures on $E \times E$ defined in (3.15) are σ-finite. This follows readily from the σ-finiteness of ν and $\hat{\nu}$ and the fact that L and $\hat{L}$ are proper kernels as defined above (3.3).

Recalling the definition of L and $\hat{L}$ and the fact that $\Gamma = \Gamma_n$ for a fixed n, we see that

$$\hat{K}(dx, y) \hat{\nu}(dy) = \nu(dx) K(x, dy) \tag{3.16}$$

as measures on $E \times E - D$ because $E \times E - D$ is the increasing union of $\{\Gamma_n\}$. But then (3.16) holds on $E \times E$ since neither side of (3.16) charges D. As before the measures defined in (3.16) are σ-finite on $E \times E$.

Eq. (3.16) is the relationship that must hold between arbitrary Lévy systems for X and $\hat{X}$. Now we can modify the given Lévy systems to obtain Theorem 3.5. Neither ν nor $\hat{\nu}$ charges semipolar sets, and $U^\alpha \nu$ and $\hat{\nu} \hat{U}^\alpha$ are bounded. Now it is easy to find a measure $\hat{\lambda} \leqq \hat{\nu}$ which is equivalent to $\hat{\nu}$ and such that $U^\alpha \hat{\lambda}$ is bounded. See [6]. Similarly let $\lambda \leqq \nu$ be a measure equivalent to ν such that $\lambda \hat{U}^\alpha$ is bounded. Now let $\mu = \lambda + \hat{\lambda}$. Then μ is equivalent to $\nu + \hat{\nu}$ and both $U^\alpha \mu$ and $\mu \hat{U}^\alpha$ are bounded. Obviously μ doesn't charge semipolar sets. Hence $\nu = \varphi \mu$ and $\hat{\nu} = \hat{\varphi} \mu$ where φ

and $\hat{\varphi}$ are finite nonnegative Borel functions. Define

$$N(x, dy) = \varphi(x) K(x, dy); \qquad \hat{N}(dy, x) = \hat{K}(dy, x)\, \hat{\varphi}(x).$$

If $f, g \in \mathscr{E}_+$, then

$$\begin{aligned}\int f(Ng)\, d\mu &= \iint f(x) K(x, dy)\, g(y)\, \nu(dx)\\ &= \iint f(x) \hat{K}(dx, y)\, g(y)\, \hat{\nu}(dy) = \int (f\hat{N})\, g\, d\mu\end{aligned}$$

and so N and $\hat{N}$ are dual kernels with respect to μ. Finally since μ doesn't charge semipolars and $U^\alpha \mu$ and $\mu \hat{U}^\alpha$ are bounded, there exist CAF's H and $\hat{H}$ of X and $\hat{X}$ respectively such that $U_H^\alpha f = U^\alpha(f\mu)$ and $f\hat{U}_H^\alpha = (f\mu)\hat{U}^\alpha$ for all $f \in \mathscr{E}_+^*$. Consequently, $U_J^\alpha 1 = U^\alpha \nu = U^\alpha(\varphi\mu) = U_H^\alpha \varphi$, and so $J = \varphi H$. Similarly $\hat{J} = \hat{\varphi} \hat{H}$. It is now clear that (N, H) and $(\hat{N}, \hat{H})$ are Lévy systems for X and $\hat{X}$, and that H and $\hat{H}$ are dual CAF's with correponding measure μ. This completes the proof of Theorem 3.5.

4. Additive Functionals

Let $\mathbf{A}^*$ and $\hat{\mathbf{A}}^*$ denote the collection of additive functionals of X and $\hat{X}$ respectively which have no infinite discontinuity. In particular $\mathbf{A} \subset \mathbf{A}^*$ and $\hat{\mathbf{A}} \subset \hat{\mathbf{A}}^*$. It is known that any $A \in \mathbf{A}^*$ may be written uniquely as

$$A = A^c + A^n + A^q \tag{4.1}$$

where A^c is continuous, A^n is a pure jump natural AF, and A^q is a pure jump AF which is quasi-left-continuous (q.l.c.) in the sense that every discontinuity of $t \to A_t^q$ is also a discontinuity of $t \to X_t$[1]. (An AF, B is *pure jump* if $B_t = \sum_{s \leq t} \Delta B_s$ where $\Delta B_s = B_s - B_{s-}$.) In [7] and [8] Revuz has given a complete description of the continuous and natural pure jump elements of $\mathbf{A}^*$. This section contains an analogous discussion of the q.l.c. pure jump elements of $\mathbf{A}^*$. The results of this section together with those Revuz give a complete description of all AF's in $\mathbf{A}^*$.

Let A in $\mathbf{A}^*$ be a pure jump q.l.c. AF of X. Then it is known, at least if X is special standard, that there exists a finite nonnegative Borel function, Φ, on $E \times E$ vanishing on the diagonal, D, such that

$$A_t = \sum_{s \leq t} \Phi(X_{s-}, X_s). \tag{4.2}$$

See [5] or [10]. The fact that this representation holds in general (for dual processes) is contained in [9]. (As usual, the equality in (4.2) means equivalence.) We are now going to characterize those Φ such that (4.2) actually defines an AF – obviously any such AF is pure jump and q.l.c. Our discussion follows that of Revuz [8] very closely. In what follows (N, H) and $(\hat{N}, \hat{H})$ are Lévy systems for X and $\hat{X}$ satisfying the conditions of Theorem 3.5, and μ denotes the measure corresponding to H and $\hat{H}$. We fix an $\alpha > 0$ and assume, as we may, that H and $\hat{H}$ have bounded α-potentials.

[1] It is an immediate consequence of the quasi-left continuity of X that if (T_n) is an increasing sequence of stopping times with limit T, then $A^q(T_n) \to A^q(T)$. Conversely if X is standard and this condition holds, then A^q is q.l.c.

We begin by characterizing the bounded Φ such that (4.2) defines an AF.

(4.3) **Theorem.** *Let Φ be a bounded nonnegative Borel function on $E\times E$ which vanishes on the diagonal. Then A defined by* (4.2) *is an* AF *of X if and only if there exists an increasing sequence $\{E_n\}$ of Borel subsets of E whose union is E and such that:*

(i) $\int u^\alpha(x,z)\int \Phi(z,y)\, I_{E_n}(y)\, N(z,dy)\,\mu(dz)$ *is bounded and integrable for each n.*

(ii) *If T_n is the hitting time of E_n^c, then $T=\lim T_n>0$.*

In this case $\{E_n\}$ may be chosen so that $T\geqq R\wedge\zeta$ where $R=\inf\{t: A_t=\infty\}$.

Proof. Suppose first of all that A is an AF. By multiplying A by a constant we may assume without loss of generality that $\Phi<1$. Now $\Delta A_t=\Phi(X_{t-},X_t)$ for each t and since $\Phi(x,x)=0$, $\Delta A_t>0$ for at most countably many t. By our assumption on Φ, $\Delta A_t<1$ for all t. Define

$$M_t=\prod_{s\leqq t}(1-\Delta A_s).$$

Then M is a MF such that $dA_t=-(M_{t-})^{-1}\,dM_t$ and $R=\inf\{t: M_t=0\}=\inf\{t: A_t=\infty\}$. Clearly $R>0$ since A is an AF, and so all points are permanent for M. Let (V^α) be the resolvent of M, then $U_A^\alpha V^\alpha g=U^\alpha g-V^\alpha g$ if $g\in b\mathscr{E}$. See [4]. Now let f be a bounded, integrable, and strictly positive Borel function on E, and define $\varphi(x)=V^\alpha f(x)$. Then φ is strictly positive and finely continuous because $E_M=E$, and

$$U_A^\alpha\varphi=U_A^\alpha V^\alpha f=U^\alpha f-V^\alpha f\leqq U^\alpha f\leqq\alpha^{-1}\|f\|.$$

Let $E_n=\{\varphi\geqq 1/n\}$. Then each E_n is a finely closed Borel set and $E_n\uparrow E$. Also $U_A^\alpha(x,E_n)\leqq n\,U_A^\alpha\varphi(x)\leqq n\,U^\alpha f(x)$. Therefore $U_A^\alpha(\cdot,E_n)$ is bounded and it is integrable because $(1,U^\alpha f)=(1\,\hat U^\alpha,f)\leqq\alpha^{-1}(1,f)<\infty$. But

$$\begin{aligned}U_A^\alpha(x,E_n)&=E^x\int_0^\infty e^{-\alpha t}\,I_{E_n}(X_t)\,dA_t\\&=E^x\sum_t e^{-\alpha t}\,\Phi(X_{t-},X_t)\,I_{E_n}(X_t)\qquad(4.4)\\&=\int u^\alpha(x,z)\int\Phi(z,y)\,I_{E_n}(y)\,N(z,dy)\,\mu(dz),\end{aligned}$$

and so the sequence $\{E_n\}$ satisfies (i). As for (ii), observe that if $\varphi(x)>1/n$, then $P^x(T_n>0)=1$ since x is not regular for $E_n^c=\{\varphi<1/n\}$. But $E=\bigcup\{\varphi>1/n\}$ and so $P^x(T>0)=1$ for all x.

Let us show next that $T\geqq R\wedge\zeta$. Since $\varphi(X_{T_n})\leqq 1/n$ we have with R defined in the preceding paragraph

$$\begin{aligned}1/n&\geqq E^x\{e^{-\alpha T_n}M_{T_n}\varphi(X_{T_n});\ T_n<R\}\\&=E^x\left\{\int_{T_n}^R e^{-\alpha t}M_t f(X_t)\,dt;\ T_n<R\right\},\end{aligned}$$

and letting $n\to\infty$ we obtain

$$E^x\left\{\int_T^R e^{-\alpha t}f(X_t)\,M_t\,dt;\ \Lambda\right\}=0$$

where $\Lambda=\{T_n<R$ for all $n\}$. But f is strictly positive and so $T\geqq R\wedge\zeta$.

Conversely suppose there exists a sequence $\{E_n\}$ satisfying (i) and (ii). Actually in the following argument we will assume only (ii) and a weaker version of (i):

$$(4.3\,\mathrm{i}') \qquad \int u^\alpha(x,z)\int \Phi(z,y)\,I_{E_n}(z)\,I_{E_n}(y)\,N(z,dy)\,\mu(dz)<\infty$$

for each n and x. Define

$$A_t^n=\sum_{s\leqq t}\Phi(X_{s-},X_s)\,I_{E_n}(X_{s-})\,I_{E_n}(X_s).$$

Then $E^x\int_0^\infty e^{-\alpha t}\,dA_t^n$ is just the integral appearing in (4.3 i′) which is finite by assumption. Therefore each A^n is an AF of X with a finite α-potential. Obviously $A_t=\lim_n A_t^n$. According to the remark at the bottom of p. 59 of [1]

$$T_n=\inf\{t>0\colon\ X_t\in E_n^c \text{ or } X_{t-}\in E_n^c\},$$

and hence, $A_t=A_t^n$ if $t<T_n$. Therefore $A_t<\infty$ if $t<T$, and since $T>0$ it follows readily that A is an AF. See, for example, Lemma III.2 of [8]. This completes the proof of Theorem 4.3.

It is natural to ask if Φ is such that (4.2) defines an AF of X, then does $\hat{A}_t=\sum_{s\leqq t}\Phi(\hat{X}_s,\hat{X}_{s-})$ define an AF of $\hat{X}$. We answer this question for bounded Φ before passing to the discussion of general Φ.

(4.5) **Theorem.** *Let Φ be as in the statement of* (4.3) *and let A be defined by* (4.2).

(a) *If A is an AF of X which is finite on* $[0,\zeta)$, *then there exists a polar set P such that $\hat{A}_t=\sum_{s\leqq t}\Phi(\hat{X}_s,\hat{X}_{s-})$ is an AF of $\hat{X}$ restricted to $E-P$ which is finite on* $[0,\hat{\zeta})$.

(b) *Suppose that semipolar sets are polar and that A is an AF of X. Then there exists a polar set P such that $\hat{A}$ is an AF of $\hat{X}$ restricted to $E-P$.*

Proof. We begin with a lemma that is of some interest in itself. It is closely related to facts contained implicitly in [1] and [8]

(4.6) **Lemma.** *There exists an increasing sequence $\{B_n\}$ of Borel subsets of E whose union is E such that if $\hat{S}_n$ is the hitting time of B_n^c by $\hat{X}$, then $\hat{S}_n\uparrow\hat{\zeta}$, and with the property that for any measure v with $U^\alpha v$ integrable, $v(B_n)<\infty$ for each n.*

Proof. Let $B_n=\{1\,\hat{U}^\alpha>1/n\}$. Clearly $B_n\uparrow E$. If v is a measure with $U^\alpha v$ integrable, then

$$v(B_n)\leqq n\int_{B_n}(1\,\hat{U}^\alpha)\,dv\leqq n(1,U^\alpha v)<\infty.$$

Let $\hat{\varphi}=1\,\hat{U}^\alpha$. Then $\hat{\varphi}(\hat{X}_{\hat{S}_n})\leqq 1/n$ and so

$$1/n\geqq\hat{E}^x\{e^{-\alpha S_n}\,\hat{\varphi}(X_{S_n});\ S_n<\zeta\}=\hat{E}^x\left\{\int_{S_n}^{\zeta}e^{-\alpha t}\,dt;\ S_n<\zeta\right\}.$$

Now letting $n\to\infty$ we see that $\hat{S}_n\uparrow\hat{\zeta}$.

We return now to the proof of Theorem 4.5. Assume that A is an AF of X that is finite on $[0,\zeta)$. Let $\{E_n\}$ be the sequence mentioned in Theorem 4.3. Define

$$v_n(dz)=\left[\int\Phi(z,y)\,I_{E_n}(y)\,N(z,dy)\right]\mu(dz).$$

Then $U^\alpha v_n$ is just the integral appearing in (4.3 i), and so it is bounded and integrable. Let $\{B_n\}$ be the sequence in Lemma 4.6 and let $C_n = B_n \cap E_n$. Then $C_n \uparrow E$, and $v_n(C_n) < \infty$ for each n. Next define

$$\hat{v}_n(\Gamma) = \int_{C_n} \mu(dz) \int_{\Gamma} \Phi(z, y)\, I_{C_n}(y)\, N(z, dy).$$

Clearly $\hat{v}_n$ is a measure carried by C_n and $\hat{v}_n(C_n) \leqq v_n(C_n) < \infty$. Therefore

$$(\hat{v}_n \hat{U}^\alpha, 1) = (\hat{v}_n, U^\alpha 1) \leqq \alpha^{-1} \hat{v}_n(C_n) < \infty$$

and so $\hat{v}_n \hat{U}^\alpha$ is integrable and hence finite except on a polar set P_n. Therefore writing $\hat{\Phi}(x, y) = \Phi(y, x)$,

$$\begin{aligned} \int \hat{v}_n(dy)\, u^\alpha(y, x) &= \iint \mu(dz)\, I_{C_n}(z)\, I_{C_n}(y)\, \Phi(z, y)\, N(z, dy)\, u^\alpha(y, x) \\ &= \iint \mu(dy)\, I_{C_n}(z)\, I_{C_n}(y)\, \hat{\Phi}(y, z)\, \hat{N}(dz, y)\, u^\alpha(y, x) \end{aligned}$$

is finite provided $x \notin P_n$. But this last integral is just the dual of (4.3 i′) with respect to the sequence $\{C_n\}$.

Let $\hat{S}_n$ and $\hat{T}_n$ be the hitting times of B_n^c and E_n^c respectively by $\hat{X}$. We know that $\hat{S}_n \uparrow \hat{\zeta}$. We claim that there exists a polar set Q such that $\hat{T}_n \uparrow \hat{\zeta}$ almost surely $\hat{P}^x$ for $x \notin Q$. Let us assume this for the moment and use it to complete the proof of (4.5 a). Let $P = Q \cup (\bigcup_n P_n)$. If $\hat{R}_n$ is the hitting time of $C_n^c = E_n^c \cup B_n^c$, then $\hat{R}_n = \min(\hat{T}_n, \hat{S}_n) \uparrow \hat{\zeta}$ almost surely $\hat{P}^x$ for $x \in E - P$. Consequently $\hat{\Phi}$ and $\{C_n\}$ satisfy the conditions (4.3 i′) and (4.3 ii) relative to $\hat{X}$ restricted $E - P$, and so

$$\hat{A}_t = \sum_{s \leqq t} \hat{\Phi}(\hat{X}_{s-}, \hat{X}_s) = \sum_{s \leqq t} \Phi(\hat{X}_s, \hat{X}_{s-})$$

is an AF of $\hat{X}$ restricted to $E - P$ which is finite on $[0, \hat{\zeta})$.

To complete the proof of (4.5 a) it remains to show that $\hat{T} = \lim \hat{T}_n = \hat{\zeta}$ almost surely $\hat{P}^x$ for x not in some polar set Q. By assumption $P^x(T = \zeta) = 1$ for all x, and the desired conclusion now follows by the argument on p. 281 of [1]. See also the proof of Theorem IV.1 of [8].

To establish (4.5 b) it suffices in view of the above to show that $\hat{P}^x(\hat{T} > 0) = 1$ for x not in some polar set under the assumptions of (4.5 b). But $T_n \leftrightarrow \hat{T}_n$ for each n, and so $T \leftrightarrow \hat{T}$. Now T is exact and $P^x(T > 0) = 1$ for all x, but $\hat{T}$ need not be exact. Let $\hat{T}^*$ be the exact regularization of $\hat{T}$. See section 6 of [2]. Thus T and $\hat{T}^*$ are dual exact terminal times, and by Theorem 3.2 of [2], $\hat{P}^x(\hat{T}^* > 0) = 1$ for x not in some semipolar, and hence polar, set. Let f be a bounded strictly positive Borel function. Then

$$f \hat{U}^\alpha \hat{P}^\alpha_{T_n} \downarrow f \hat{U}^\alpha \hat{P}^\alpha_T \geqq f \hat{U}^\alpha \hat{P}^\alpha_{T^*}, \tag{4.7}$$

and it is immediate that $f \hat{U}^\alpha \hat{P}^\alpha_{T^*}$ is the regularization of $f \hat{U}^\alpha \hat{P}^\alpha_T$. Therefore by Doob's Theorem and the assumption of (4.5 b) the last two terms in (4.7) agree except on a polar set. Consequently $\hat{P}^x(\hat{T} = \hat{T}^*) = 1$ except for x in a polar set, and so $\hat{P}^x(\hat{T} > 0) = 1$ except for x in a polar set. This completes the proof of Theorem 4.5.

The case in which Φ is unbounded is reduced to the bounded case by the following result. Although the proof is exactly the same as in [8] we include it for completeness. In particular, it follows that the assumption that Φ is bounded may be dropped in the hypotheses of Theorem 4.5.

(4.8) **Proposition.** *Let Φ be a nonnegative finite Borel function which vanishes on the diagonal. Let $\Psi = 1 - e^{-\Phi}$. Then $A_t = \sum_{s \leqq t} \Phi(X_{s-}, X_s)$ is an AF if and only if $B_t = \sum_{s \leqq t} \Psi(X_{s-}, X_s)$ is, and when this is the case*

$$\inf\{t\colon A_t = \infty\} = \inf\{t\colon B_t = \infty\}.$$

Proof. Obviously $0 \leqq \Psi < 1$. Let $M_t = \prod_{s \leqq t} [1 - \Psi(X_{s-}, X_s)]$. Then $A_t = -\log M_t$. If B is an AF, then M is a MF and $R = \inf\{t\colon M_t = 0\} = \inf\{t\colon B_t = \infty\} > 0$. Consequently A is an AF and $R = \inf\{t\colon A_t = \infty\}$. If A is an AF, then so is B because $1 - e^{-t} \leqq t$ for $t \geqq 0$.

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