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A Limit Theorem Related to a New Class of Self Similar Processes

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To Leo Schmetterer on his 60^{th} anniversary

Summary. We study partial sums of a stationary sequence of dependent random variables of the form $W_n = \sum_1^n \xi(S_k)$. Here $S_k = X_1 + \ldots + X_k$ where the X_i are i.i.d. integer valued, and $\xi(n)$, $n \in \mathbb{Z}$ are also i.i.d. and independent of the X's. It is assumed that the X's and ξ's belong to the domains of attraction of different stable laws of indices $1 < \alpha \leqq 2$ and $0 < \beta \leqq 2$. It is shown that for some $\delta > \frac{1}{2}$, $n^{-\delta} W_{[nt]}$ converges weakly as $n \to \infty$ to a self similar process with stationary increments, which depends on α and β. The constant δ is related to α and β via $\delta = 1 - \alpha^{-1} + (\alpha\beta)^{-1}$.

1. Introduction

As a starting point for our theory consider the following simple problem of "random walk in random scenery"[1]. Let $S_n = X_1 + X_2 + \ldots + X_n$, $n \geqq 0$ be simple random walk on $\mathbb{Z}$, starting at $S_0 = 0$, and define the random scenery $\xi(x)$, $x \in \mathbb{Z}$, as a sequence of i.i.d. random variables (independent also of the random walk), taking the values ± 1 with probability 1/2. Our problem then concerns the asymptotic behavior, as $n \to \infty$, of the "cumulative random scenery" defined by

$$W_n = \sum_{k=0}^{n} \xi(S_k) = \sum_{x \in \mathbb{Z}} \xi(x) N_n(x). \tag{1.1}$$

Here $N_n(x)$ is the number of visits of the random walk to the point x in the time interval $[0, n]$. The first sum in (1.1) exhibits W_n as the n-th partial sum of a stationary sequence of random variables with mean 0. Next we calculate its variance.

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[1] For this name we thank Paul Shields. It suggests less interaction between the random walk and its surroundings, than is present in the model known under the name "random walk in random environment" [11]

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$$\begin{aligned}\sigma^2(W_n) &= E\Big(\sum_{x\in\mathbb{Z}} \xi(x)\,N_n(x)\Big)^2\\ &= E\sum_{x\in\mathbb{Z}} N_n^2(x) = E\sum_{x\in\mathbb{Z}}\Big(\sum_{k=0}^{n} I[S_k=x]\Big)^2\\ &= E\sum_{x\in\mathbb{Z}}\sum_{k=0}^{n}\sum_{l=0}^{n} I[S_k=S_l=x] = E\sum_{k=0}^{n}\sum_{l=0}^{n} I[S_k=S_l]\\ &= \sum_{k=0}^{n}\sum_{l=0}^{n} P[S_{|k-l|}=0].\end{aligned} \tag{1.2}$$

Since $P[S_{2n}=0]\sim(\pi n)^{-\frac{1}{2}}$ as $n\to\infty$, one obtains the asymptotic behavior $\sigma^2(W_n)\sim\text{constant}\cdot n^{\frac{3}{2}}$. That suggests looking for the limiting distribution of $W_n/n^{\frac{3}{4}}$, and finally for a weak limit of the sequence of stochastic processes

$$D_t^n = n^{-\frac{3}{4}}\, W_{nt}, \qquad t\geqq 0, \qquad n=1,2,3,\dots \tag{1.3}$$

where W_s is defined as the linear interpolation

$$W_s = W_n + (s-n)(W_{n+1}-W_n) \qquad \text{when } n\leqq s\leqq n+1. \tag{1.4}$$

We shall show that $\{D_t^n\}$ indeed converges weakly, in $C[0,\infty)$, to a process Δ_t, $t\geqq 0$, defined by (1.5) below. Intuitively this process may be described as the process obtained from the random walk in a random scenery when $\mathbb{Z}$ is changed into $\mathbb{R}$, the simple random walk $\{S_n\}$ into a Brownian motion $\{b_t\}$, and the random scenery $\{\xi(x)\}$ into a white noise process $w(x)$, independent of $\{b_t\}$. Then intuitively

$$\Delta_t = \int_0^t w(b_s)\,ds, \qquad t\geqq 0,$$

which does not make sense. But imitating the last term in (1.1), and replacing $N_n(x)$ by $L_t(x)$, the local time at x of the Brownian motion b_t, we get

$$\Delta_t = \int_{-\infty}^{\infty} L_t(x)\,dZ(x), \qquad t\geqq 0. \tag{1.5}$$

This does make sense as a stochastic integral if $Z(x)$ is a Brownian motion with time $-\infty<x<\infty$ (or we can use a pair of independent Brownian motions $Z_+(x)$, $Z_-(x)$, $x\geqq 0$, to write the integral (1.5) as in (1.21) below).

The asymptotic behavior $\sigma^2(W_n)\sim\text{const}\cdot n^{\frac{3}{2}}$ suggests strong dependence between remote terms of the stationary sequence $\xi(S_k)$, $k\geqq 0$. Therefore it is no surprise that the limit process Δ_t is *not Gaussian*. (As we shall see the distribution of Δ_t for each t is a convex combination of Gaussian distributions with mean 0 and variance σ^2 – the probability measure governing σ^2 being that of the integral $\int_{-\infty}^{\infty} L_t^2(x)\,dx$.) On the other hand Δ_t evidently has the following two properties:

(i) Δ_t *has stationary increments;*

(ii) Δ_t *is self-similar, i.e. there is an index* $\delta > 0$ *such that* Δ_{ct} *is equivalent to the process* $c^\delta \Delta_t, t \geqq 0$

Under the influence of questions in theoretical physics there is presently great interest in finding all processes with the two properties (i) and (ii). If one replaces the simple random walk S_n by another random walk, and the random scenery $\xi(x)$ by another sequence of i.i.d. random variables, and if in this new set up the analogue of D_t^n again converges, then the limit will automatically have properties (i) and (ii) (cf. Lamperti [12]). By carrying out this program we shall obtain a large class of self similar processes Δ_t with all indices of similarity δ in the range $\frac{1}{2} < \delta < \infty$. These do not seem to be contained in the lists of Lamperti [12], Dobrushin [4], Dobrushin and Major [5], or Taqqu [16].

For the random walk we shall take the X_i satisfying

$$EX_i = 0, \tag{1.6}$$

$$P[n^{-\frac{1}{\alpha}} S_n \leqq x] \to F_\alpha(x), \tag{1.7}$$

where F_α is a stable distribution with index $1 < \alpha \leqq 2$. F_α is not necessarily symmetric, but in view of (1.6) it has zero mean. When $\alpha < 2$ its characteristic function must be of the form

$$\varphi(\theta) = \exp[-|\theta|^\alpha (C_1 + i C_2 \operatorname{sgn} \theta)] \tag{1.8}$$

for some $0 < C_1 < \infty$, $|C_1^{-1} C_2| \leqq \tan \frac{\pi}{2} \alpha$. From the known characterization of the domain of attraction of F_α (Gnedenko-Kolmogorov, Th. 35.2 [9] or Feller, II, Chap. 17 [6]) it follows that, for $\alpha < 2$, (1.7) and (1.8) are equivalent to (1.6) and

$$\lim_{\rho \to \infty} \rho^\alpha P[X_1 \geqq \rho] = D_1$$

$$\lim_{\rho \to \infty} \rho^\alpha P[X_1 \leqq -\rho] = D_2, \tag{1.9}$$

where

$$C_1 = (D_1 + D_2) \int_0^\infty \frac{\sin t}{t^\alpha} dt$$

$$C_2 = (D_1 - D_2) \int_0^\infty \frac{1 - \cos t}{t^\alpha} dt. \tag{1.10}$$

In turn (1.6), (1.9), and (1.10) are equivalent to

$$1 - \psi(\theta) \sim |\theta|^\alpha (C_1 + i C_2 \operatorname{sgn} \theta), \qquad \theta \to 0, \tag{1.11}$$

where ψ is the characteristic function of X_1. When $\alpha = 2$ of course $D_1 = D_2 = 0$ in (1.9) while $C_2 = 0$ in (1.11).

Concerning the random scenery $\{\xi(x)\}$, $x \in \mathbb{Z}$, we shall assume that

$$\lim_{n\to\infty} P\left[n^{-\frac{1}{\beta}} \sum_{k=1}^{n} \xi(k) \leqq x\right] = G_\beta(x), \tag{1.12}$$

where G_β is stable of index $0<\beta\leqq 2$, and with characteristic function

$$\zeta(\theta) = \exp[-|\theta|^\beta (A_1 + i A_2 \operatorname{sgn}\theta)] \tag{1.13}$$

for some $0<A_1<\infty$, $|A_1^{-1} A_2| \leqq \tan\frac{\pi}{2}\beta$. Note that (1.12) and (1.13) imply

$$E[\xi(x)] = 0 \quad \text{if } \beta > 1. \tag{1.14}$$

For $\beta=1$ we impose an additional symmetry condition (stronger than (1.12) and (1.13)), namely that for some K

$$|E[\xi(x); |\xi(x)| \leqq \rho]| \leqq K < \infty \quad \text{for all } \rho > 0. \tag{1.15}$$

Note that just as in (1.9), if $\beta \neq 1, 2$, then (1.12) and (1.13) are equivalent to

$$\begin{aligned} &\lim_{\rho\to\infty} \rho^\beta P[\xi(0) \geqq \rho] = B_1 \\ &\lim_{\rho\to\infty} \rho^\beta P[\xi(0) \leqq -\rho] = B_2 \end{aligned} \tag{1.16}$$

for suitable B_1, B_2. If $\beta=1$, then (1.12) and (1.15) imply

$$\lim_{\rho\to\infty} \rho P[\xi(0) \geqq \rho] = \lim_{\rho\to\infty} \rho P[\xi(0) \leqq -\rho] = \frac{1}{\pi} A_1. \tag{1.17}$$

Finally, if $\lambda(\theta) = E\exp[i\theta\xi(x)]$, then (1.12) and (1.13) are equivalent to

$$1 - \lambda(\theta) \sim |\theta|^\beta (A_1 + i A_2 \operatorname{sgn}\theta), \quad \theta \to 0. \tag{1.18}$$

It would be possible to weaken the above hypotheses concerning the random walk and the random scenery by allowing slowly varying functions in (1.7) and (1.12). However the computation then becomes much more elaborate while no new limiting processes would be obtained.

To describe the results consider two right continuous stable processes $\{Z_+(t); t \geqq 0\}$ and $\{Z_-(t); t \geqq 0\}$ both with characteristic functions

$$E[e^{i\theta Z_\pm(t)}] = \exp[-t|\theta|^\beta (A_1 + i A_2 \operatorname{sgn}\theta)]. \tag{1.19}$$

Let $\{Y(t); t \geqq 0\}$ be a right continuous stable process of index α with characteristic function

$$E[e^{i\theta Y(t)}] = \exp[-t|\theta|^\alpha (C_1 + i C_2 \operatorname{sgn}\theta)]. \tag{1.20}$$

We assume these processes to be defined on one probability space, and to be independent of each other. Then $Z_\pm(t)$ is also independent of $L_t(x)$, the *local time* at x of $Y(\cdot)$. Since $x \to L_t(x)$ is continuous with probability one ([2, 7]) and $Z_\pm(x)$ is a semimartingale ([13] Sect. IV 15) the stochastic integrals

$$\Delta_t = \int_0^\infty L_t(x)\, dZ_+(x) + \int_0^\infty L_t(-x)\, dZ_-(x) \tag{1.21}$$

can be defined as in [13], Chap. IV. (see also proof of Lemma 5). These integrals will turn out to be the proper generalizations of (1.5).

Now let W_n be the cumulative sums in (1.1) and W_t for real $t \geqq 0$ the process defined in (1.4). Select a random walk and random scenery subject to the assumptions in (1.6) through (1.18), and define $Z_{\pm}(\cdot)$, $Y(\cdot)$ and hence $\{\Delta_t\}$ with the parameters α, β, A_1, A_2, C_1, C_2 in (1.6)–(1.18). Then with

$$D_t^n = n^{-\delta} W_{nt}, \quad \delta = 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta} \tag{1.22}$$

we shall prove the

Theorem 1.1. $\{D_t^n; t \geqq 0\}$ *converges weakly in* $C[0, \infty)$ *to the process* $\{\Delta_t; t \geqq 0\}$ *defined in* (1.21). *Thus this process has a continuous version which is of course self similar with index* δ, *and has stationary increments.*

Remarks. 1. In view of (1.22) we obtain a limiting process Δ_t for any δ in the semi-infinite interval $(\frac{1}{2}, \infty)$. But each δ can in general be obtained from many different pairs (α, β), and in general the limiting processes are different for different choices of parameters. E.g. it follows from (3.1) that

$$1 - E[e^{i\theta \Delta_t}] \sim |\theta|^\beta (A_1 + iA_2 \operatorname{sgn} \theta) E\left[\int_{-\infty}^{\infty} L_t^\beta(y)\, dy\right], \qquad \theta \to 0,$$

so that two processes Δ_t can be the same only if they correspond to the same β, A_2/A_1, and of course also the same δ. For $\beta \neq 1$ this means that they must correspond to the same α. For $\beta = 1$, however, $\delta = 1$ for all $\alpha \in (1, 2]$. In this case Δ_t *has exactly the same marginal distributions as a Cauchy process* η_t *with*

$$E[e^{i\theta\eta_t}] = \exp\{-t|\theta|(A_1 + iA_2 \operatorname{sgn} \theta)\}.$$

Indeed, assume for simplicity that the $\xi(x)$ have a Cauchy distribution. Then $W_n/(n+1)$ has exactly the same Cauchy distribution for each n, quite regardless of the distribution of the random walk S_n. Nevertheless Δ_t cannot be a Cauchy process, because Δ_t has a continuous version. We believe that Δ_t processes with $\beta = 1$ but different α are different processes, even though their one-dimensional marginals coincide.

2. Until now we avoided discussing the situation when the random walk is asymptotically stable of index α in the range $0 < \alpha \leqq 1$. The case $\alpha = 1$ is the most difficult. Let $\beta = 2$ and $\alpha = 1$, in fact let $\{X_k\}$ be i.i.d. symmetric Cauchy random variables. Then we believe that

$$D_t^n = \frac{1}{\sqrt{n \log n}} W_{nt} \Rightarrow \Delta_t \tag{1.23}$$

where Δ_t is ordinary Brownian motion. A similar result should hold when $W_n = \sum_0^n \xi(\vec{S}_k)$, where $\vec{S}_k$ is simple random walk in the planar lattice $\mathbb{Z}_2$.

3. The case $0 < \alpha < 1$, however, is easy and has been treated before (see [14], p. 53, problems 14, 15, for the case $\xi(x) = \pm 1$ with probability $\frac{1}{2}$, and S_n any transient random walk on $\mathbb{Z}$. Indeed we get, when $\alpha < 1$, β arbitrary,

$$\chi_n(t) \equiv n^{-\frac{1}{\beta}} \sum_{k=1}^{[nt]} \xi(S_k) \Rightarrow \text{a stable process with index } \beta. \tag{1.24}$$

Here is a sketch of the proof that the limiting characteristic function of $\chi_n(t)$ in (1.24) is $\exp[-t|\theta|^\beta c(A_1+iA_2 \operatorname{sgn}\theta)]$ where c is a positive constant depending on β and on the random walk S_n. To simplify calculations we assume that the random scenery $\xi(x)$ has exactly the characteristic function $\exp[-|\theta|^\beta]$. (A similar argument will show that the increments of $\chi_n(t)$ are independent in the limit, so that (1.24) holds.)

We may write

$$\chi_n(t) = n^{-\frac{1}{\beta}} \sum_x N_{[nt]}(x)\, \xi(x),$$

$$E \exp[i\chi_n(t)] = E \exp\left[-\frac{|\theta|^\beta}{n} \sum_x N_{[nt]}^\beta(x)\right].$$

Therefore it will suffice to show that

$$\frac{1}{n} \sum_x N_{[nt]}^\beta(x) \to c\,t \qquad \text{a.s.}$$

This follows from the identity

$$\frac{1}{n} \sum_x N_{[nt]}^\beta(x) = \frac{1}{n} \sum_{k=0}^{[nt]} N_{[nt]}^{\beta-1}(S_k), \tag{1.25}$$

and the fact that we can apply Birkhoff's ergodic theorem to a slight modification of the right hand side in (1.25). Replace $N_k(x)$ by $N_\infty(x)$ defined as the total number of visits to x of a random walk S_n, with time $-\infty < n < \infty$, $S_0 = 0$, and $\{S_k, k \leqq -1\}$ the reversed random walk of $\{S_k, k \geqq 1\}$. Then, in view of $\alpha < 1$, which implies that the random walk is transient, it can be shown that the error due to the replacement of $N_{[nt]}(S_k)$ by $N_\infty(S_k)$ contributes nothing in the limit. (See [14], pp. 38–40 for similar calculations.) The sequence $N_\infty(S_k)$ is stationary and ergodic. Hence the limit of (1.25) is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{[nt]} N_\infty^{\beta-1}(S_k) = t E N_\infty^{\beta-1}(0) = ct.$$

In the next section we derive some useful information concerning the asymptotic behavior of the sequence of occupation times $N_n(x)$, as $n \to \infty$. Then, in the last section, this information will be used to complete the proof of Theorem 1.1.

2. Properties of Occupation Times

Let $Y(t)$ be a right continuous stable process of index α with characteristic function

$$E[e^{i\theta Y(t)}]=\exp[-t|\theta|^{\alpha}(C_1+iC_2\operatorname{sgn}\theta)]. \tag{2.1}$$

Then the process

$$n^{-\frac{1}{\alpha}}S_{[nt]}, \qquad t\geqq 0 \tag{2.2}$$

converges weakly in $D([0,\infty))$ to the process $Y(t)$, $t\geqq 0$, in (1.20) (see [8], Theorem 2, p. 480). We shall use this fact to derive the convergence in distribution of certain functionals of the process (2.2).

Consider the occupation times $N_n(x)$ of the random walk, as defined in (1.1) and their linear interpolation

$$N_s(x)=N_n(x)+(s-n)(N_{n+1}(x)-N_n(x)), \qquad n\leqq s\leqq n+1. \tag{2.3}$$

For $-\infty<a<b<\infty$ we set

$$T_t^n(a,b)=\frac{1}{n}\sum_{a\leqq n^{-\frac{1}{\alpha}}x<b} N_{nt}(x). \tag{2.4}$$

This is the fraction of time, during the time interval $[0,nt]$, that the process in (2.2) spends in the interval $[a,b)$. The analogue of this quantity for the process $Y(\cdot)$ is the occupation time of $[a,b)$ during $[0,t]$, i.e.,

$$\Lambda_t(a,b)=\int_0^t 1_{[a,b)}(Y(\sigma))\,d\sigma. \tag{2.5}$$

It is known [2, 7], that the process $Y(\cdot)$ possesses a local time $L_t(x)$ which is jointly continuous in t and x, such that Λ_t as defined in (2.5) is a.s. equal to

$$\Lambda_t(a,b)=\int_a^b L_t(x)\,dx. \tag{2.6}$$

The weak convergence of (2.2) to the process $Y(\cdot)$ implies that the distribution of (2.4) converges to that of (2.5). To see this we merely have to show that the map $Y(\cdot)\to\Lambda_t(a,b)$ is continuous in the J_1-topology on $D([0,T])$ for any $T\geqq t$ at almost all sample points of the Y-process. But if $Y_n(\cdot)\to Y(\cdot)$ in $D([0,T])$ then ([1], Sect. 14) there exist continuous increasing one to one maps λ_n from $[0,T]$ onto itself such that

$$\sup_{0\leqq s\leqq T}|Y_n(\lambda_n(s))-Y(s)|\to 0$$

and such that each λ_n is a Lipschitz function, absolutely continuous and satisfying

$$\sup|\lambda_n'(s)-1|\to 0,$$

where the last sup is only over those $s\in[0,T]$ at which λ_n is differentiable (but this only excludes a Lebesgue null set). Consequently, if Λ_t^n corresponds to Y_n, then

$$\Lambda_t^n(a,b)=\int_0^t 1_{[a,b)}(Y_n(s))\,ds$$

and

$$\Lambda_t^n(a,b)-\int_0^t 1_{[a,b)}(Y_n(\lambda_n(s))\,ds=\int_0^t 1_{[a,b)}(Y_n(s))\left[1-\frac{1}{\lambda_n'(\lambda_n^{-1}(s))}\right]ds+0|\lambda_n(t)-t|\to 0.$$

Furthermore,

$$\int_0^t 1_{[a,b)}(Y_n(\lambda_n(s))\,ds\to\int_0^t 1_{[a,b)}(Y(s))\,ds$$

whenever $Y(s)\neq a,b$ outside a Lebesgue null set. The latter is true for almost all sample points of the Y-process by virtue of (2.6). This proves the required continuity of the map $Y(\cdot)\to\Lambda_t(a,b)$ and hence the convergence in distribution of (2.4) to (2.5). More generally, the joint distribution of

$$T_{t_i}^n(a_i,b_i),\qquad 1\leqq i\leqq k,\tag{2.7}$$

converges to the joint distribution of

$$\Lambda_{t_i}(a_i,b_i),\qquad 1\leqq i\leqq k.\tag{2.8}$$

For each $x\in\mathbb{Z}$ we define

$$\tau(x)=\inf\{n\geqq 0: S_n=x\}.\tag{2.9}$$

Lemma 1. *For all* $x\in\mathbb{Z}$, $r\geqq 0$, $s\geqq 0$,

$$P[N_s(x)\geqq r]\leqq P[N_{[s+1]}(0)\geqq r]\,P[\tau(x)\leqq s+1],\tag{2.10}$$

$P[N_s(x)>0$ *for some* x *with* $|x|>As^{\frac{1}{\alpha}}]\leqq\varepsilon(A)$ *for* $s\geqq 1$, *where* $\varepsilon(A)\to 0$ *as* $A\to\infty$, *and* $\varepsilon(A)$ *is independent of* s. (2.11)

There exists a constant $C_3>0$ *such that*

$$E[N_s^\nu(0)]\sim\frac{\nu!\,C_3^\nu}{\Gamma\left(1+\nu-\frac{\nu}{\alpha}\right)}s^{\nu-\frac{\nu}{\alpha}},\qquad s\to\infty,\ \nu=1,2,3,\ldots.\tag{2.12}$$

Finally, for some $C_4>0$ *and all* $s\geqq 1$,

$$\sum_{x\in\mathbb{Z}}EN_s^2(x)\sim C_4 s^{2-\frac{1}{\alpha}}.\tag{2.13}$$

Proof. One has

$$N_s(x)\leqq\sum_{j=0}^{[s+1]}I[\tau(x)=j]\sum_{m=j}^{[s+1]}I[S_m=x].$$

Hence

$$P[N_s(x)\geqq r]\leqq\sum_{j=0}^{[s+1]}P\left\{\tau(x)=j;\ \sum_{m=j}^{[s+1]}I[S_m=x]\geqq r\right\}.$$

Now it is clear that $\{\tau(x)=j\}$ is measurable with respect to

$$\mathfrak{F}_j=\sigma\{X_1, X_2, \ldots, X_j\}, \tag{2.14}$$

while on $\{\tau(x)=j\}$

$$\sum_{m=j}^{[s+1]} I[S_m=x]=\sum_{m=0}^{[s+1]-j} I\left[\sum_{i=j+1}^{j+m} X_i=0\right]. \tag{2.15}$$

The right hand side is independent of $\mathfrak{F}_j$ and has the same distribution as

$$\sum_{m=0}^{[s+1]-j} I[S_m=0]=N_{[s+1]-j}(0)\leqq N_{[s+1]}(0). \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$P[N_s(x)\geqq r]\leqq \sum_{j=0}^{[s+1]} P[\tau(x)=j]\, P[N_{[s+1]}(0)\geqq r],$$

which is (2.10).

As for (2.11) note that $N_s(x)=0$ unless $S_n=x$, and a fortiori $|S_n|\geqq |x|$, for some $\leqq s+1$. Thus the left hand side of (2.11) is bounded by $\varepsilon(A)$, defined as

$$\varepsilon(A)=\sup_{s\geqq 1} P[|S_n|>A s^{\frac{1}{\alpha}} \text{ for some } n\leqq s+1]$$

$$=\sup_{s\geqq 1} P[\max_{n\leqq s+1} |S_n|>A s^{\frac{1}{\alpha}}]. \tag{2.17}$$

By the weak convergence of (2.2) to $\{Y(t); t\geqq 0\}$,

$$\lim_{s\to\infty} P[\max_{n\leqq s+1} |S_n|>A s^{\frac{1}{\alpha}}]=P[\sup_{t\leqq 1} Y(t)>A],$$

which tends to 0 as $A\to\infty$. From this one easily deduces that $\varepsilon(A)\to 0$ as $A\to\infty$, and hence (2.11) holds.

Next (2.12) can be proved from the local limit theorem (see Stone [15])

$$P[S_n=0]\sim C_5 n^{-\frac{1}{\alpha}}, \qquad n\to\infty \tag{2.18}$$

either directly or as in Darling and Kac [3]. (Actually $P[S_n=0]$ has to be replaced by $P[S_{nd}=0]$ unless the random walk is strongly aperiodic ([14], Chap. I)).

Finally (2.13) follows from (2.18) applied to (1.2). □

Now we need some additional notation. Let

$$0\leqq \tau_1(x)\leqq \tau_2(x)<\ldots$$

be the successive times at which S_n visits x. Thus $\tau_1(x)=\tau(x)$, and $S_n=x$ if and only if $n=\tau_k(x)$ for some k. For $y\neq x$, let

$$M_j(x,y)=\sum_{\tau_j(x)<n\leqq\tau_{j+1}(x)} I(S_n=y) \tag{2.19}$$

be the number of visits to y between the j^{th} and $(j+1)^{\text{st}}$ visit to x. Also let

$$\begin{aligned} p(x,y)&=P[M_j(x,y)\neq 0]\\ &=P[S_n=y \text{ for some } n \text{ such that } \tau_j(x)<n\leqq\tau_{j+1}(x)]. \end{aligned} \tag{2.20}$$

One easily sees from the strong Markov property that $p(x,y)$ is independent of j and a function of $y-x$ only. In fact ([14], Chap. II and VII)

$$p(x,y)=[a(x-y)+a(y-x)]^{-1}, \tag{2.21}$$

where $a(\cdot)$ is the potential kernel of the random walk S_n,

$$a(z)=\sum_{n=0}^{\infty}\{P[S_n=0]-P[S_n=z]\}, \qquad z\in\mathbb{Z}. \tag{2.22}$$

It follows that

$$p(x,y)=p(y,x)=p(0,x-y). \tag{2.23}$$

Finally, define the σ-fields

$$\mathfrak{G}_j(x)=\mathfrak{F}_{\tau_j(x)},$$

where $\mathfrak{F}_k$ was defined in (2.14).

Lemma 2.

$$E[M_j(x,y)|\mathfrak{G}_j(x)]=1, \tag{2.24}$$

$$E[M_j^\nu(x,y)|\mathfrak{G}_j(x)]\leqq K_\nu[1+a(x-y)+a(y-x)]^{\nu-1} \tag{2.25}$$

for some constant K_ν *independent of* x,y,j.

$$a(z)+a(-z)\sim C_6|z|^{\alpha-1}, |z|\to\infty. \tag{2.26}$$

Proof. Again the strong Markov property and the fact that $S_{\tau_j(x)}=x$ show that $M_j(x,y)$ is independent of $\mathfrak{G}_j(x)$ and that

$$P[M_j(x,y)=k|\mathfrak{G}_j(x)]=\begin{cases}1-p(x,y) & \text{if } k=0\\ p(x,y)\,[1-p(y,x)]^{k-1}\,p(y,x) & \text{if } k\geqq 1.\end{cases}$$

Equations (2.24) and (2.25) are now immediate from (2.21) and (2.23).

To prove (2.26) observe that (see [14], proof of P28.4)

$$a(z)+a(-z)=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-\cos\theta z}{1-\psi(\theta)}\,d\theta, \qquad z\in\mathbb{Z}. \tag{2.27}$$

The result (2.26) now follows from (2.27) and (1.11) by standard asymptotic analysis. □

Lemma 3. *For some $C_7>0$, independent of $x, y\in\mathbb{Z}$ and $s\geqq 1$,*

$$E[|N_s(x)-N_s(y)|^2]\leqq C_7[1+a(x-y)+a(y-x)]\,s^{1-\frac{1}{\alpha}}.$$

Proof. It suffices to take for s an integer. Also we only estimate

$$E\{|N_n(x)-N_n(y)|^2\, I[\tau(x)<\tau(y)]\},$$

the other half, with x and y interchanged, being analogous. On the set $\{\tau(x)<\tau(y)\}$,

$$\begin{aligned} N_n(x)-N_n(y) &= \sum_{j=1}^{N_n(x)} \{1-\sum_{\tau_j(x)<k\leqq\tau_{j+1}(x)\wedge n} I[S_k=y]\} \\ &= \sum_{j=1}^{N_n(x)} \{1-M_j(x,y)\} + \sum_{n<k\leqq\tau_{N_n+1(x)}(x)} I[S_k=y]. \end{aligned}$$

(Recall that $\tau_j(x)\leqq n$ exactly for $j\leqq N_n(x)$.) Moreover

$$0\leqq \sum_{n<k\leqq\tau_{N_n+1(x)}(x)} I[S_k=y]\leqq M_{N_n(x)}(x,y).$$

Thus

$$\begin{aligned} &E\{|N_n(x)-N_n(y)|^2\, I[\tau(x)<\tau(y)]\} \\ &\leqq 2E\left\{\left|\sum_{j=1}^{N_n(x)}[1-M_j(x,y)]\right|^2\right\}+2E\{M^2_{N_n(x)}(x,y)\} \\ &\leqq 2E\left\{\left|\sum_{j=1}^{n+1}[1-M_j(x,y)]\,I[\tau_j(x)\leqq n]\right|^2\right\} \\ &\quad +2E\left\{\sum_{j=1}^{n+1} M_j^2(x,y)\,I[\tau_j(x)\leqq n]\right\}. \end{aligned} \tag{2.29}$$

By (2.24) the random variables

$$[1-M_j(x,y)]\,I[\tau_j(x)\leqq n]$$

have mean zero and are orthogonal. Therefore we can resume the estimation in (2.29) to get the upper bound

$$2E\sum_{j=1}^{n+1}[1-M_j(x,y)]^2\, I[\tau_j(x)\leqq n]+2E\sum_{j=1}^{n+1} M_j^2\, I[\tau_j(x)\leqq n]. \tag{2.30}$$

But conditioned on $\mathfrak{G}_j(x)$

$$E[(1-M_j(x,y))^2|\mathfrak{G}_j(x)]\leqq E[(M_j(x,y))^2|\mathfrak{G}_j(x)]=E[M_1^2(x,y)].$$

Therefore the estimate in (2.30) is bounded above by

$$4EM_1^2(x,y)\,E\sum_{j=1}^{n+1} I[\tau_j(x)\leqq n]$$

$$=4EM_1^2(x,y)\sum_{j=1}^{n+1} P[N_n(x)\geqq j]$$
$$=4EM_1^2(x,y)\,EN_n(x).$$

By use of (2.25) and (2.12) we complete the proof of (2.28). □

We need one final occupation time estimate before combining the random walk with the random scenery.

Lemma 4. *Let* $\delta=1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}$. *Then*

$$\lim_{s\to\infty} s^{-\delta}\sup_{x\in\mathbb{Z}} N_s(x)=0 \quad \textit{in probability.} \tag{2.31}$$

Proof. By Lemma 1

$$E[N_s^\nu(x)]\leqq EN_{[s+1]}^\nu(0)=0(s^{\nu-\frac{\nu}{\alpha}}).$$

Thus, also by Lemma 1,

$$P[\sup_x s^{-\delta}N_s(x)>\varepsilon]$$
$$\leqq P[N_s(x)>0 \text{ for some } |x|\geqq As^{\frac{1}{\alpha}}]+\sum_{|x|\leqq As^{\frac{1}{\alpha}}}\varepsilon^{-\nu}s^{-\nu\delta}E[N_s^\nu(x)]$$
$$\leqq\varepsilon(A)+O(s^{\frac{1}{\alpha}-\nu\delta+\nu\left(1-\frac{1}{\alpha}\right)}). \tag{2.32}$$

But if we choose ν such that $\frac{1}{\alpha}-\nu\,\delta+\nu\left(1-\frac{1}{\alpha}\right)=\frac{1}{\alpha}-\frac{\nu}{\alpha\beta}<0$, then the last term in (2.32) tends to zero as $s\to\infty$ for each fixed ε and A. (2.31) now follows from the fact that $\varepsilon(A)\to 0$ as $A\to\infty$. □

3. The Random Scenery

Let Δ_t be the process defined in (1.21) by means of stochastic integrals. It follows from the definition of these integrals that one can write down the characteristic function of Δ_t, and in fact all joint characteristic functions. We now do so as a preliminary to the proof that they are the limits of the joint characteristic functions of the processes D_t^m defined in (1.22).

Lemma 5. *The joint (k-fold) distributions of* Δ_t *are given for distinct* $t_1, t_2, \ldots, t_k\geqq 0$ *and* $\theta_1, \theta_2, \ldots, \theta_k\in\mathbb{R}$, *by*

$$E\exp\left[i\sum_{j=1}^k\theta_j\Delta_{t_j}\right]=E\exp\left[-A_1\int_{-\infty}^{\infty}\left|\sum_{j=1}^k\theta_jL_{t_j}(x)\right|^\beta dx\right.$$
$$\left.-iA_2\int_{-\infty}^{\infty}\left|\sum_{j=1}^k\theta_jL_{t_j}(x)\right|^\beta \operatorname{sgn}\left(\sum_{j=1}^k\theta_jL_{t_j}(x)\right)dx\right]$$

$$= \int_{v\in\mathbb{R}} \int_{u\in\mathbb{R}^+} P\left[\int_{-\infty}^{\infty} \left|\sum_{j=1}^{k} \theta_j L_{t_j}(x)\right|^{\beta} dx \in du;\right.$$

$$\left.\int_{-\infty}^{\infty} \left|\sum_{j=1}^{k} \theta_j L_{t_j}(x)\right|^{\beta} \operatorname{sgn}\left(\sum_{j=1}^{k} \theta_j L_{t_j}(x)\right) dx \in dv\right] \exp(-A_1 u - iA_2 v). \tag{3.1}$$

Proof. By means of the Ito-Lévy representation for processes with independent increments (see [10], Chap. 1) we can write $Z_+(t) = M(t) + A(t) + \Gamma t$ for some constant Γ and independent processes $M(\cdot)$ and $A(\cdot)$ of independent increments. $M(\cdot)$ is a martingale which has characteristic function

$$E[e^{i\theta M(t)}] = \exp[t \int_{|y|\leqq 1} [e^{i\theta y} - 1 - i\theta y]\, v_1(dy)] \tag{3.2}$$

where v_1 is the restriction of the Lévy measure of Z_+ to $[-1, +1]$. The jumps of $M(\cdot)$ are precisely the jumps of $Z_+(\cdot)$ of size $\leqq 1$. Similarly, $A(t)$ is the sum of all jumps of Z_+ during $[0, t]$ of size > 1. If v_2 is the restriction of the Lévy measure of Z_+ to $\mathbb{R}\setminus[-1, +1]$, then $A(t)$ has characteristic function

$$E[e^{i\theta A(t)}] = \exp t \int_{|y|>1} [e^{i\theta y} - 1]\, v_2(dy).$$

One easily sees from (3.2) that M has all moments, and since it has independent increments

$$\langle M, M\rangle_t = t \int_{|y|\leqq 1} y^2\, v_1(dy)$$

(see [13], Chap. II. 20). Consequently, (see [13] Chap. II. 23) for any fixed sample path of the Y process on which $(x, t) \to L_t(x)$ is continuous and has compact support, we have

$$\int L_t(x)\, dM(x) = \lim_{n\to\infty} \sum_{l=0}^{\infty} L_t(x_l^n)\,[M(x_{l+1}^n) - M(x_l^n)], \tag{3.3}$$

where the lim in (3.3) is an L^2 limit with respect to the probability measure governing Z_+, and where $0 = x_0^n < x_1^n <$ are any sequences which satisfy

$$\lim_{l\to\infty} x_l^n = \infty, \qquad \lim_{n\to\infty} \max_l (x_{l+1}^n - x_l^n) = 0.$$

It is easy to see that we can then also choose x_k^n such that (3.3) holds as an almost everywhere limit with respect to the (product) probability measure governing $Y(\cdot)$ and $Z_+(\cdot)$ jointly. Since $A(\cdot)$ changes over each finite interval merely by a finite number of jumps it is even easier to see that

$$\int L_t(x)\, dA(x) = \lim_{n\to\infty} \sum_{l=0}^{\infty} L_t(x_l^n)\,[A(x_{l+1}^n) - A(x_l^n)] \qquad \text{w.p.l.}$$

Thus, for suitable x_l^n

$$\int L_t(x)\, dZ_+(x) = \lim_{n\to\infty} \sum_{l=0}^{\infty} L_t(x_l^n)\,[Z_+(x_{l+1}^n) - Z_+(x_l^n)] \qquad \text{w.p.l.}$$

On the other hand (see (1.19)),

$$Z_+(x_{l+1}^n) - Z_+(x_l^n), \qquad l = 0, 1, \ldots$$

are independent with characteristic function

$$\exp[-(x_{k+1}^n - x_k^n)|\theta|^\beta (A_1 + iA_2 \operatorname{sgn}\theta)]$$

so that

$$\begin{aligned}
&E\left[\exp i \sum_{j=1}^k \theta_j \int L_{t_j}(x)\, dZ_+(x)\right] \\
&\quad = \lim_{n\to\infty} E \exp\left[-\sum_l (x_{l+1}^n - x_l^n)\left\{A_1 \left|\sum_{j=1}^k \theta_j L_{t_j}(x_l^n)\right|^\beta \right.\right. \\
&\qquad \left.\left. + iA_2 \left|\sum_{j=1}^k \theta_j L_{t_j}(x_l^n)\right|^\beta \operatorname{sgn}\left(\sum_{j=1}^k \theta_j \mathrm{L}_{t_j}(x_l^n)\right)\right\}\right] \\
&\quad = E \exp\left[-A_1 \int_0^\infty \left|\sum_{j=1}^k \theta_j L_{t_j}(x)\right|^\beta dx \right. \\
&\qquad \left. - iA_2 \int_0^\infty \left|\sum_{j=1}^k \theta_j L_{t_j}(x)\right|^\beta \operatorname{sgn}\left(\sum_{j=1}^k \theta_j L_{t_j}(x)\right) dx\right].
\end{aligned}$$

By treating Z_- in the same manner as Z_+ on arrives at (3.1). □

The next step is to show how the joint distributions in (3.1) arise as limits of certain joint distributions of random walk occupation times.

Lemma 6. *For any distinct $t_1, t_2, \ldots t_k \geqq 0$ and $\theta_1, \theta_2, \ldots \theta_k \in \mathbb{R}$, the joint distribution of*

$$n^{-\delta\beta} \sum_x \left|\sum_{j=1}^k \theta^j N_{nt_j}(x)\right|^\beta$$

and

$$n^{-\delta\beta} \sum_x \left|\sum_{j=1}^k \theta_j N_{nt_j}(x)\right|^\beta \operatorname{sgn}\left(\sum_{j=1}^k \theta_j N_{nt_j}(x)\right) \tag{3.4}$$

converges, as $n \to \infty$ to the joint distribution of

$$\int_{-\infty}^\infty \left|\sum_{j=1}^k \theta_j L_{t_j}(x)\right|^\beta dx$$

and

$$\int_{-\infty}^\infty \left|\sum_{j=1}^k \theta_j L_{t_j}(x)\right|^\beta \operatorname{sgn}\left(\sum_{j=1}^k \theta_j L_{t_j}(x)\right) dx. \tag{3.5}$$

Proof. For simplicity we only prove that

$$n^{-\delta\beta} \sum_x \left|\sum_{j=1}^k \theta_j N_{nt_j}(x)\right|^\beta \to \int_{-\infty}^\infty \left|\sum_{j=1}^k \theta_j L_{t_j}(x)\right|^\beta dx \tag{3.6}$$

in distribution, as $n \to \infty$. The method of proof is to approximate the left side of (3.6) by a finite combination of $T^n_{t_j}$, of which we know that it converges in distribution to the corresponding combination of the Λ_{t_j} (see (2.4) through (2.8)). Then, in turn, this combination of the Λ_{t_j} will be shown to approximate the right hand side in (3.6).

To follow this plan we define, for some small $\tau > 0$ and large M,

$$a(l,n) = \tau\, l\, n^{\frac{1}{\alpha}}, \qquad l \in \mathbb{Z},$$

$$\begin{aligned} T(l,n) &= \sum_{j=1}^{k} \theta_j T^n_{t_j}(l\tau,(l+1)\tau) \\ &= \frac{1}{n} \sum_{j=1}^{k} \theta_j \sum_{a(l,n) \leqq y < a(l+1,n)} N_{nt_j}(y), \end{aligned}$$

$$U = U(\tau, M, n) = n^{-\delta\beta} \sum_{\substack{x < -M\tau n^{\frac{1}{\alpha}} \\ \text{or } x \geqq M\tau n^{\frac{1}{\alpha}}}} \left| \sum_{j=1}^{k} \theta_j\, N_{nt_j}(x) \right|^{\beta},$$

$$V = V(\tau, M, n) = \tau^{1-\beta} \sum_{|l| \leqq M} |T(l,n)|^{\beta}.$$

Then

$$\begin{aligned} & n^{-\delta\beta} \sum_x \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right|^{\beta} - U(\tau, M, n) - V(\tau, M, n) \\ &= \sum_{|l| \leqq M} \sum_{a(l,n) \leqq x < a(l+1,n)} n^{-\delta\beta} \left\{ \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right|^{\beta} \right. \\ &\quad \left. - n^{\beta}[a(l+1,n) - a(l,n)]^{-\beta} |T(l,n)|^{\beta} \right\} \\ &\quad + \sum_{|l| \leqq M} \{ n^{\beta-\delta\beta}[a(l+1,n) - a(l,n)]^{1-\beta} - \tau^{1-\beta} \} |T(l,n)|^{\beta}. \end{aligned} \tag{3.7}$$

Since

$$n^{\beta-\delta\beta}[a(l+1,n) - a(l,n)]^{1-\beta} - \tau^{1-\beta} \to 0,$$

and $T(l,n)$ converges in distribution to

$$\sum_{j=1}^{k} \theta_j \Lambda_{t_j}(l\tau, (l+1)\tau),$$

the second sum over l in the right hand side of (3.7) tends to zero in probability as $n \to \infty$. We now show that the first sum over l in the right hand side of (3.7) is small in probability when τ is small. We use the following inequality, valid for any $a \geqq 0$, $b \geqq 0$

$$|a^{\beta} - b^{\beta}| \leqq \begin{cases} |a-b|^{\beta} & \text{if } \beta \leqq 1 \\ \beta |a-b| (a^{\beta-1} + b^{\beta-1}) & \text{if } \beta > 1, \end{cases}$$

to estimate the sum over x. We only carry out the remaining details when $\beta \leqq 1$.

For such β

$$E\left\{\left|\left|\sum_{j=1}^{k}\theta_j N_{nt_j}(x)\right|^{\beta}-n^{\beta}[a(l+1,n)-a(l,n)]^{-\beta}|T(l,n)|^{\beta}\right|\right\}$$
$$\leqq E\left\{\left|\left|\sum_{j=1}^{k}\theta_j N_{nt_j}(x)\right|-n[a(l+1,n)-a(l,n)]^{-1}|T(l,n)|\right|^{\beta}\right\}$$
$$\leqq\left[E\left\{\left|\sum_{j=1}^{k}\theta_j N_{nt_j}(x)-n[a(l+1.n)-a(l,n)]^{-1}\,T(l,n)\right|^2\right\}\right]^{\beta/2}. \tag{3.8}$$

In turn,

$$E\left\{\left|\sum_{j=1}^{k}\theta_j N_{nt_j}(x)-n[a(l+1,n)-a(l,n)]^{-1}\,T(l,n)\right|^2\right\}$$
$$=[a(l+1,n)-a(l,n)]^{-2}\,E\left\{\left|\sum_{j=1}^{k}\sum_{a(l,n)\leqq y<a(l+1,n)}\theta_j[N_{nt_j}(x)-N_{nt_j}(y)]\right|^2\right\}$$
$$\leqq[a(l+1,n)-a(l,n]^{-1}\sum_{i=1}^{k}\theta_i^2\sum_{j=1}^{k}\sum_{a(l,n)\leqq y<a(l+1,n)}E\{|N_{nt_j}(x)-N_{nt_j}(y)|^2\}$$
$$\leqq C_8(\theta_1,\theta_2,\ldots,\theta_k;\,t_1,t_2,\ldots,t_k)$$
$$\cdot n^{1-\frac{1}{\alpha}}\max_{a(l,n)\leqq y<a(l+1,n)}[1+a(y-x)+a(x-y)]. \tag{3.9}$$

(The last majorization was obtained from Lemma 3.) By virtue of (2.26), the last member of (3.9) is dominated, for $a(l,n)\leqq x<a(l+1,n)$ and large n, by

$$C_9\,n^{1-\frac{1}{\alpha}}|a(l+1,n)-a(l,n)|^{\alpha-1}\leqq C_{10}\,\tau^{\alpha-1}\,n^{2-\frac{2}{\alpha}}.$$

Combining these estimates we obtain

$$E\left\{\left|\sum_{|l|\leqq M}\sum_{a(l,n)\leqq x<a(l+1,n)}n^{-\delta\beta}\right.\right.$$
$$\left.\left.\cdot\left[\left|\sum_{j=1}^{k}\theta_j N_{nt_j}(x)\right|^{\beta}-n^{\beta}[a(l+1,n)-a(l,n)]^{-\beta}|T(l,n)|^{\beta}\right]\right|\right\}$$
$$\leqq(2M+1)\,\tau\,n^{\frac{1}{\alpha}}\,n^{-\delta\beta}\{C_{10}\,\tau^{\alpha-1}\,n^{2-\frac{2}{\alpha}}\}^{\beta/2}$$
$$\leqq C_{11}(2M+1)\,\tau^{1+\frac{\beta}{2}(\alpha-1)}. \tag{3.10}$$

This completes the estimate of (3.7).

Finally using (2.11), we observe that for large n

$$P[U(\tau,M,n)\neq 0]\leqq\varepsilon\left(\tfrac{1}{2}M\tau(\max_j t_j)^{-\frac{1}{\alpha}}\right).$$

Thus for each $\eta>0$ we can first take $M\tau$ so large that

$$\varepsilon\left(\tfrac{1}{2}M\tau(\max_j t_j)^{-\frac{1}{\alpha}}\right)\leqq\eta,$$

and then τ so small that

$$C_{11}(2M+1)\tau^{1+\frac{\beta}{2}(\alpha-1)} \leqq \eta^2.$$

Then, by (3.7) and (3.10), for such $\tau, M,$ and large n

$$P\left[\left|n^{-\delta\beta}\sum_x \left|\sum_{j=1}^k \theta_j N_{nt_j}(x)\right|^\beta - V(\tau, M, n)\right| > \eta\right] \leqq 3\eta. \tag{3.11}$$

Next we observe that the convergence in distribution of (2.7) to (2.8) (as $n \to \infty$) implies the convergence in distribution as $n \to \infty$ of $V(\tau, M, n)$ to

$$\tau^{1-\beta} \sum_{|l| \leqq M} \left|\sum_{j=1}^k \theta_j \Lambda_{t_j}(l\tau, (l+1)\tau)\right|^\beta$$

$$= \tau^{1-\beta} \sum_{|l| \leqq M} \left|\sum_{j=1}^k \theta_j \int_{l\tau}^{(l+1)\tau} L_{t_j}(x)\,dx\right|^\beta.$$

Finally, the continuity of $x \to \sum_{j=1}^k \theta_j L_{t_j}(x)$, and the fact that $L_{t_j}(\cdot)$ has a.s. compact support imply that

$$\tau^{1-\beta} \sum_{|l| \leqq M} \left|\int_{l\tau}^{(l+1)\tau} \sum_{j=1}^k \theta_j L_{t_j}(x)\,dx\right|^\beta$$

$$\to \int_{-\infty}^{\infty} \left|\sum_{j=1}^k \theta_j L_{t_j}(x)\right|^\beta dx,$$

as $\tau \to 0$, $M\tau \to \infty$. Together with (3.11) this finally proves the convergence in distribution as claimed in (3.6), and hence Lemma 6. □

Recall now the definitions of W_n, W_t, and D_t^n in (1.1), (1.4), and (1.22). We are ready to prove

Proposition 1. *The finite dimensional distributions of D_t^n converge to those of Δ_t, as $n \to \infty$.*

Proof. By the definition of N_t and W_t for $t \geqq 0$ we have

$$W_t = \sum_{x \in \mathbb{Z}} N_t(x)\,\xi(x), \qquad t \geqq 0. \tag{3.12}$$

Thus

$$\sum_{j=1}^k \theta_j D_{t_j}^n = n^{-\delta} \sum_x \sum_{j=1}^k \theta_j N_{nt_j}(x)\,\xi(x) \tag{3.13}$$

and

$$E\left\{\exp\left[i \sum_{j=1}^k \theta_j D_{t_j}^n\right]\right\} = E\left[\prod_{x \in \mathbb{Z}} \lambda\left(n^{-\delta} \sum_{j=1}^k \theta_j N_{nt_j}(x)\right)\right]. \tag{3.14}$$

By virtue of (1.18) and (2.31) the limit as $n \to \infty$ of (3.14) equals

$$\lim_{n\to\infty} E\left\{\exp\left[-\sum_{x \in \mathbb{Z}} n^{-\delta\beta} \left|\sum_{j=1}^k \theta_j N_{nt_j}(x)\right|^\beta \left[A_1 + iA_2 \operatorname{sgn} \sum_{j=1}^k \theta_j N_{nt_j}(x)\right]\right\}, \tag{3.15}$$

provided the last limit exists. Moreover, by Lemma 6 this limit equals

$$E\exp\left\{-\int_{-\infty}^{\infty} dx\left|\sum_{j=1}^{k}\theta_j L_{t_j}(x)\right|^{\beta}\left[A_1+iA_2\operatorname{sgn}\sum_{j=1}^{k}\theta_j L_{t_j}(x)\right]\right\}$$
$$=E\exp\left\{i\sum_{j=1}^{k}\theta_j\Delta_{t_j}\right\}. \quad \square$$

In order to prove weak convergence of $\{D_t^n; t\geqq 0\}$ to $\{\Delta_t; t\geqq 0\}$ in $C[0,\infty)$ we still have to prove tightness of the family $\{D_t^n\}_{t\geqq 0}$, $n=1,2,3,\ldots$ The necessary estimate is supplied by

Lemma 7. *For each $T<\infty$, $\eta>0$,*

$$\limsup_{n\to\infty}\lim_{\delta\downarrow 0}\sup_{\substack{0\leqq t_1,t_2\leqq T\\ |t_2-t_1|\leqq\delta}} P[|D_{t_1}^n-D_{t_2}^n|\geqq\eta]=0. \tag{3.16}$$

Proof. Let $\varepsilon>0$. We first approximate D_t^n by a process $\bar{D}_t^n$ (obtained by certain truncations), plus a linear function $E_n t$ such that E_n is a bounded sequence and

$$\limsup_{n\to\infty} P[\sup_{t\leqq T}|D_t^n-\bar{D}_t^n-E_n t|>\tfrac{1}{2}\eta]\leqq\frac{\varepsilon}{2}. \tag{3.17}$$

After that we show that $\bar{D}_t^n$ satisfies Kolmogorov's criterion: For some $K_0=K_0(T)<\infty$

$$E\{|\bar{D}_{t_2}^n-\bar{D}_{t_1}^n|^2\}\leqq K_0(t_2-t_1)^{2-\frac{1}{\alpha}} \tag{3.18}$$

Equation (3.16) then follows from (3.17) and (3.18) by Theorem 12.3 in Billingsley [1] $\left(\text{note that } 2-\frac{1}{\alpha}>1\right)$.

To obtain $\bar{D}_t^n$ first choose A such that $\varepsilon(\frac{1}{2}AT^{-\frac{1}{\alpha}})\leqq\varepsilon/4$ (see (2.11)). Then (see (3.13))

$$P\Big[D_t^n\neq n^{-\delta}\sum_{|x|\leqq An^{\frac{1}{\alpha}}} N_{nt}(x)\,\xi(x)\quad\text{for some } t\leqq T\Big]$$
$$\leqq P[N_{nt}(x)>0\quad\text{for some } |x|>An^{\frac{1}{\alpha}} \text{ and } t\leqq T]$$
$$\leqq\varepsilon(\tfrac{1}{2}AT^{-\frac{1}{\alpha}})\leqq\frac{\varepsilon}{4}. \tag{3.19}$$

Next we choose ρ_1 and ρ_2 such that for all n

$$3An^{\frac{1}{\alpha}}\{1-P[-\rho_1 n^{\frac{1}{\alpha\beta}}\leqq\xi(0)\leqq\rho_2 n^{\frac{1}{\alpha\beta}}]\}\leqq\frac{\varepsilon}{4}. \tag{3.20}$$

Such (finite) ρ_1 and ρ_2 can be chosen, as functions of A and ε only, by virtue of (1.16). Indeed, for $\rho_1=\rho_2=\rho$ the left hand side of (3.20) is for large ρ at most

$$4A(B_1+B_2)\rho^{-\beta}. \tag{3.21}$$

For $\beta<1$ we indeed choose $\rho_1=\rho_2=\rho$. In this case

$$|E[\xi(0);\ |\xi(0)|\leqq\rho\, n^{\frac{1}{\alpha\beta}}]|\leqq \int_0^{\rho n^{\frac{1}{\alpha\beta}}} P[|\xi(0)|\geqq x]\,dx$$

$$\sim(B_1+B_2)(1-\beta)^{-1}[\rho\, n^{\frac{1}{\alpha\beta}}]^{1-\beta}. \tag{3.22}$$

For $1<\beta\leqq 2$ the left hand side of (3.22) is also of the order $O(n^{(1-\beta)/\alpha\beta})$, now by virtue of (1.14) and (1.16) with a trivial modification if $\beta=2$ (in which case we may take $B_1=B_2=0$ in (1.16)). If $\beta=1$ we shall choose ρ_1 and ρ_2 such that the left hand side of (3.22) is at most equal to $3K$. To do this we first choose ρ such that (3.21) is no more than $\varepsilon/4$. Assume that

$$F\equiv E[\xi(0);\ |\xi(0)|\leqq\rho\, n^{\frac{1}{\alpha}}]\leqq 0 \tag{3.23}$$

(the case $F\geqq 0$ is handled similarly). Then take $\rho_1=\rho$ and $\rho_2\geqq\rho$ such that

$$0\leqq E[\xi(0);\ +\rho\, n^{\frac{1}{\alpha}}\leqq\xi(0)\leqq\rho_2\, n^{\frac{1}{\alpha}}]\leqq 3K.$$

Since $F\geqq -K$ (by (1.15)) it suffices for this to take $\rho_2\geqq\rho$ such that (see (1.17))

$$E[\xi(0);\ \rho\, n^{\frac{1}{\alpha}}<\xi(0)\leqq\rho_2\, n^{\frac{1}{\alpha}}]\sim\frac{A_1}{\pi}\int_{\rho n^{\frac{1}{\alpha}}}^{\rho_2 n^{\frac{1}{\alpha}}}\frac{dx}{x}=\frac{A_1}{\pi}\log\frac{\rho_2}{\rho_1}=2K.$$

We now set

$$\bar{\xi}(x)=\xi(x)\,I[-\rho_1\, n^{\frac{1}{\alpha\beta}}\leqq\xi(x)\leqq\rho_2\, n^{\frac{1}{\alpha\beta}}],$$

and

$$E_n=n^{-\delta}E[\sum_x N_n(x)\,\bar{\xi}(x)]=n^{-\delta}E[\sum_x N_n(x)\,E[\bar{\xi}(x)]],$$

and

$$\bar{D}_t^n=n^{-\delta}\sum_x N_{nt}(x)\,\{\bar{\xi}(x)-E[\bar{\xi}(x)]\}.$$

To verify the boundedness of E_n and (3.17) and (3.18) observe first that

$$\sum_x N_{nt}(x)\,E[\bar{\xi}(x)]=E[\bar{\xi}(0)]\sum_x N_{nt}(x)=E[\bar{\xi}(0)]\,(n\,t+1).$$

This, together with

$$|E[\bar{\xi}(0)]|=O[n^{(1-\beta)/\alpha\beta}]$$

in all cases, proves that E_n is bounded. Also it shows that

$$D_t^n-\bar{D}_t^n-E_n\,t=n^{-\delta}\sum_x N_{nt}(x)\,[\xi(x)-\bar{\xi}(x)]+O\left(\frac{1}{n}\right). \tag{3.24}$$

By (3.19) and (3.20)

$$P\{\sum_x N_{nt}(x)\,[\xi(x)-\bar{\xi}(x)]\neq 0 \ \text{ for some } t\leqq T\}$$

$$\leqq \frac{\varepsilon}{4} + P[\xi(x) \neq \bar{\xi}(x) \text{ for some } |x| \leqq A n^{\frac{1}{\alpha}}]$$

$$\leqq \frac{\varepsilon}{4} + 3 A n^{\frac{1}{\alpha}} P[\xi(0) \neq \bar{\xi}(0)] \leqq \frac{\varepsilon}{2}.$$

This proves (3.17).

Finally we turn to the proof of (3.18). By definition of $\bar{D}_t^n$ and the independence of all the occupation times $N_{nt}(x)$ from the random scenery $\{\xi(x)\}$ we obtain

$$\begin{aligned} E[(\bar{D}_{t_1}^n - \bar{D}_{t_2}^n)^2] &= n^{-2\delta} \sum_x E[(N_{nt_2}(x) - N_{nt_1}(x))^2]\, \sigma^2[\bar{\xi}(x)] \\ &\leqq n^{-2\delta} E[\bar{\xi}^2(0)]\, E\{\sum_x [N_{nt_2}(x) - N_{nt_1}(x)]^2\}. \end{aligned} \tag{3.25}$$

By virtue of (1.16)

$$E[\bar{\xi}^2(0)] \leqq C_{12}\, n^{(2-\beta)/\alpha\beta}. \tag{3.26}$$

Moreover one easily sees from the strong Markov property that for $k \leqq j$

$$\sum_x [N_k(x) - N_j(x)]^2$$

has the same distribution as

$$\sum_x \mathrm{N}_{j-k}^2(x)^{-1}.$$

Therefore, if $t_1 \leqq t_2$, we obtain from (2.13)

$$\begin{aligned} E\{\sum_x [N_{nt_2}(x) - N_{nt_1}(x)]^2\} &\leqq E\{\sum_x N_{[(t_2-t_1)n+2]}^2(x)\} \\ &\leqq 2 C_4 [(t_2 - t_1) n + 2]^{2-\frac{1}{\alpha}}. \end{aligned} \tag{3.27}$$

If $(t_2 - t_1) n \geqq 1$, then (3.18) follows, by combining (3.25), (3.26) and (3.27). If $0 \leqq (t_2 - t_1) n < 1$ we must replace (3.27) by the following observation: $N_{k+1}(x) - N_k(x) = 0$ for all but one x, namely $x = S_{k+1}$. For this x, $N_{k+1}(x) - N_k(x) = 1$. Therefore, for $k \leqq s_1 < s_2 < k+1$,

$$\sum_x [N_{s_2}(x) - N_{s_1}(x)]^2 = (s_2 - s_1)^2,$$

and if $0 \leqq (t_2 - t_1) n \leqq 1$, then

$$\begin{aligned} &E\{\sum_x [N_{nt_2}(x) - N_{nt_1}(x)]^2\} \\ &\quad \leqq 2(t_2 - t_1)^2 n^2 \leqq 2[(t_2 - t_1) n]^{2-\frac{1}{\alpha}} \end{aligned} \tag{3.28}$$

With this replacement for (3.27) we find from (3.25) and (3.26) that

$$E\{[\bar{D}_{t_2}^n - D_{t_1}^n]^2\} \leqq 2 C_{13} (t_2 - t_1)^{2-\frac{1}{\alpha}},$$

which again gives (3.18) for $n(t_2 - t_1) < 1$. □

Proposition 1 and Lemma 7 together imply Theorem 1.1.

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