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Convergence of Weighted Averages of Independent Random Variables

By

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1. Introduction

Let $\{X_k\}$ be a sequence of independent, identically distributed random variables and $\{w_k\}$ a sequence of positive numbers. Define $S_n = \sum_1^n w_k X_k$ and $W_n = \sum_1^n w_k$. The purpose of this paper is to study the convergence properties of S_n/W_n. We will say that the weak (strong) law holds for $\{X_k, w_k\}$ if and only if $S_n/W_n \to c$ in probability (almost surely) for some constant c.

Let X be a random variable with the same distribution as the X_k's. The trivial case when X is degenerate (almost surely a constant) will be omitted from consideration. Suppose $\sum w_k < \infty$. Then the convergence of S_n/W_n and the convergence of the series $\sum_1^\infty w_k X_k$ are equivalent so that S_n/W_n either fails to converge in probability or else converges almost surely to a non-degenerate limit. Thus even the weak law does not hold for $\{X_k, w_k\}$. We assume from now on that $\sum w_k = \infty$.

The identity

$$\frac{S_n}{W_n} - \frac{S_{n-1}}{W_{n-1}} = \left(\frac{W_{n-1}}{W_n} - 1\right)\frac{S_{n-1}}{W_{n-1}} + \left(\frac{w_n}{W_n}\right)X_n, \tag{1.1}$$

$n = 2, 3, \ldots$ makes it evident that the weak law does not hold for $\{X_k, w_k\}$ unless $\lim\,(w_n/W_n) = 0$. This condition is also assumed throughout the paper. (We note for future reference that $w_n/W_n \to 0$ and $\sum w_k = \infty$ if and only if $\max_{1\leq k\leq n}(w_k/W_n) \to 0$.) In the next section it is shown that if the weak law holds for $\{X_k, 1\}$ then it holds for all $\{X_k, w_k\}$ for which $\sum w_k = \infty$ and $w_n/W_n \to 0$. In particular this is the case if $E|X|$ is finite.

The strong law is studied in section 3. The class of sequences of weights which gives the strong law for all X with $E|X| < \infty$ is characterized in Theorem 3. This is a much smaller class than the one studied for the weak law so there is some interest in considering other classes of weights. Bounded sequences of weights are investigated and even here the strong law does not hold for all X with $E|X| < \infty$. A sufficient condition (Theorem 4) is that $E|X|\log^+|X| < \infty$. Some examples are given in section 4.

Baxter [1], [2] has recently obtained some very general pointwise ergodic theorems for sequences of weighted averages. The special feature of the present

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paper is that the restriction of the type of process permits consideration of a much larger class of weights.

2. The weak law

Let F be the distribution function of the random variable X.

Theorem 1. *The weak law holds for all divergent sequences* $\{w_k\}$ *such that* $w_n/W_n \to 0$ *if and only if*

$$\lim_{T\to\infty} T\,P[|X| \geqq T] = 0 \quad \text{and} \quad \lim_{T\to\infty} \int_{|x|<T} x\,dF \tag{2.1}$$

exists.

Proof: It follows from the classical degenerate convergence criterion [3, p. 278] that the conditions (2.1) are a consequence of the classical weak law ($w_k = 1$, all k). Now suppose (2.1) is true. Let X_{nk} be X_k truncated at W_n/w_k and $S_{nn} = \sum_1^n w_k X_{nk}$. Then for all n sufficiently large, since $\max_{1\leqq k\leqq n} w_k/W_n \to 0$, we have

$$P[S_{nn} \neq S_n] \leqq \sum_{k=1}^n P[X_{nk} \neq X_k] = \sum_{k=1}^n P\left[|X| \geqq \frac{W_n}{w_k}\right] \leqq \varepsilon \sum_{k=1}^n \frac{w_k}{W_n} = \varepsilon\,.$$

Therefore it will suffice to consider S_{nn} instead of S_n. Now

$$E\,\frac{S_{nn}}{W_n} = \frac{1}{W_n} \sum_{k=1}^n w_k \int_{|x|<W_n/w_k} x\,dF \to \mu$$

where μ is the second limit in (2.1). Since

$$\frac{1}{T} \int_{|x|<T} x^2\,dF = \frac{1}{T}\left\{- T^2 P[|X| \geqq T] + 2\int_{0\leqq x<T} x\,P[|X| \geqq x]\,dx\right\} \to 0\,,$$

it follows that for n sufficiently large

$$\operatorname{Var} \frac{S_{nn}}{W_n} = \frac{1}{W_n^2} \sum_{k=1}^n w_k^2 \operatorname{Var} X_{nk} \leqq \frac{1}{W_n^2} \sum_{k=1}^n w_k^2 \int_{|x|<W_n/w_k} x^2\,dF \leqq \frac{1}{W_n^2} \sum_{k=1}^n w_k^2\, \varepsilon\, \frac{W_n}{w_k} = \varepsilon\,.$$

An application of Chebyshev's inequality completes the proof.

Condition (2.1) is equivalent to the existence of the derivative of the characteristic function of X at zero [4] and is clearly weaker than $E|X| < \infty$.

3. The strong law

For a given sequence of positive weights $\{w_k\}$, define for each $x > 0$ $N(x)$ as the number of n such that $W_n/w_n \leqq x$. The rate of growth of this function is the critical factor in establishing the strong law. First we prove the fundamental

Theorem 2.* *If* $E|X| < \infty$, $EN(|X|) < \infty$, *and*

$$\int x^2 \int_{y\geqq|x|} \frac{N(y)}{y^3}\,dy\,dF(x) < \infty\,, \tag{3.1}$$

then the strong law obtains.

Proof. Let Y_k be X_k truncated at W_k/w_k and $T_n = \sum_1^n w_k Y_k$. Since

$$\sum_k P[Y_k \neq X_k] = \sum_k \int_{|x|\geqq W_k/w_k} dF = \int N(|x|)\,dF < \infty\,,$$

* The condition $EN(|X|) < \infty$ may be omitted, as it is a consequence of (3.1).

T_n/W_n and S_n/W_n converge on the same set and to the same limit almost surely. Furthermore, if $\mu = EX$, then $E(T_n/W_n) \to \mu$. It will suffice to prove that $(T_n - E T_n)/W_n \to 0$ almost surely, or, by [3, p. 238], that $\sum_k w_k^2 \operatorname{Var} Y_k/W_k^2$ converges. This sum is bounded by

$$\sum_k \frac{w_k^2}{W_k^2} \int_{|x| < W_k/w_k} x^2\, dF = \int x^2 \sum_{\{k : W_k/w_k > |x|\}} \frac{w_k^2}{W_k^2}\, dF(x)\,.$$

To estimate the sum, observe that

$$\sum_{\{k : |x| < W_k/w_k \leqq z\}} \frac{w_k^2}{W_k^2} = \int_{|x| < y \leqq z} \frac{dN(y)}{y^2} = \frac{N(z)}{z^2} - \frac{N(|x|)}{x^2} + 2 \int_{|x| < y \leqq z} \frac{N(y)}{y^3}\, dy\,,$$

and

$$\frac{N(z)}{z^2} \leqq 2 \int_z^\infty \frac{N(y)}{y^3}\, dy \to 0 \quad \text{as} \quad z \to \infty\,,$$

where the integral converges as a result of (3.1). Thus

$$\sum_{\{k : W_k/w_k > |x|\}} \frac{w_k^2}{W_k^2} \leqq 2 \int_{y \geqq |x|} \frac{N(y)}{y^3}\, dy$$

which completes the proof.

The next step will be to find the class of weights which gives the strong law for every X with finite expectation.

Lemma 1. *The strong law for* $\{X_k, w_k\}$ *implies* $w_n X_n/W_n \to 0$ *almost surely and the latter condition is equivalent to* $E\, N(c|X|) < \infty$ *for every* $c > 0$.

Proof: The identity (1.1) shows that if the strong law holds then $w_n X_n/W_n$ converges to zero almost surely. Now this is equivalent to

$$\int N\left(\frac{|x|}{\varepsilon}\right) dF = \sum_{n=1}^{\infty} \int_{|x| \geqq \varepsilon W_n/w_n} dF = \sum_{n=1}^{\infty} P\left[\left|\frac{w_n X_n}{W_n}\right| \geqq \varepsilon\right] < \infty$$

for every $\varepsilon > 0$.

Theorem 3. *For a given sequence of weights* $\{w_k\}$, *the strong law holds for all* X *with* $E|X| < \infty$ *if and only if* $\limsup N(x)/x < \infty$ *as* $x \to \infty$.

Proof: First suppose that $\limsup N(x)/x < \infty$ so that $N(x) < Mx$ for all $x > 0$. Then $E|X| < \infty$ implies $E\, N(|X|) < \infty$ and

$$\int x^2 \int_{y \geqq |x|} \frac{N(y)}{y^3}\, dy\, dF(x) \leqq \int x^2 \frac{M}{|x|}\, dF = M\, E|X|$$

so the strong law applies by Theorem 2. On the other hand, if $\limsup N(x)/x = \infty$, let x_k be a sequence such that $N(x_k)/x_k \to \infty$. Then a sequence $\{f_k\}$ can be chosen with sum one and $\sum f_k x_k < \infty$, $\sum f_k N(x_k) = \infty$. This is a distribution such that $E|X| < \infty$, but $EN(|X|) = \infty$ so that the strong law does not obtain by Lemma 1.

The first example of the next section will show that $N(x)$ may grow arbitrarily fast even when $w_n/W_n \to 0$ so that it is in order to consider other conditions on the weights. One interesting possibility is a boundedness condition. The bound may be taken as one so we assume that $0 < w_k \leqq 1$ and $W_n \to \infty$.

Lemma 2. *For a divergent sequence of positive weights bounded by one,*

$$\limsup N(x)/x \log x \leqq 2 \quad as \quad x \to \infty .$$

Proof: Let $B_n = \{k : n < W_k \leqq n + 1\}$ and v_n be the number of k in B_n such that $W_k/w_k \leqq x$. Then

$$\frac{n}{x} v_n \leqq \sum_{k \in B_n} w_k \leqq n + 1 - (n - 1) = 2$$

so that $v_n \leqq 2x/n$, and

$$N(x) = \sum_{n=0}^{[x]} v_n \leqq v_0 + 2x \sum_{n=1}^{[x]} \frac{1}{n} \leqq v_0 + 2x(1 + \log x).$$

The second example of the next section will give a bounded sequence of weights with $\limsup N(x)/x \log x \geqq \frac{1}{2}$ so that even some bounded sequences fall outside the domain of Theorem 3. However, inserting the bound obtained in the preceding lemma in Theorem 2 yields

Theorem 4. *If* $E|X| \log^+ |X| < \infty$, *then the strong law holds for all bounded divergent sequences of weights.*

4. Examples

The first example will be to show that $N(x)$ may grow arbitrarily fast. Let $\{n_m\}$ be any increasing sequence of positive integers with $n_1 = 1$. Take $w_1 = 1$ and for $n_m < k \leqq n_{m+1}$, let

$$w_k = W_{n_m}\left(\frac{1}{m+1}\right)\left(\frac{m+1}{m}\right)^{k-n_m} \quad \text{so that} \quad W_k = W_{n_m}\left(\frac{m+1}{m}\right)^{k-n_m} = (m+1)\, w_k .$$

Therefore, for $n_m < k \leqq n_{m+1}$, $W_k/w_k = m + 1$ and $N(m) = n_m$ for all integral m. This shows that $N(x)$ can grow as fast as desired even if $w_n/W_n \to 0$.

The other example is of a bounded sequence of weights with

$$\limsup N(x)/x \log x \geqq \frac{1}{2} .$$

The idea of the construction is the same as in the first example but the boundedness condition makes the task more difficult. In this case, let $n_0 = 1$, $w_1 = 1$, and define for $n_m < k \leqq n_{m+1}$

$$w_k = W_{n_m}(1 + x_m^{-1})^{k-n_m}(1 + x_m)^{-1}$$

where $x_m = e^{2^{m+1}}$ and the n_m are defined by $n_{m+1} - n_m = [2^m e^{2^{m+1}}]$. Then

$$W_k = W_{n_m}(1 + x_m^{-1})^{k-n_m}$$

so that $W_k/w_k = 1 + x_m$. Furthermore

$$\frac{W_{n_{m+1}}}{W_{n_m}} = (1 + x_m^{-1})^{n_{m+1}-n_m} \leqq (1 + x_m^{-1})^{2^m x_m} \leqq e^{2^m}$$

and

$$W_{n_{m+1}} = \prod_{k=0}^{m} \frac{W_{n_{k+1}}}{W_{n_k}} \leqq e^{2^{m+1}}$$

which implies that

$$w_k = (1 + x_m)^{-1} W_k \leqq (1 + x_m)^{-1} W_{n_{m+1}} \leqq x_m (1 + x_m)^{-1} \leqq 1 .$$

Hence the sequence is bounded and

$$N(x_m + 1) = n_{m+1} \geqq n_{m+1} - n_m \geqq 2^m x_m - 1 = \tfrac{1}{2} x_m \log x_m - 1 .$$

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