Z. Wahrscheinlichkeitstheorie verw. Gebiete
68, 557–578 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Stochastic Calculus for Continuous Additive Functionals of Zero Energy

Shintaro Nakao*

Department of Mathematics, Osaka University, Toyonaka Osaka 560, Japan

1. Introduction

In the study of multidimensional diffusion processes and related problems, one of the main tools is the stochastic calculus of Itô which, nowadays, has been formulated in a very general framework of semimartingales, cf. [2, 11, 17]. This stochastic calculus can be applied to diffusion processes with sufficiently smooth coefficients but there is an important class of diffusion processes which cannot be covered by this approach; for example, a class of diffusion processes corresponding to uniformly elliptic second order differential operators with measurable coefficients and of the divergence form, analytical theory of which has been developed by De Giorgi, Nash, Stampacchia, Aronson, etc. cf. [7, 15]. These diffusion processes, when they are symmetrizable with respect to some measure, can be treated in the framework of Fukushima's theory of Dirichlet spaces: Actually Fukushima [4, 5] developed another stochastic calculus in this framework which is well suited for the study of this class of diffusion processes. In this study one is naturally led to a class of additive functionals which are no longer semimartingales but retain a similar property with the part of the process of bounded variation being replaced by a process of zero energy, or equivalently, a process of zero quadratic variation. The purpose of the present paper is to develop a systematic stochastic calculus for such additive functionals of a symmetric Markov process.

More specifically, let $\mathbf{M}=(X_t, P_x)$ be a nice symmetric Markov process on a state space X with respect to a measure m on X and $(\mathscr{F}, \mathscr{E})$ be the associated Dirichlet space (cf. [6] for details). Fukushima showed that, for every $u \in \mathscr{F}$, the corresponding additive functional (abbreviated as a.f.) $A_t^{[u]}=\tilde{u}(X_t)-\tilde{u}(X_0)$ ($\tilde{u}$ is a quasi-continuous version of u) admits the decomposition

$$A_t^{[u]}=M_t^{[u]}+N_t^{[u]},$$

where $M^{[u]}$ is a martingale a.f. of finite evergy and $N^{[u]}$ is a continuous a.f. of zero energy. We shall introduce the following class of continuous a.f.'s of zero

* Research supported in part by Grant-in-aid for Scientific Research.

energy:

$$\tilde{\mathcal{N}_c}=\left\{N_t^{[u]}+\int_0^t g(X_s)\,ds;\, u\in\mathcal{F}, g\in L^2(X;m)\right\}. \tag{1.1}$$

In the forthcoming paper [14], we show the importance of this class and indeed $\tilde{\mathcal{N}_c}$ exhausts all continuous a.f.'s of zero energy if the process $\mathbf{M}$ is conservative (see Remark 3.3 in Sect. 3). In this connection we mention a recent work of Ôshima-Yamada [16] on related representations.

The main problem treated in this paper is to define an integral $\int_0^t v(X_s)\,dN_s$, $N\in\tilde{\mathcal{N}_c}$, for a suitable class of functions v on X. Since $N\in\tilde{\mathcal{N}_c}$ is, usually, not of bounded variation, this integral cannot be defined as an ordinary Stieltjes integral. By assuming $v\in\mathcal{F}_b$ ($\mathcal{F}_b=\mathcal{F}\cap L^\infty(X;m)$), however, we can define it for $N\in\tilde{\mathcal{N}_c}$ to be a continuous a.f. expressible as a sum of an element in $\tilde{\mathcal{N}_c}$ and a continuous a.f. of bounded variation. If this integral is once introduced, we can define stochastic integrals of the form $\int_0^t v(X_s)\,dA_s^{[u]}$ (Itô-type) and $\int_0^t v(X_s)\circ dA_s^{[u]}$ (Stratonovich-type) for $u\in\mathcal{F}$ and $v\in\mathcal{F}_b$. Itô formulae will then be obtained in Sect. 4 in context of these stochastic integrals. In this connection, we note that Föllmer [3] introduced a class of processes which are called Dirichlet processes (the sum of a martingale and a process of zero quadratic variation) and developed a similar calculus for these processes.

As an application of the Stratonovich-type integral in the above sense, we define in Sect. 5 stochastic line integrals of 1-forms along the paths of general symmetric diffusion processes on manifolds. The stochastic line integral was introduced by Ikeda-Manabe [10] for a diffusion process on a manifold whose generator has smooth coefficients, and consequently whose coordinate processes are semimartingales. Kusuoka then considered the stochastic line integral for a certain class of symmetric diffusion processes whose coordinate processes are no more semimartingales in general (see Remark 5.1 and Example 5.1). Our stochastic line integral turns out to be an extension of Kusuoka's one.

2. Preliminary Facts

Let X be a locally compact separable metric space and m be an everywhere dense positive Radon measure on X. Consider an m-symmetric Hunt process $\mathbf{M}=(\Omega,\mathcal{B},X_t,\zeta,P_x)$ on X whose Dirichlet space $(\mathcal{F},\mathcal{E})$ on $L^2(X;m)$ is C_0-regular. In this paper we treat a.f.'s admitting exceptional sets of $\mathbf{M}$, the precise meaning being appeared in [4, 6]. Two a.f.'s $A^{(1)}$ and $A^{(2)}$ are said to be equivalent if for each $t>0$ $P_x(A_t^{(1)}=A_t^{(2)})=1$ (q.e. $x\in X$) and then we denote by $A^{(1)}=A^{(2)}$.

We use the following notations. For a signed Borel measure ν on X, a Borel function g on X and a random variable Λ on Ω, we let $\langle\nu,g\rangle=\int_X g(x)\,d\nu(x)$, $p_t g(x)=E_x[g(X_t)]$ and $E_{g\cdot\nu}[\Lambda]=\int_X E_x[\Lambda]\,g(x)\,d\nu(x)$. Let $\mathbf{S}$ be the set of all smooth measures of $\mathbf{M}$ and $\mathbf{A}_c^+$ be the family of all positive continuous a.f.'s of $\mathbf{M}$. Fukushima [4] showed that $\mathbf{A}_c^+$ and $\mathbf{S}$ are in one to one correspondence and this correspondence is characterized by the following formula; for $A \in \mathbf{A}_c^+$ and $\mu \in \mathbf{S}$

$$E_{h \cdot m}\left[\int_0^t g(X_s)\, dA_s\right] = \int_0^t \langle g \cdot \mu, p_s h\rangle\, ds \qquad (g, h \in \mathscr{B}^+ \text{ and } t > 0), \tag{2.1}$$

where $\mathscr{B}^+$ is the set of all non-negative Borel functions on X.

We denote by $\mathbf{A}$ the space of all a.f.'s of $\mathbf{M}$ and define for $A \in \mathbf{A}$

$$e(A) = \lim_{t \to 0} \frac{1}{2t} E_m[A_t^2],$$

whenever the limit exists. $e(A)$ is called the energy of A. Now we exhibit two important subclasses of $\mathbf{A}$. One is

$$\mathring{\mathscr{M}} = \left\{ M \in \mathbf{A};\ \begin{array}{l} e(M) \text{ is finite. For q.e. } x \in X\ E_x[M_t^2] < \infty \\ \text{and } E_x[M_t] = 0\ (t > 0). \end{array} \right\}$$

and the other is

$$\mathscr{N}_c = \left\{ N \in \mathbf{A};\ \begin{array}{l} N \text{ is a continuous a.f. such that for q.e.} \\ x \in X\ E_x[|N_t|] < \infty\ (t > 0) \text{ and } e(N) = 0 \end{array} \right\}.$$

We call $M \in \mathring{\mathscr{M}}$ a martingale a.f. of finite energy and $N \in \mathscr{N}_c$ a continuous a.f. of zero energy. Since $\sqrt{e}$ is a Hilbertian norm on $\mathring{\mathscr{M}}$, we define an inner product

$$e(M_1, M_2) = \tfrac{1}{2}\{e(M_1 + M_2) - e(M_1) - e(M_2)\} \qquad \text{for } M_1, M_2 \in \mathring{\mathscr{M}}.$$

Then $(\mathring{\mathscr{M}}, e)$ is a Hilbert space. The smooth measure corresponding to the quadratic variation $\langle M\rangle$ of $M \in \mathring{\mathscr{M}}$ is denoted by $\mu_{\langle M\rangle}$. Similarly we define

$$\mu_{\langle M_1, M_2\rangle} = \tfrac{1}{2}\{\mu_{\langle M_1 + M_2\rangle} - \mu_{\langle M_1\rangle} - \mu_{\langle M_2\rangle}\} \qquad \text{for } M_1, M_2 \in \mathring{\mathscr{M}}.$$

Obviously it holds that $e(M_1, M_2) = \frac{1}{2}\mu_{\langle M_1, M_2\rangle}(X)$ for $M_1, M_2 \in \mathring{\mathscr{M}}$.

Fukushima [5] showed that the a.f. $A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$ $(u \in \mathscr{F})$ can be decomposed in the following manner;

$$A^{[u]} = M^{[u]} + N^{[u]}, \qquad M^{[u]} \in \mathring{\mathscr{M}} \quad \text{and} \quad N^{[u]} \in \mathscr{N}_c.$$

This decomposition is unique according to $\mathring{\mathscr{M}} \cap \mathscr{N}_c = \{0\}$. Moreover $M^{[u]}$ admits a decomposition

$$M^{[u]} = \overset{c}{M}{}^{[u]} + \overset{j}{M}{}^{[u]} + \overset{k}{M}{}^{[u]},$$

where $\overset{c}{M}{}^{[u]}$ is a continuous martingale a.f. of finite energy

$$\overset{j}{M}{}^{[u]}_t = \widehat{\sum_{0 < s \leq t} (\tilde{u}(X_s) - \tilde{u}(X_{s-}))\, I_{\{t < \zeta\}}} \in \mathring{\mathscr{M}} \quad \text{and} \quad \overset{k}{M}{}^{[u]}_t = \widehat{-\tilde{u}(X_{\zeta-})\, I_{\{\zeta \leq t\}}} \in \mathring{\mathscr{M}},$$

cf. [9, 13]. Here $\hat{A}=A-A^{p}$ (A^{p} is the predictable dual projection of A) and I_B is the indicator function of a set B. It is known that this is an orthogonal decomposition in $(\mathring{\mathscr{M}}, e)$. The smooth measures $\mu_{\langle M^{[u]}\rangle}$, $\mu_{\langle \overset{c}{M}{}^{[u]}\rangle}$, $\mu_{\langle \overset{j}{M}{}^{[u]}\rangle}$ and $\mu_{\langle \overset{k}{M}{}^{[u]}\rangle}$ are simply denoted by $\mu_{\langle u\rangle}$, $\overset{c}{\mu}_{\langle u\rangle}$, $\overset{j}{\mu}_{\langle u\rangle}$ and $\overset{k}{\mu}_{\langle u\rangle}$ respectively. The notations $\mu_{\langle u,v\rangle}$, $\overset{c}{\mu}_{\langle u,v\rangle}$, $\overset{j}{\mu}_{\langle u,v\rangle}$ and $\overset{k}{\mu}_{\langle u,v\rangle}$ $(u, v\in\mathscr{F})$ are used in an analogous sense.

We state fundamental properties of $\overset{c}{\mu}_{\langle u,v\rangle}$, $\overset{j}{\mu}_{\langle u,v\rangle}$ and $\overset{k}{\mu}_{\langle u,v\rangle}$, cf. [9, 12]. $\overset{c}{\mu}_{\langle u,v\rangle}$ possesses the derivation property; for $u, v\in\mathscr{F}_b$ and $w\in\mathscr{F}$

$$d\overset{c}{\mu}_{\langle uv,w\rangle}(x)=\tilde{u}(x)\,d\overset{c}{\mu}_{\langle v,w\rangle}(x)+\tilde{v}(x)\,d\overset{c}{\mu}_{\langle u,w\rangle}(x). \tag{2.2}$$

Let $(H, N(x, dy))$ be a Lévy system of the Hunt process $\mathbf{M}$ (cf. [1]) and ν be the smooth measure corresponding to $H\in\mathbf{A}_c^+$. We put

$$J(dx\,dy)=\tfrac{1}{2}N(x,dy)\,\nu(dx), \qquad k(dx)=N(x,\{\Delta\})\,\nu(dx) \tag{2.3}$$

and call them the jumping measure and the killing measure of $\mathbf{M}$ respectively. J is a symmetric measure on the product space $X\times X$ such that $J(d)=0$ and J is a Radon measure on $X\times X-d$, where d is the diagonal set of $X\times X$. k is a Radon measure on X and moreover k is the smooth measure whose associated positive continuous a.f. is $I_{\{\zeta\le t,\, X_{\zeta-}\in X\}}{}^{p}$. It is known that for $u, v\in\mathscr{F}$

$$\overset{j}{\mu}_{\langle u,v\rangle}(dx)=2\int_X(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))\,J(dx\,dy) \tag{2.4}$$

and

$$\overset{k}{\mu}_{\langle u,v\rangle}(dx)=\tilde{u}(x)\,\tilde{v}(x)\,k(dx). \tag{2.5}$$

Setting $\mathscr{E}^{(c)}(u,v)=\frac{1}{2}\overset{c}{\mu}_{\langle u,v\rangle}(X)$, $\mathscr{E}^{(j)}(u,v)=\frac{1}{2}\overset{j}{\mu}_{\langle u,v\rangle}(X)$ and $\mathscr{E}^{(k)}(u,v)=\overset{k}{\mu}_{\langle u,v\rangle}(X)$ for $u, v\in\mathscr{F}$, we have

$$\mathscr{E}(u,v)=\mathscr{E}^{(c)}(u,v)+\mathscr{E}^{(j)}(u,v)+\mathscr{E}^{(k)}(u,v). \tag{2.6}$$

The derivation property (2.2) implies a local property of the symmetric form $(\mathscr{E}^{(c)}, \mathscr{F})$ and thus the decomposition (2.6) of $(\mathscr{F}, \mathscr{E})$ is just the Beurling-Deny formula, cf. [6].

Next we state a necessary and sufficient condition for which $N^{[u]}(u\in\mathscr{F})$ is of bounded variation on each compact interval of $[0,\zeta)$. This result is obtained by Fukushima (Theorem 5.3.2 in [6]).

Definition 2.1. We say that $\{K_l\}_{l=1,2,\dots}$ is a nest of a smooth measure μ if $\{K_l\}_{l=1,2,\dots}$ satisfies the following conditions.

(i) $\{K_l\}_{l=1,2,\dots}$ is an increasing sequence of compact subsets of X such that $\mu\left(X-\bigcup_{l=1}^{\infty}K_l\right)=0$.

(ii) For each $l=1,2,\dots, I_K\cdot\mu$ is of finite energy integral and $\lim_{l\to\infty}\mathrm{Cap}(K-K_l)=0$ for any compact subset K of X.

For $B\subset X$ define $\mathscr{F}_B=\{u\in\mathscr{F}\,;\,\tilde{u}(x)=0$ q.e. on $X-B\}$.

Theorem 2.1 (M. Fukushima). *Consider $u \in \mathscr{F}$ and $\mu^{(1)}$, $\mu^{(2)} \in \mathbf{S}$. Let $A^{(1)}$ and $A^{(2)}$ be the positive continuous a.f.'s corresponding to $\mu^{(1)}$ and $\mu^{(2)}$ respectively. Then the following conditions are equivalent to each other.*

(i) $N_t^{[u]} = -A_t^{(1)} + A_t^{(2)}$ *for* $t < \zeta$.

(ii) *For each common nest* $\{K_l\}_{l=1,2,\dots}$ *of* $\mu^{(1)}$ *and* $\mu^{(2)}$

$$\mathscr{E}(u, v) = \langle I_{K_l} \cdot (\mu^{(1)} - \mu^{(2)}), \tilde{v} \rangle$$

for $v \in \mathscr{F}_{K_l}$ *and* $l = 1, 2, \dots$.

(iii) *There exists a common nest* $\{K_l\}_{l=1,2,\dots}$ *of* $\mu^{(1)}$ *and* $\mu^{(2)}$ *such that*

$$\mathscr{E}(u, v) = \langle I_{K_l} \cdot (\mu^{(1)} - \mu^{(2)}), \tilde{v} \rangle$$

for $v \in \mathscr{F}_{K_l} \cap \mathscr{F}_b$ *and* $l = 1, 2, \dots$.

We now introduce a subclass $\tilde{\mathbf{A}}_c^+$ of $\mathbf{A}_c^+$ by $\tilde{\mathbf{A}}_c^+ = \{A \in \mathbf{A}_c^+;$ the smooth measure corresponding to A is finite.$\}$.

Lemma 2.1. *Let $A \in \tilde{\mathbf{A}}_c^+$. Then we have for $h \in \mathscr{F}_b$*

$$\lim_{t \to 0} \frac{1}{t} E_{h \cdot m}[A_t] = \langle \mu, \tilde{h} \rangle,$$

where μ is the smooth measure corresponding to A.

Proof. Obviously we may assume that h is quasi-continuous and Borel measurable. From (2.1), we get

$$E_{h \cdot m}[A_t] = \int_0^t \langle \mu, p_s h \rangle \, ds.$$

$\langle \mu, p_s h \rangle$ is continuous at $s = 0$, because $p_s h$ converges to h q.e. on X as $s \to 0$, cf. [8]. Thus the proof is completed. ◻

Lemma 5.1.9 in [6] implies that, for $A \in \tilde{\mathbf{A}}_c^+$, $P_x(A_t < \infty$ for $0 \leqq t < \infty) = 1$ (q.e. $x \in X$) and so the following space is well defined;

$$\mathscr{N}_c^* = \{N + A_1 + A_2;\ N \in \tilde{\mathscr{N}_c} \text{ and } A_1, A_2 \in \tilde{\mathbf{A}}_c^+\},$$

where $\tilde{\mathscr{N}_c}$ is the subspace of $\mathscr{N}_c$ defined by (1.1). This space plays an important role in the next section.

Theorem 2.2. *Let $C^{(1)}$, $C^{(2)} \in \mathscr{N}_c^*$. Suppose that*

$$\lim_{t \to 0} \frac{1}{t} E_{h \cdot m}[C_t^{(1)}] = \lim_{t \to 0} \frac{1}{t} E_{h \cdot m}[C_t^{(2)}] \qquad \text{for } h \in \mathscr{F}_b.$$

Then we have $C^{(1)} = C^{(2)}$.

Proof. By the definition of $\mathscr{N}_c^*$, we can find $u_i \in \mathscr{F}$, $f_i \in L^2(X; m)$ and $A^{(i)}$, $B^{(i)} \in \tilde{\mathbf{A}}_c^+$ such that

$$C_t^{(i)} = N_t^{[u_i]} + \int_0^t f_i(X_s) \, ds + A_t^{(i)} - B_t^{(i)} \qquad \text{for } i = 1, 2.$$

We put

$$d\delta^{(1)} = (f_1^- + f_2^+)\,dm + d\nu^{(1)} + d\mu^{(2)}$$

and

$$d\delta^{(2)} = (f_1^+ + f_2^-)\,dm + d\mu^{(1)} + d\nu^{(2)},$$

where $\mu^{(i)}$ and $\nu^{(i)}$ are the smooth measures which correspond to $A^{(i)}$ and $B^{(i)}$ respectively for $i=1,2$. Then it is easy to see that $\delta^{(1)}$ and $\delta^{(2)}$ are smooth measures. From Theorem 5.3.1 in [6] we obtain

$$\lim_{t\to 0}\frac{1}{t}E_{h\cdot m}[N_t^{[u_1-u_2]}] = -\mathscr{E}(u_1-u_2,h) \qquad \text{for } h\in\mathscr{F}.$$

On the other hand (2.1) and Lemma 2.1 imply

$$\lim_{t\to 0}\frac{1}{t}E_{h\cdot m}\left[\int_0^t(-f_1+f_2)(X_s)\,ds - A_t^{(1)} + B_t^{(1)} + A_t^{(2)} - B_t^{(2)}\right]$$
$$= \langle \delta^{(1)} - \delta^{(2)}, \tilde{h}\rangle \qquad \text{for } h\in\mathscr{F}_b.$$

Thus we obtain $\mathscr{E}(u_1-u_2,h) = -\langle\delta^{(1)}-\delta^{(2)},\tilde{h}\rangle$ for $h\in\mathscr{F}_b$ and, applying Theorem 2.1, we conclude $C^{(1)} = C^{(2)}$. □

The following two lemmas are used in Sect. 5. We use the following notation; for Borel measures μ and ν on X $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν.

Lemma 2.2. *There exists a finite Borel measure σ on X such that*

$$\mu_{\langle M\rangle} \ll \sigma \ll \sum_{L\in\mathring{\mathscr{M}}}\mu_{\langle L\rangle} \qquad \textit{for } M\in\mathring{\mathscr{M}}. \tag{2.7}$$

Proof. We denote by $(\mathscr{F},\mathscr{E}_1)$ the Hilbert space endowed the inner product $\mathscr{E}_1(u,v) = \mathscr{E}(u,v) + \int_X u(x)v(x)\,dm(x)$ $(u,v\in\mathscr{F})$. We can find a countable dense subset $\mathscr{F}' = \{u_n; n=1,2,\dots\}$ of $\mathscr{F}$, because this Hilbert space is separable. Setting

$$\sigma = \sum_{n=1}^{\infty}\frac{1}{2^n(1+\mu_{\langle u_n\rangle}(X))}\mu_{\langle u_n\rangle},$$

it is obvious that σ is a finite Borel measure on X such that $\sigma \ll \sum_{L\in\mathring{\mathscr{M}}}\mu_{\langle L\rangle}$. For any $u\in\mathscr{F}$, there exists a sequence $\{v_n\}_{n=1,2,\dots}$ $(v_n\in\mathscr{F}')$ which converges to u in $(\mathscr{F},\mathscr{E}_1)$. It is easy to see that for a Borel set A of X with $\sigma(A)=0$

$$\mu_{\langle u\rangle}(A) = \mu_{\langle v_n-u\rangle}(A) \leqq \mu_{\langle v_n-u\rangle}(X).$$

Letting $n\to\infty$ we obtain $\mu_{\langle u\rangle}(A)=0$ and thus $\mu_{\langle u\rangle}$ is absolutely continuous with respect to σ. Combining this result and Lemma 5.4.5 in [6], we get $\mu_{\langle M\rangle} \ll \sigma$ for any $M\in\mathring{\mathscr{M}}$ in the same way as above. □

We note that σ-finite measures on X satisfying (2.7) are mutually absolutely continuous.

Lemma 2.3. *Let* $M_n \in \mathring{\mathcal{M}}$ $(n=1,2,\dots)$, $M \in \mathring{\mathcal{M}}$ *and* $L \in \mathring{\mathcal{M}}$. *If* $\{M_n\}_{n=1,2,\dots}$ *converges to* M *in* $(\mathring{\mathcal{M}}, e)$, *then the total variation of* $\mu_{\langle M_n - M, L\rangle}$ *converges to* 0 *as* $n \to \infty$.

Proof. Let $X = \sum_{j=1}^{\infty} X_j$ (X_j is a Borel set of X for $j=1,2,\dots$) be a countable partition of X. Using Lemma 5.4.3 in [6] and the Schwarz inequality we get

$$\sum_{j=1}^{\infty} |\mu_{\langle M_n - M, L\rangle}(X_j)| \leqq \left\{\sum_{j=1}^{\infty} \mu_{\langle M_n - M\rangle}(X_j)\right\}^{1/2} \left\{\sum_{j=1}^{\infty} \mu_{\langle L\rangle}(X_j)\right\}^{1/2}$$
$$\leqq \sqrt{2e(M_n - M)\,\mu_{\langle L\rangle}(X)}.$$

This completes the proof. □

3. Definition of an Integral with Respect to $N_t^{[u]} + \int_0^t g(X_s)\,ds$

First we define an a.f. $Q_t = \int_0^t (v(X_s) + C)\,dN_s^{[u]}$ to be an element of $\mathcal{N}_c^*$ for $C \in \mathbf{R}$ and $u, v \in \mathscr{F}$ with $\tilde{v} \in L^2(X; \mu_{\langle u\rangle})$. If Q is defined to be an element of $\mathcal{N}_c^*$, Theorem 2.2 assures that Q is uniquely determined by the quantities

$$\lim_{t\to 0} \frac{1}{t} E_{h\cdot m}[Q_t] \qquad (h \in \mathscr{F}_b).$$

Denote by $\mathscr{L}$ the generator of the Hunt process $\mathbf{M}$. If u is in the domain of $\mathscr{L}$ and $v \in \mathscr{F}_b$, then $dN_s^{[u]} = \mathscr{L}u(X_s)\,ds$ and we ought to let $Q_t = \int_0^t (v(X_s) + C)\,\mathscr{L}u(X_s)\,ds$. Consequently

$$\lim_{t\to 0} \frac{1}{t} E_{h\cdot m}[Q_t] = (h, (v+C)\,\mathscr{L}u) = -\mathscr{E}(h(v+C), u).$$

In order to construct Q satisfying the above formula, it is therefore necessary to analyze the quantity $\mathscr{E}(h\,v, u)$.

Lemma 3.1. *We have for* $u \in \mathscr{F}$ *and* $h, v \in \mathscr{F}_b$

$$\mathscr{E}(h\,v, u) = \tfrac{1}{2}\int_X \tilde{h}(x)\,d(\overset{c}{\mu}_{\langle v,u\rangle} + \overset{j}{\mu}_{\langle v,u\rangle})(x) + \tfrac{1}{2}\int_X \tilde{v}(x)\,d(\mu_{\langle h,u\rangle} + \overset{k}{\mu}_{\langle h,u\rangle})(x).$$

Proof. It holds from (2.2), (2.5) and (2.6)

$$\begin{aligned}\mathscr{E}(h\,v, u) &= \tfrac{1}{2}\overset{c}{\mu}_{\langle hv,u\rangle}(X) + \tfrac{1}{2}\overset{j}{\mu}_{\langle hv,u\rangle}(X) + \overset{k}{\mu}_{\langle hv,u\rangle}(X)\\ &= \tfrac{1}{2}\int_X \tilde{h}(x)\,d\overset{c}{\mu}_{\langle v,u\rangle}(x) + \tfrac{1}{2}\int_X \tilde{v}(x)\,d\overset{c}{\mu}_{\langle h,u\rangle}(x)\\ &\quad + \tfrac{1}{2}\overset{j}{\mu}_{\langle hv,u\rangle}(X) + \int_X \tilde{v}(x)\,d\overset{k}{\mu}_{\langle h,u\rangle}(x).\end{aligned}$$

Using the symmetry of J and (2.4), we have

$$\overset{j}{\mu}_{\langle hv,u\rangle}(X)=2\int_{X\times X}(\tilde{h}(x)\tilde{v}(x)-\tilde{h}(y)\tilde{v}(y))(\tilde{u}(x)-\tilde{u}(y))\,J(dx\,dy)$$
$$=2\int_{X\times X}\{\tilde{h}(y)(\tilde{v}(x)-\tilde{v}(y))+v(x)(\tilde{h}(x)-\tilde{h}(y))\}(\tilde{u}(x)-\tilde{u}(y))\,J(dx\,dy)$$
$$=\int_X \tilde{h}(x)\,d\overset{j}{\mu}_{\langle v,u\rangle}(x)+\int_X \tilde{v}(x)\,d\overset{j}{\mu}_{\langle h,u\rangle}(x)$$

and thus the proof is completed. □

We introduce the functional λ defined by

$$\lambda(h;M)=\tfrac{1}{2}\,\mu_{\langle M^{[h]}+\overset{k}{M}{}^{[h]},M\rangle}(X)\qquad\text{for } h\in\mathscr{F},\ M\in\mathring{\mathscr{M}}. \tag{3.1}$$

Then by the definition of the stochastic integral $v\cdot M^{[u]}$ (see (5.4.14) in [6]) it holds that

$$\lambda(h;v\cdot M^{[u]})=\tfrac{1}{2}\int_X \tilde{v}(x)\,d(\mu_{\langle h,u\rangle}+\overset{k}{\mu}_{\langle h,u\rangle})(x). \tag{3.2}$$

Lemma 3.2. *For $M\in\mathring{\mathscr{M}}$ there exists a unique function $w\in\mathscr{F}$ such that $\lambda(h;M)=\mathscr{E}_1(w,h)$ for $h\in\mathscr{F}$.*

Proof. We can easily check that

$$\lambda(h;M)^2\leqq\tfrac{1}{2}\,\mu_{\langle M\rangle}(X)(\mu_{\langle h\rangle}(X)+\overset{k}{\mu}_{\langle h\rangle}(X))\leqq\mu_{\langle M\rangle}(X)\,\mathscr{E}_1(h,h)$$

and thus $\lambda(\cdot\,;M)$ is a continuous linear functional defined on the Hilbert space $(\mathscr{F},\mathscr{E}_1)$. □

From now on we denote the function $w\in\mathscr{F}$ in Lemma 3.2 by $\gamma(M)$. It is easy to see that γ is a mapping from $\mathring{\mathscr{M}}$ to $\mathscr{F}$ satisfying

$$\gamma(a_1M_1+a_2M_2)=a_1\gamma(M_1)+a_2\gamma(M_2)\qquad(a_1,a_2\in\mathbf{R},\ M_1,M_2\in\mathring{\mathscr{M}}) \tag{3.3}$$

and

$$\mathscr{E}_1(\gamma(M),\gamma(M))\leqq\mu_{\langle M\rangle}(X)=2e(M)\qquad(M\in\mathring{\mathscr{M}}). \tag{3.4}$$

Using this bounded operator γ, we introduce the linear operator Γ from $\mathring{\mathscr{M}}$ to $\tilde{\mathscr{N}}_c$ in the following manner;

$$\Gamma(M)_t=N_t^{[\gamma(M)]}-\int_0^t\gamma(M)(X_s)\,ds\qquad\text{for } M\in\mathring{\mathscr{M}}. \tag{3.5}$$

We then have

$$\lim_{t\to 0}\frac{1}{t}E_{h\cdot m}[\Gamma(M)_t]=-\lambda(h;M)\qquad\text{for } h\in\mathscr{F}_b. \tag{3.6}$$

In particular we see

$$\Gamma(M^{[u]})=N^{[u]}\qquad\text{for } u\in\mathscr{F}, \tag{3.7}$$

because $\lambda(h;M^{[u]})=\mathscr{E}(h,u)$.

Remark 3.1. Assume that $k=0$ and $\mathbf{M}$ is transient. We denote by $(\mathscr{F}_e, \mathscr{E})$ the extended Dirichlet space associated with $\mathbf{M}$. It is known that $\{M^{[u]}; u \in \mathscr{F}_e\}$ is a closed subspace of $\mathring{\mathscr{M}}$ and so we can consider the projection $\mathscr{P}_{\mathscr{F}_e}$ upon $\{M^{[u]}; u \in \mathscr{F}_e\}$. Then it is easy to see that $\Gamma(M) = N^{[w]}$ for $M \in \mathring{\mathscr{M}}$, where w is a unique element in $\mathscr{F}_e$ such that $M^{[w]} = \mathscr{P}_{\mathscr{F}_e} M$.

Remark 3.2. There are many cases in which $k=0$ and $\{M^{[u]}; u \in \mathscr{F}\}$ is a closed subspace of $\mathring{\mathscr{M}}$. In these cases we can easily check that $\Gamma(M) = N^{[w]}$ for $M \in \mathring{\mathscr{M}}$, where w is an element in $\mathscr{F}$ satisfying $M^{[w]} = \mathscr{P}_{\mathscr{F}} M$ and $\mathscr{P}_{\mathscr{F}}$ is the projection upon $\{M^{[u]}; u \in \mathscr{F}\}$. Though w is not unique but $N^{[w]}$ is determined uniquely.

Definition 3.1. Let $C \in \mathbf{R}$ and $u, v \in \mathscr{F}$ such that $\tilde{v} \in L^2(X; \mu_{\langle u \rangle})$. The a.f.

$$\Gamma((v+C)\cdot M^{[u]}) - \tfrac{1}{2}\langle \overset{c}{M}{}^{[v]} + \overset{j}{M}{}^{[v]}, \overset{c}{M}{}^{[u]} + \overset{j}{M}{}^{[u]} \rangle \tag{3.8}$$

is called the integral of $v(X_{\cdot}) + C$ with respect to $N^{[u]}$. We shall often denote it by $\int_0^t (v(X_s) + C)\, dN_s^{[u]}$.

Obviously this integral is linear in u and $v+C$. Moreover, in case that $v \in \mathscr{F}_b$, we can combine (3.2), (3.6) and (3.7) with Lemma 3.1 to see that the a.f. $\int_0^t (v(X_s) + C)\, dN_s^{[u]}$ satisfies

$$\begin{aligned} &\lim_{t \to 0} \frac{1}{t} E_{h \cdot m}\left[\int_0^t (v(X_s) + C)\, dN_s^{[u]}\right] \\ &\qquad = -\mathscr{E}((v+C)h, u) \qquad \text{for } h \in \mathscr{F}_b, \end{aligned} \tag{3.9}$$

as was expected at the beginning of this section. We further show that, if $N^{[u]}$ is of bounded variation, the a.f. in Definition 3.1 is the same as the a.f. which is defined as the ordinary Stieltjes integral.

Lemma 3.3. *Consider $\mu_1, \mu_2 \in \mathbf{S}$ such that the supports of μ_1 and μ_2 are contained in a compact set K of X. If $u \in \mathscr{F}$ satisfies*

$$\mathscr{E}(h, u) = \langle \mu_1 - \mu_2, \tilde{h} \rangle \qquad \textit{for } h \in \mathscr{F}_b \cap \mathscr{F}_K,$$

then we have

$$\tfrac{1}{2}\int_X \tilde{v}(x)\, d\mu_{\langle h, u \rangle}(x) + \tfrac{1}{2}\int_X \tilde{h}(x)\, d\mu_{\langle v, u \rangle}(x) = \langle \mu_1 - \mu_2, \tilde{h}\tilde{v} \rangle \tag{3.10}$$

for $h \in \mathscr{F}_b \cap \mathscr{F}_K$ and $\tilde{v} \in \mathscr{F} \cap L^2(X; \mu_{\langle u \rangle})$ such that $\tilde{v}$ is bounded on K.

Proof. If $\tilde{v}$ is bounded on X, (3.10) is obvious. Consider the truncations $\tilde{v}_n = (\tilde{v} \vee (-n)) \wedge n$ for $n = 1, 2, \ldots$. Since $\{\tilde{v}_n\}_{n=1,2,\ldots}$ converges to $\tilde{v}$ in $(\mathscr{F}, \mathscr{E}_1)$, we have

$$\begin{aligned} &\lim_{n \to \infty} \left\{\int_X \tilde{v}_n(x)\, d\mu_{\langle h, u \rangle}(x) + \int_X \tilde{h}(x)\, d\mu_{\langle v_n, u \rangle}(x)\right\} \\ &\qquad = \int_X \tilde{v}(x)\, d\mu_{\langle h, u \rangle}(x) + \int_X \tilde{h}(x)\, d\mu_{\langle v, u \rangle}(x). \end{aligned}$$

On the other hand the boundedness of $\tilde{v}$ on K implies

$$\lim_{n\to\infty}\langle\mu_1-\mu_2,\tilde{h}\tilde{v}_n\rangle=\langle\mu_1-\mu_2,\tilde{h}\tilde{v}\rangle.$$

and this completes the proof. □

Theorem 3.1. *Let $C\in\mathbf{R}$, $u\in\mathscr{F}$ and $v\in\mathscr{F}$ such that $\tilde{v}\in L^2(X;\mu_{\langle u\rangle})$. Suppose that $N^{[u]}$ is of bounded variation on each compact interval of $[0,\zeta)$. Then we have for $t<\zeta$*

$$\int_0^t (v(X_s)+C)\,dN_s^{[u]}=\int_0^t(\tilde{v}(X_s)+C)\,d(-A_s^{(1)}+A_s^{(2)}),$$

where $A^{(1)}$, $A^{(2)}\in\mathbf{A}_c^+$ such that $N_t^{[u]}=-A_t^{(1)}+A_t^{(2)}$ $(t<\zeta)$.

Proof. We may assume that $C=0$. Let $\mu^{(i)}$ be the smooth measure corresponding to $A^{(i)}$ for $i=1,2$. According to Theorem 2.1, $\mu^{(1)}$ and $\mu^{(2)}$ satisfy

$$\mathscr{E}(h,u)=\langle I_{K_l}\cdot(\mu^{(1)}-\mu^{(2)}),\tilde{h}\rangle \qquad \text{for } h\in\mathscr{F}_b\cap\mathscr{F}_{K_l} \quad \text{and} \quad l=1,2,\ldots,$$

where $\{K_l\}_{l=1,2,\ldots}$ is a common nest of $\mu^{(1)}$ and $\mu^{(2)}$. It is easy to see that there is a common nest $\{K_l'\}_{l=1,2,\ldots}$ of $\mu^{(1)}$ and $\mu^{(2)}$ such that $K_l'\subset K_l$ and $\tilde{v}|_{K_l'}$ is continuous on K_l' for $l=1,2,\ldots$. Putting $w=\gamma(v\cdot M^{[u]})$ we let

$$dv^{(1)}=\tilde{v}^+\,d\mu^{(1)}+\tilde{v}^-\,d\mu^{(2)}+\tfrac{1}{8}\,d\overset{c}{\mu}_{\langle v-u\rangle}+\tfrac{1}{8}\,d\overset{j}{\mu}_{\langle v-u\rangle}+w^-\,dm$$

and

$$dv^{(2)}=\tilde{v}^-\,d\mu^{(1)}+\tilde{v}^+\,d\mu^{(2)}+\tfrac{1}{8}\,d\overset{c}{\mu}_{\langle v+u\rangle}+\tfrac{1}{8}\,d\overset{j}{\mu}_{\langle v+u\rangle}+w^+\,dm,$$

then $v^{(1)}$ and $v^{(2)}$ are smooth measures.

We have from the definition of $w=\gamma(v\cdot M^{[u]})$

$$\mathscr{E}(h,w)=\tfrac{1}{2}\int_X\tilde{v}(x)\,d(\mu_{\langle h,u\rangle}+\overset{k}{\mu}_{\langle h,u\rangle})(x)-\int_X h(x)\,w(x)\,dm(x) \qquad \text{for } h\in\mathscr{F}.$$

Lemma 3.3 implies for $h\in\mathscr{F}_b\cap\mathscr{F}_{K_l'}$ and $l=1,2,\ldots$

$$\tfrac{1}{2}\int_X\tilde{v}(x)\,d\mu_{\langle h,u\rangle}(x)+\tfrac{1}{2}\int_X\tilde{h}(x)\,d\mu_{\langle v,u\rangle}(x)=\langle I_{K_l'}\cdot(\mu^{(1)}-\mu^{(2)}),\tilde{h}\tilde{v}\rangle.$$

Consequently we obtain for $h\in\mathscr{F}_b\cap\mathscr{F}_{K_l'}$ and $l=1,2,\ldots$

$$\begin{aligned}\mathscr{E}(h,w)&=\langle I_{K_l'}\cdot(\mu^{(1)}-\mu^{(2)}),\tilde{h}\tilde{v}\rangle-\tfrac{1}{2}\int_X\tilde{h}(x)\,d(\overset{c}{\mu}_{\langle v,u\rangle}+\overset{j}{\mu}_{\langle v,u\rangle})(x)\\&\quad-\int_X h(x)\,w(x)\,dm(x)\\&=\langle I_{K_l'}\cdot(v^{(1)}-v^{(2)}),\tilde{h}\rangle.\end{aligned}$$

Therefore Theorem 2.1 implies the desired equality. □

Next we define an integral with respect to $N\in\tilde{\mathscr{N}}_c$.

Definition 3.2. Let $N_t = N_t^{[u]} + \int_0^t g(X_s)\,ds$ ($u \in \mathscr{F}$, $g \in L^2(X;m)$). For $C \in \mathbf{R}$ and $v \in \mathscr{F}_b$, we define the a.f. $\int_0^t (v(X_s)+C)\,dN_s$ by

$$\int_0^t (v(X_s)+C)\,dN_s = \int_0^t (v(X_s)+C)\,dN_s^{[u]} + \int_0^t (v(X_s)+C)\,g(X_s)\,ds \qquad \text{for } t \geqq 0.$$

It follows from Theorem 3.1 that the above integral does not depend on a choice of $u \in \mathscr{F}$ and $g \in L^2(X;m)$ which represent N.

Remark 3.3 (see [14]). If $\mathbf{M}$ is conservative, then $\mathscr{N}_c = \tilde{\mathscr{N}_c}$. So in this case the integral with respect to any continuous a.f. of zero energy is now defined. But $\tilde{\mathscr{N}_c}$ is a proper subspace of $\mathscr{N}_c$ if, for example, $\mathbf{M}$ is an absorbing Brownian motion on $(0,\infty)$. In general a necessary and sufficient condition for which $N \in \mathscr{N}_c$ belongs to $\tilde{\mathscr{N}_c}$ is $\int_{0+} \int_X E_x[N_t]^2/t^2\,dm(x)\,dt < \infty$.

Using the integral with respect to an element in $\tilde{\mathscr{N}_c}$ we can define the following stochastic integral of Itô type.

Definition 3.3. Let $A = M + N$ ($M \in \mathring{\mathscr{M}}$, $N \in \tilde{\mathscr{N}_c}$). For $C \in \mathbf{R}$ and $v \in \mathscr{F}_b$, we define the a.f. $\int_0^t (v(X_s)+C)\,dA_s$ by

$$\int_0^t (v(X_s)+C)\,dA_s = \int_0^t (v(X_s)+C)\,dM_s + \int_0^t (v(X_s)+C)\,dN_s \qquad \text{for } t \geqq 0.$$

The above stochastic integral of Itô type is determined uniquely because $\mathring{\mathscr{M}} \cap \mathscr{N}_c = \{0\}$. Now we state an important property of $\Gamma(M)$ of (3.5) which plays a fundamental role in the above integrals.

Theorem 3.2. *Let $M_n \in \mathring{\mathscr{M}}$ $(n=1,2,\dots)$ and $M \in \mathring{\mathscr{M}}$. If $\{M_n\}_{n=1,2,\dots}$ converges to M in $(\mathring{\mathscr{M}}, e)$, then there exists a subsequence $\{n'\}$ such that for q.e. $x \in X$ $P_x(\lim_{n'\to\infty} \Gamma(M_{n'})_t = \Gamma(M)_t$ uniformly on any finite interval of $t) = 1$.*

Proof. Set $w_n = \gamma(M_n)$ $(n=1,2,\dots)$ and $w = \gamma(M)$. In view of (3.3) and (3.4) $\{w_n\}_{n=1,2,\dots}$ converges to w in $(\mathscr{F}, \mathscr{E}_1)$. We have for any measure ν of finite energy integral

$$\begin{aligned} E_\nu\left[\sup_{0 \leqq s \leqq t}\left|\int_0^s (w_n - w)(X_u)\,du\right|\right] &\leqq \int_0^t \langle \nu, p_s|w_n - w|\rangle\,ds \\ &= \int_0^t \mathscr{E}_1(U_1\nu, p_s|w_n - w|)\,ds \\ &\leqq \mathscr{E}_1(U_1\nu, U_1\nu)^{1/2} \int_0^t \left(\frac{1}{2s}+1\right)^{1/2} ds\,(w_n - w, w_n - w)^{1/2}, \end{aligned}$$

where $U_1\nu$ is the 1-potential of ν and $(\ ,\ )$ is the inner product on $L^2(X;m)$. Therefore, using the same method as in the proof of Lemma 5.1.2 in [6], we can find a subsequence $\{n_l\}$ such that for q.e. $x \in X$

$$P_x\left(\lim_{n_l\to\infty} \int_0^t w_{n_l}(X_s)\,ds = \int_0^t w(X_s)\,ds \text{ uniformly on any finite interval of } t\right) = 1.$$

Theorem 5.2.1 in [6] assures the existence of a subsequence satisfying the above property for $N^{[w_n]}$. □

Finally we want to define a stochastic integral of Stratonovich type but, in doing so, we must assume that **M** is a diffusion. So in the remainder of this section, we assume that $J=k=0$.

Definition 3.4. Assume that $J=k=0$. Let $u\in\mathscr{F}$ and $v\in L^2(X;\mu_{\langle u\rangle})$. The a.f.

$$v\cdot M^{[u]}+\Gamma(v\cdot M^{[u]}) \tag{3.11}$$

is called the stochastic integral of Stratonovich type of $v(X.)$ with respect to $A^{[u]}$. We denote it by $\int_0^t v(X_s)\circ dA_s^{[u]}$.

We discuss a local extension of the stochastic integral of Stratonovich type. Denote by τ_B the leaving time of a Borel set B of X.

Lemma 3.4. *Assume that $J=k=0$. Let $M^{(1)}$, $M^{(2)}\in\mathring{\mathscr{M}}$ and G be a relatively compact open set of X. If $\mu_{\langle M^{(1)}-M^{(2)}\rangle}(G)=0$, then we have for $t\leqq\tau_G$ $\Gamma(M^{(1)})_t=\Gamma(M^{(2)})_t$ and $M_t^{(1)}+\Gamma(M^{(1)})_t=M_t^{(2)}+\Gamma(M^{(2)})_t$.*

Proof. Consider $h\in\mathscr{F}\cap C_0(X)$ such that $\mathrm{supp}(h)\subset G$. We have from Lemma 2.2 in [9] $\mu_{\langle h\rangle}(X-G)=0$ and this implies

$$\begin{aligned}&\mu_{\langle M^{[h]},M^{(1)}-M^{(2)}\rangle}(X)^2\\ &\quad\leqq 2\mu_{\langle h\rangle}(X)\,\mu_{\langle M^{(1)}-M^{(2)}\rangle}(G)+2\mu_{\langle M^{(1)}-M^{(2)}\rangle}(X)\,\mu_{\langle h\rangle}(X-G)\\ &\quad=0.\end{aligned}$$

Consequently we obtain that $\mathscr{E}_1(\gamma(M^{(1)}),h)=\mathscr{E}_1(\gamma(M^{(2)}),h)$, namely, G is a 1-regular set of $\gamma(M^{(1)})-\gamma(M^{(2)})$. By the same argument as in the proof of Theorem 5.3.4 in [6], we can conclude that $\Gamma(M^{(1)})_t=\Gamma(M^{(2)})_t$ for $t\leqq\tau_G$. Combining this with Lemma 5.4.6 in [6], the second assertion is also proved. □

The following lemma is immediate from Lemma 3.4.

Lemma 3.5. *Assume that $J=k=0$. Consider $u_1,u_2\in\mathscr{F}$ and bounded Borel functions v_1 and v_2 on X. Let G be a relatively compact open set of X. If $u_1=u_2$ m-a.e. on G and $v_1=v_2$ on G, then we have for $t\leqq\tau_G$*

$$\int_0^t v_1(X_s)\circ dA_s^{[u_1]}=\int_0^t v_2(X_s)\circ dA_s^{[u_2]}.$$

For a relatively compact open set G of X define $\mathscr{F}(G)=\{u|_G;u\in\mathscr{F}\}$. By virtue of Lemma 3.5, for $w\in\mathscr{F}(G)$ and a bounded Borel function g on G, we can define an integral A which is defined locally;

$$A_t=\int_0^t g(X_s)\circ dA_s^{[w]}\quad \text{for } 0\leqq t\leqq\tau_G. \tag{3.12}$$

We say that u is locally in $\mathscr{F}$ ($u\in\mathscr{F}_{\mathrm{loc}}$ in notation) if $u|_G\in\mathscr{F}(G)$ for any relatively compact open set G of X. Lemma 3.5 enables us to extend Definition 3.4 as follows:

Definition 3.5. Assume that $J=0$ and $\zeta=\infty$. Let v be a locally bounded Borel function on X and $u\in\mathscr{F}_{\mathrm{loc}}$. There exists the unique a.f. A such that for any relatively compact open set G of X

$$A_t=\int_0^t v|_G(X_s)\circ dA_s^{[u|_G]} \qquad \text{for } t\leqq\tau_G. \tag{3.13}$$

A is denoted by $\int_0^t v(X_s)\circ dA_s^{[u]}$.

4. Itô Formula

In this section we show that the usual Itô formula for the semimartingales remains valid in our situation if we use the integral $\int_0^t v(X_s)\,dN_s^{[u]}$ of Sect. 3. In this formula $N^{[u]}$ plays the same role as a process of bounded variation in the usual Itô formula, cf. [2]. Consider $\Psi\in C^1(\mathbf{R}^n)$ such that $\Psi(0)=0$. Here $C^l(\mathbf{R}^n)$ is the space of all l-th continuously differentiable functions on $\mathbf{R}^n$. It is known that $\Psi(u_1,u_2,\ldots,u_n)\in\mathscr{F}_b$ for $u_1,u_2,\ldots,u_n\in\mathscr{F}_b$. First we present a Itô formula for the stochastic integral of Itô type.

Theorem 4.1. *Let $\Phi\in C^2(\mathbf{R}^n)$ and $u_1,u_2,\ldots,u_n\in\mathscr{F}_b$. Then we have for $t\geqq 0$*

$$\begin{aligned}\Phi(\tilde u(X_t))-\Phi(\tilde u(X_0)) &=\sum_{i=1}^n\int_0^t \Phi_i(u(X_s))\,dM_s^{[u_i]}+\widehat{V_t}\\ &\quad+\tfrac12\sum_{i,j=1}^n\int_0^t\Phi_{ij}(\tilde u(X_s))\,d\langle \overset{c}{M}{}^{[u_i]},\overset{c}{M}{}^{[u_j]}\rangle_s+V_t^{\not{p}}\\ &\quad+\sum_{i=1}^n\int_0^t\Phi_i(u(X_s))\,dN_s^{[u_i]},\end{aligned} \tag{4.1}$$

where

$$V_t=\sum_{0<s\leqq t}\left\{\Phi(\tilde u(X_s))-\Phi(\tilde u(X_{s-}))-\sum_{i=1}^n\Phi_i(\tilde u(X_{s-}))(\tilde u_i(X_s)-\tilde u_i(X_{s-}))\right\},$$

$$\Phi_i(x)=\frac{\partial\Phi}{\partial x_i}(x),\qquad \Phi_{ij}(x)=\frac{\partial^2\Phi}{\partial x_i\,\partial x_j}(x)\qquad (i,j=1,2,\ldots,n)$$

and $u=(u_1,u_2,\ldots,u_n)$.

Proof. We may assume $\Phi(0)=0$. Put $w_i=\gamma(\Phi_i(\tilde u)\cdot M^{[u_i]})$ for $i=1,2,\ldots,n$. For $h\in\mathscr{F}_b$ we have from (2.3), (2.4) and Theorem 5.4.3 in [6]

$$\begin{aligned}\sum_{i=1}^n\mathscr{E}_1(w_i,h)&=\tfrac12\sum_{i=1}^n\int_X\Phi_i(\tilde u(x))\,d\mu_{\langle M^{[h]}+\overset{k}{M}{}^{[h]},\,M^{[u_i]}\rangle}(x)\\ &=\mathscr{E}_1(\Phi(u),h)-\lambda_1(h)-\lambda_2(h)-\lambda_3(h),\end{aligned}$$

where

$$\lambda_1(h)=\int_{X\times X}\left\{\Phi(\tilde{u}(x))-\Phi(\tilde{u}(y))-\sum_{i=1}^{n}\Phi_i(\tilde{u}(x))(\tilde{u}_i(x)-\tilde{u}_i(y))\right\}\cdot(\tilde{h}(x)-\tilde{h}(y))\,J(dx\,dy),$$

$$\lambda_2(h)=\int_X\left\{\Phi(\tilde{u}(x))-\sum_{i=1}^{n}\Phi_i(\tilde{u}(x))\,\tilde{u}_i(x)\right\}\tilde{h}(x)\,dk(x)$$

and

$$\lambda_3(h)=\int_X \Phi(u(x))\,h(x)\,dm(x).$$

It is easy to see that for $i=1,2,3$ $\lambda_i(h)$ is a continuous linear functional on the Hilbert space $(\mathscr{F},\mathscr{E}_1)$. By virtue of the Riesz theorem there exists a unique $g_i\in\mathscr{F}$ such that $\lambda_i(h)=\mathscr{E}_1(g_i,h)$ for $h\in\mathscr{F}$ $(i=1,2,3)$. Obviously we have

$$\sum_{i=1}^{n} w_i=\Phi(u)-g_1-g_2-g_3 \quad\text{and}\quad N_t^{[g_3]}=-\int_0^t\Phi(u(X_s))\,ds+\int_0^t g_3(X_s)\,ds.$$

Therefore we obtain

$$\sum_{i=1}^{n}\int_0^t\Phi_i(u(X_s))\,dN_s^{[u_i]}=N_t^{[\Phi(u)]}-N_t^{[g_1+g_2]}+\int_0^t(g_1+g_2)(X_s)\,ds$$
$$-\tfrac{1}{2}\sum_{i=1}^{n}\langle \overset{c}{M}{}^{[\Psi_i(u)]}+\overset{j}{M}{}^{[\Psi_i(u)]},\ \overset{c}{M}{}^{[u_i]}+\overset{j}{M}{}^{[u_i]}\rangle_t,$$

where $\Psi_i(x)=\Phi_i(x)-\Phi_i(0)$ for $i=1,2,\dots,n$. This equality implies that the right hand side of (4.1) is equal to

$$\overset{c}{M}{}_t^{[\Phi(u)]}+N_t^{[\Phi(u)]}+\left\{\sum_{i=1}^{n}\int_0^t\Phi_i(u(X_s))\,d\overset{j}{M}{}_s^{[u_i]}-N_t^{[g_1]}+\int_0^t g_1(X_s)\,ds\right.$$
$$-\tfrac{1}{2}\sum_{i=1}^{n}\langle\overset{j}{M}{}^{[\Psi_i(u)]},\overset{j}{M}{}^{[u_i]}\rangle_t+\sum_{\substack{0<s\leq t\\ t<\zeta}}(\Phi(\tilde{u}(X_s))-\Phi(\tilde{u}(X_{s-}))$$
$$\left.-\sum_{i=1}^{n}\Phi_i(\tilde{u}(X_{s-}))(\tilde{u}_i(X_s)-\tilde{u}_i(X_{s-})))\right\}+\left\{\sum_{i=1}^{n}\int_0^t\Phi_i(u(X_s))\,d\overset{k}{M}{}_s^{[u_i]}-N_t^{[g_2]}\right.$$
$$\left.+\int_0^t g_2(X_s)\,ds+\sum_{\zeta\leq t}(-\Phi(\tilde{u}(X_{\zeta-}))+\sum_{i=1}^{n}\Phi_i(\tilde{u}(X_{\zeta-}))\,\tilde{u}_i(X_{\zeta-}))\right\}$$
$$=\overset{c}{M}{}_t^{[\Phi(u)]}+N_t^{[\Phi(u)]}+J_t+K_t.$$

Therefore it is sufficient to prove $J_t=\overset{j}{M}{}_t^{[\Phi(u)]}$ and $K_t=\overset{k}{M}{}_t^{[\Phi(u)]}$.

We show $J_t=\overset{j}{M}{}_t^{[\Phi(u)]}$. Let $\{K_l\}_{l=1,2,\dots}$ be an increasing sequence of compact sets of X such that $\lim_{l\to\infty}K_l=X$ and ρ be the metric of X. We put for $i=1,2,\dots,n$ and $l=1,2,\dots$.

$$V_t^{i(l)} = \sum_{0<s\leq t} (\tilde{u}_i(X_s) - \tilde{u}_i(X_{s-})) I_{\{(X_s, X_{s-})\in D_l\}},$$

$$V_t^{(l)} = \sum_{0<s\leq t} (\Phi(\tilde{u}(X_s)) - \Phi(\tilde{u}(X_{s-}))) I_{\{(X_s, X_{s-})\in D_l\}},$$

$$W_t^{i(l)} = \sum_{0<s\leq t} (\Psi_i(\tilde{u}(X_s)) - \Psi_i(\tilde{u}(X_{s-}))) I_{\{(X_s, X_{s-})\in D_l\}},$$

$$\overset{j}{M}{}_t^{[u_i](l)} = \hat{V}_t^{i(l)}, \quad \overset{j}{M}{}_t^{[\Phi(u)](l)} = \hat{V}_t^{(l)} \quad \text{and} \quad \overset{j}{M}{}_t^{[\Psi_i(u)](l)} = \hat{W}_t^{i(l)},$$

where $D_l = \left\{(x, y)\in K_l \times K_l;\ \rho(x, y) \geq \frac{1}{l}\right\}$. By virtue of the Riesz theorem there exists a unique $f^{(l)}\in\mathscr{F}$ $(l=1,2,\dots)$ such that for $h\in\mathscr{F}$

$$\mathscr{E}_1(f^{(l)}, h) = \int_{D_l} \left\{\Phi(\tilde{u}(x)) - \Phi(\tilde{u}(y)) - \sum_{i=1}^n \Phi_i(\tilde{u}(x))(\tilde{u}_i(x) - \tilde{u}_i(y))\right\}(\tilde{h}(x) - \tilde{h}(y))\, J(dx\, dy).$$

Putting for $l=1,2,\dots$

$$\begin{aligned} J_t^{(l)} = &\sum_{i=1}^n \int_0^t \Phi_i(\tilde{u}(X_{s-}))\, d\overset{j}{M}{}_s^{[u_i](l)} - N_t^{[f^{(l)}]} + \int_0^t f^{(l)}(X_s)\, ds \\ &- \tfrac{1}{2} \sum_{i=1}^n \langle \overset{j}{M}{}^{[\Psi_i(u)](l)}, \overset{j}{M}{}^{[u_i](l)} \rangle_t + \sum_{0<s\leq t} \left\{\Phi(\tilde{u}(X_s)) - \Phi(\tilde{u}(X_{s-}))\right. \\ &\left. - \sum_{i=1}^n \Phi_i(\tilde{u}(X_{s-}))(\tilde{u}_i(X_s) - \tilde{u}_i(X_{s-}))\right\} I_{\{(X_s, X_{s-})\in D_l\}}, \end{aligned}$$

we have from the symmetry of J for $h\in\mathscr{F}_b$

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{t} E_{h\cdot m}[J_t^{(l)} - \overset{j}{M}{}_t^{[\Phi(u)](l)}] \\ &\quad = 2 \sum_{i=1}^n \int_{D_l} \tilde{h}(x)\, \Phi_i(\tilde{u}(x))(\tilde{u}_i(x) - \tilde{u}_i(y))\, J(dx\, dy) + \mathscr{E}_1(f^{(l)}, h) \\ &\qquad - \sum_{i=1}^n \int_{D_l} \tilde{h}(x)(\Phi_i(\tilde{u}(x)) - \Phi_i(\tilde{u}(y)))(\tilde{u}_i(x) - \tilde{u}_i(y))\, J(dx\, dy) \\ &\qquad - 2 \int_{D_l} \tilde{h}(x)(\Phi(\tilde{u}(x)) - \Phi(\tilde{u}(y)))\, J(dx\, dy) \\ &\quad = 0. \end{aligned}$$

By virtue of Theorem 2.2 we arrive at $J_t^{(l)} = \overset{j}{M}{}_t^{[\Phi(u)](l)}$ for $l=1,2,\dots$. Since $\{K_l\}_{l=1,2,\dots}$ converges to X and $\{f^{(l)}\}_{l=1,2,\dots}$ converges to g_1 in $(\mathscr{F}, \mathscr{E}_1)$, there exists a subsequence $\{l'\}$ satisfying that for q.e. $x\in X$, $P_x(J_t^{(l')}$ converges to J_t on each finite interval of $[0,\infty)) = 1$ and $P_x(\overset{j}{M}{}_t^{[\Phi(u)](l')}$ converges to $\overset{j}{M}{}_t^{[\Phi(u)]}$ on each finite interval of $[0,\infty)) = 1$. Therefore $J_t = \overset{j}{M}{}_t^{[\Phi(u)]}$.

We can prove $K_t = \overset{k}{M}{}_t^{[\Phi(u)]}$ in the same way. □

Now we discuss a Itô formula involving the stochastic integral of Stratonovich type.

Theorem 4.2. *Assume that $J=k=0$. Let $\Phi\in C^1(\mathbf{R}^n)$, $u_1, u_2, \dots, u_n \in \mathscr{F}_b$ and g be a bounded Borel function on X. Then we have for $t\geqq 0$*

$$\int_0^t g(X_s)\circ dA_s^{[\Phi(u)]} = \sum_{i=1}^n \int_0^t g(X_s)\,\Phi_i(u(X_s))\circ dA_s^{[u_i]}, \tag{4.2}$$

where $u=(u_1, u_2, \dots, u_n)$ and $\Phi_i(x)=\frac{\partial \Phi}{\partial x_i}(x)$ $(i=1, 2, \dots, n)$.

Proof. It is known that $g\cdot M^{[\Phi(u)]}=\sum_{i=1}^n g\,\Phi_i(u)\cdot M^{[u_i]}$. Since the mapping Γ is linear we obtain $\Gamma(g\cdot M^{[\Phi(u)]})=\sum_{i=1}^n \Gamma(g\,\Phi_i(u)\cdot M^{[u_i]})$. Thus (4.2) follows from the definition of the stochastic integral of Stratonovich type. □

We can easily obtain the following corollary.

Corollary 4.1. *Assume that $J=k=0$. Let G be a relatively compact open set of X. Consider $\Phi\in C^1(\mathbf{R}^n)$, $u_1, u_2, \dots, u_n \in \mathscr{F}_b(G)$ and a bounded Borel function g on G, where $\mathscr{F}_b(G)=\{v|_G;\ v\in\mathscr{F}_b\}$. Then it holds that for $t\leqq\tau_G$*

$$\int_0^t g(X_s)\circ dA_s^{[\Phi(u)]} = \sum_{i=1}^n \int_0^t g(X_s)\,\Phi_i(u(X_s))\circ dA_s^{[u_i]}, \tag{4.3}$$

where $u=(u_1, u_2, \dots, u_n)$ and $\Phi_i(x)=\frac{\partial \Phi}{\partial x_i}(x)$ $(i=1, 2, \dots, n)$.

5. Stochastic Line Integral of Differential Form Along Paths of Diffusion Process

The purpose of this section is, using the stochastic calculus in Sect. 3 and Sect. 4, to define the stochastic line integral for symmetric diffusion processes on manifolds in a systematic way.

Consider an m-symmetric diffusion process $\mathbf{M}=(\Omega, \mathscr{B}, \zeta, X_t, P_x)$ on a σ-compact d-dimensional manifold X such that $\zeta=\infty$. The Dirichlet space associated with $\mathbf{M}$ is denoted by $(\mathscr{F}, \mathscr{E})$ and in this section we assume that $C_0^\infty(X)$ is a dense subspace of $(\mathscr{F}, \mathscr{E}_1)$, where $C_0^\infty(X)$ is the space of all C^∞-functions with compact support. Let $\Lambda_{\text{l.b.}}(X)$ be the space of all differentiable 1-forms on X with locally bounded Borel measurable coefficients.

Definition 5.1. Let $\alpha\in\Lambda_{\text{l.b.}}(X)$. A continuous a.f. satisfying the following condition is called a stochastic line integral of α along X_t $(0\leqq t<\infty)$; for any local coordinate neighbourhood $(U, \phi=(x^1, x^2, \dots, x^d))$ of X such that U is relatively compact, it holds that

$$P_x\left(A_t=\sum_{i=1}^{d}\int_0^t \alpha_i(X_s)\circ dA_s^{[x^i]} \quad \text{for } t\leqq \tau_U\right)=1 \qquad (\text{q.e. } x\in X), \tag{5.1}$$

where $\alpha=\sum_{i=1}^{d}\alpha_i(x)\,dx^i$ on U. A_t is often denoted by $\int_{X[0,t]}\alpha$.

Lemma 5.1. *For any $\alpha\in\Lambda_{\text{l.b.}}(X)$ the stochastic line integral of α along X_t $(0\leqq t<\infty)$ is determined uniquely.*

Proof. Assume that continuous a.f.'s $A^{(1)}$ and $A^{(2)}$ satisfy (5.1). For relatively compact local coordinate neighbourhoods $(U,\phi=(x^1,x^2,\dots,x^d))$ and $(V,\psi=(y^1,y^2,\dots,y^d))$, applying Corollary 4.1, we obtain

$$P_x\left(\sum_{i=1}^{d}\int_0^t \alpha_i(X_s)\circ dA_s^{[x^i]}=\sum_{i=1}^{d}\int_0^t \beta_i(X_s)\circ dA_s^{[y^i]} \text{ for } t\leqq\tau_{U\cap V}\right)=1 \qquad (\text{q.e. } x\in X),$$

where $\alpha=\sum_{i=1}^{d}\alpha_i(x)\,dx^i$ on U and $\alpha=\sum_{i=1}^{d}\beta_i(y)\,dy^i$ on V. Consequently there exists a stopping time η such that $P_x(\eta>0)=1$ and $P_x(A^{(1)}(t)=A^{(2)}(t)$ for $t\leqq\eta)=1$ for q.e. $x\in X$. Thus, using the additivity of a.f., we can conclude that $A^{(1)}=A^{(2)}$. □

Now we discuss the existence of the stochastic line integral of $\alpha\in\Lambda_{\text{l.b.}}(X)$ along X_t $(0\leqq t<\infty)$. To prove this, first of all, we construct a Hilbert space of 1-forms on X with Borel measurable coefficients which is isomorphic to the Hilbert space $(\mathring{\mathcal{M}},e)$ in a natural way. We choose $\mathscr{U}=\{(U_n,\phi_n);n=1,2,\dots\}$ satisfying the following conditions;

(i) for each $n=1,2,\dots$, U_n is a relatively compact open set of X and all components of $\phi_n(x)=(x_n^1(x),\ x_n^2(x),\ \dots,\ x_n^d(x))$ belong to $C_0^\infty(X)$,

(ii) for each $n=1,2,\dots,(U_n,\phi_n|_{U_n})$ is a local coordinate neighbourhood of X and $\{U_n\}_{n=1,2,\dots}$ is an open covering of X.

Put $V_1=U_1$ and $V_n=U_n-\bigcup_{j=1}^{n-1}U_j\,(n\geqq 2)$ and fix a σ-finite Borel measure σ on X satisfying (2.7). For $i,j=1,2,\dots,d$ and $n=1,2,\dots$ the Radon-Nikodym derivative $\dfrac{d\mu_{\langle x_n^i,x_n^j\rangle}}{2d\sigma}(x)$ is denoted by $a_n^{ij}(x)$. Setting $a^{ij}(x)=a_n^{ij}(x)$ $(x\in V_n,\ n=1,2,\dots)$ and $A(x)=(a^{ij}(x))_{i,j=1,2,\dots,d}$, then it is easy to see that for σ-a.e. $x\in X, A(x)$ is symmetric and non-negative definite. A 1-form $\alpha=\sum_{i=1}^{d}\alpha_i(x)dx_n^i$ $(x\in V_n, n=1,2,\dots)$ on X is denoted by $\alpha=\sum_{i=1}^{d}\alpha_i(x)dx^i$ simply. For 1-forms $\alpha=\sum_{i=1}^{d}\alpha_i(x)dx^i$ and $\beta=\sum_{i=1}^{d}\beta_i(x)dx^i$ on X, define

$$\langle\alpha,\beta\rangle(x)=\sum_{i,j=1}^{d}\alpha_i(x)\beta_j(x)a^{ij}(x) \qquad (x\in X)$$

and

$$\mathbb{E}(\alpha,\beta)=\int_X\langle\alpha,\beta\rangle(x)\,d\sigma(x).$$

Let $\mathbb{F}^*$ be the space of all 1-forms with Borel measurable coefficients α on X such that $\mathbb{E}(\alpha, \alpha)$ is finite. $(\mathbb{F}^*, \mathbb{E})$ is independent of a choice of $\mathscr{U}$ ans σ. Two 1-forms α and β in $\mathbb{F}^*$ are said to be equivalent if $\mathbb{E}(\alpha-\beta, \alpha-\beta)=0$ and the set of all equivalence classes of $\mathbb{F}^*$ is denoted by $\mathbb{F}$. We shall show that $(\mathbb{F}, \mathbb{E})$ is the desired Hilbert space. Set for $\mathscr{M} \in \mathring{\mathscr{M}}$ and $i=1,2,\ldots,d$, $\beta_i(x; M) = \dfrac{d\mu_{\langle M, M^{[x_n^i]}\rangle}}{d\sigma}(x)$ $(x \in V_n,\ n=1,2,\ldots)$.

Lemma 5.2. *Let $M \in \mathring{\mathscr{M}}$. Then it holds that ${}^t\beta(x; M)={}^t(\beta_1(x; \mathrm{M}), \beta_2(\mathrm{x}; M), \ldots, \beta_d(x; M)) \in \operatorname{Range} A(x)$ for σ-a.e. $x \in X$, where ${}^ta={}^t(a_1, \ldots, a_d)$ is the transposed vector of $a \in \mathbf{R}^d$.*

Proof. First consider the case where $M=f \cdot M^{[u]}$ $(f \in C_0(X),\ u \in C_0^\infty(X))$. The assertion is clear. By virtue of Lemma 5.4.5 in [6], for any $M \in \mathring{\mathscr{M}}$, we can find sequences $f_n \in C_0(X)$ and $u_n \in C_0^\infty(X)$ $(n=1,2,\ldots)$ such that $\{M_n = f_n \cdot M^{[u_n]}\}_{n=1,2\ldots}$ converges to M in $(\mathring{\mathscr{M}}, e)$. Then applying Lemma 2.3 we obtain $\lim_{n\to\infty} \int_X |\beta_i(x; M_n)-\beta_i(x; M)| d\sigma(x)=0$ for $i=1,2,\ldots,d$. This implies ${}^t\beta(x; M) \in \operatorname{Range} A(x)$ for σ-a.e. $x \in X$. □

Lemma 5.3. *Let $c(x)$ be a Borel measurable symmetric nonnegative definite (d, d)-matrix valued function on X and $f(x)$ be a Borel measurable $\mathbf{R}^d$-valued function on X. If ${}^tf(x) \in \operatorname{Range} c(x)$ for any $x \in X$, then there exists a Borel measurable $\mathbf{R}^d$-valued function $g(x)$ on X such that ${}^tf(x)=c(x){}^tg(x)$ for any $x \in X$.*

Proof. Put $c_\varepsilon(x)=c(x)+\varepsilon I$ $(\varepsilon>0)$, where I is the identity matrix. Then we can easily check that $\{c_\varepsilon(x)^{-1}\,{}^tf(x)\}$ is convergent as $\varepsilon \to 0$ and ${}^tg(x) = \lim_{\varepsilon\to 0} c_\varepsilon(x)^{-1}\,{}^tf(x)$ satisfies ${}^tf(x)=c(x){}^tg(x)$ for any $x \in X$. □

Lemma 5.4. *Consider $M \in \mathring{\mathscr{M}}$.*

(i) *There exists a Borel measurable $\mathbf{R}^d$-valued function $\alpha(x; M)$ on X such that*

$$ {}^t\beta(x; M)=A(x){}^t\alpha(x; M) \quad \text{for } \sigma\text{-a.e. } x \in X. \tag{5.2}$$

(ii) *If a Borel measurable $\mathbf{R}^d$-valued function*

$$\alpha(x; M)=(\alpha_1(x; M), \alpha_2(x; M), \ldots, \alpha_d(x; M))$$

on X satisfies (5.2), *then we have*

$$e(M)=\int_X \left\{ \sum_{i,j=1}^d \alpha_i(x; M)\alpha_j(x; M)a^{ij}(x) \right\} d\sigma(x).$$

Proof. (i) is immediate from Lemma 5.2 and Lemma 5.3. To prove (ii) we choose sequences

$$f_m \in C_0(X) \quad \text{and} \quad u_m \in C_0^\infty(X) \quad (m=1,2,\ldots)$$

such that $\{M_m=f_m \cdot M^{[u_m]}\}$ converges to M. Now fix $n=1,2,\ldots$ and set

$$X_N=\left\{x \in V_n;\ \sum_{i=1}^d \alpha_i(x; M)^2 \leqq N\right\} \text{ for } N=1,2,\ldots.$$

In view of (2.2) and (5.2) we can obtain

$$\begin{aligned}\tfrac{1}{2}\mu_{\langle M,M_m\rangle}(X_N)&=\tfrac{1}{2}\sum_{i=1}^{d}\int_{X_N} f_m(x)\frac{\partial u_m}{\partial x_n^i}(x)d\mu_{\langle M,x_n^i\rangle}(x)\\&=\sum_{i,j=1}^{d}\int_{X_N} f_m(x)\frac{\partial u_m}{\partial x_n^i}(x)\alpha_j(x;M)a^{ij}(x)d\sigma(x)\\&=\sum_{j=1}^{d}\int_{X_N}\alpha_j(x;M)d\mu_{\langle x_n^j,M_m\rangle}(x)\\&=\sum_{j=1}^{d}\int_{X_N}\alpha_j(x;M)\beta_j(x;M_m)d\sigma(x).\end{aligned}$$

Therefore, letting $m\to\infty$, Lemma 2.3 provides us with

$$\tfrac{1}{2}\mu_{\langle M,M\rangle}(X_N)=\int_{X_N}\left\{\sum_{i,j=1}^{d}\alpha_i(x;M)\alpha_j(x;M)a^{ij}(x)\right\}d\sigma(x)$$

and then, letting $N\to\infty$, we have by Lebesgue's monotone convergence theorem

$$\tfrac{1}{2}\mu_{\langle M,M\rangle}(V_n)=\int_{V_n}\left\{\sum_{i,j=1}^{d}\alpha_i(x;M)\alpha_j(x;M)a^{ij}(x)\right\}d\sigma(x).$$

The proof is thus completed. □

The above lemma provides us with the following theorem.

Theorem 5.1. $(\mathbb{F},\mathbb{E})$ *is a Hilbert space which is isomorphic to the Hilbert space* $(\mathring{\mathscr{M}},e)$ *by the isomorphism* Ξ *characterized by*

$$\Xi(f\,du)=f\cdot M^{[u]}\quad \text{for } f\in C_0(X) \text{ and } u\in C_0^\infty(X).$$

Proof. For $\alpha=\sum_{i=1}^{d}\alpha_i(x)dx^i\in\mathbb{F}$ and $n=1,2,\ldots$ we denote by $M_n^{[\alpha]}$ the limit in $\mathring{\mathscr{M}}$ of the sequence

$$\left\{\sum_{i=1}^{d}(I_{X_{n,N}}\alpha_i)\cdot M^{[x_n^i]}\right\}_{N=1,2\ldots},$$

where

$$X_{n,N}=\left\{x\in V_n;\ \sum_{i=1}^{d}\alpha_i(x)^2\leqq N\right\}.$$

Since

$$e(M_n^{[\alpha]},M_m^{[\alpha]})=\delta_{n,m}\int_{V_n}\langle\alpha,\alpha\rangle(x)d\sigma(x),\qquad M^{[\alpha]}=\sum_{n=1}^{\infty}M_n^{[\alpha]}$$

is convergent in $(\mathring{\mathscr{M}},e)$ and $e(M^{[\alpha]})=\mathbb{E}(\alpha,\alpha)$. Setting $\Xi(\alpha)=M^{[\alpha]}$ it is clear that Ξ is a linear operator from $\mathbb{F}$ to $\mathring{\mathscr{M}}$. Ξ is injective from $e(M^{[\alpha]})=\mathbb{E}(\alpha,\alpha)$ $(\alpha\in\mathbb{F})$ and, applying Lemma 5.4, Ξ is surjective. □

Now we show the existence of the stochastic line integral of $\alpha\in\Lambda_{\text{l.b.}}(X)$ along $X_t(0\leqq t<\infty)$.

Theorem 5.2. *The stochastic line integral of* $\alpha \in A_{\mathrm{l.b.}}(X)$ *along* $X_t (0 \leqq t < \infty)$ *exists. In particular if* $\alpha \in A_{\mathrm{l.b.}}(X) \cap \mathbb{F}$, *then the stochastic line integral of* α *along* $X_t (0 \leqq t < \infty)$ *satisfies*

$$\int_{X[0,t]} \alpha = \Xi(\alpha)_t + \Gamma(\Xi(\alpha))_t \qquad \text{for } t \geqq 0.$$

Proof. First consider the case where $\alpha \in A_{\mathrm{l.b.}}(X) \cap \mathbb{F}$ and put $A = \Xi(\alpha) + \Gamma(\Xi(\alpha))$. It is sufficient to prove that A satisfies (5.1), because A is a continuous a.f. already. Let $(U, \phi = (x^1, \dots, x^d))$ be a relatively compact local coordinate neighbourhood of X and $\alpha = \sum_{i=1}^{d} \alpha_i(x) dx^i$ on U. For any bounded Borel measurable $\mathbf{R}^d$-valued function $(\bar{\alpha}_1(x), \bar{\alpha}_2(x), \dots, \bar{\alpha}_d(x))$ on X with $(\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_d)|_U = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and any $(u_1, u_2, \dots, u_d)$ $(u_i \in \mathscr{F}\ i = 1, 2, \dots, d)$ with $(u_1, u_2, \dots, u_d)| = (x^1, x^2, \dots, x^d)$ m-a.e. on U, we can easily check that

$$\mu_{\langle \sum_{i=1}^{d} \bar{\alpha}_i \cdot M^{[u_i]} - \Xi(\alpha) \rangle}(U) = 0.$$

So Lemma 3.4 implies for q.e. $x \in X$

$$P_x\left(A_t = \sum_{i=1}^{d} \int_0^t \alpha_i(X_s) \circ dA_s^{[x^i]} \text{ for } t \leqq \tau_U\right) = 1.$$

Namely A is the stochastic line integral of α.

Using Lemma 3.4 again, we can see the existence of the stochastic line integral of $\alpha \in A_{\mathrm{l.b.}}(X)$. □

Remark 5.1. For a certain class of symmetric diffusion processes on a manifold such that $\{M^{[u]}; u \in \mathscr{F}\}$ is a closed subspace of $\mathring{\mathscr{M}}$, Kusuoka has defined the stochastic line integral of $\alpha \in \mathbb{F}$ by $\Xi(\alpha) + N^{[w]}$, where $w \in \mathscr{F}$ is an element with $M^{[w]} = \mathscr{P}_{\mathscr{F}} \Xi(\alpha)$ and $\mathscr{P}_{\mathscr{F}}$ is the projection in Remark 3.2. Our present definition is an extension of his to a more general situation (see Remark 3.2).

Finally we state an application of the stochastic line integral. This is due to Kusuoka (a communication at a conference on Markov processes, held in Japan 1981).

Example 5.1. Let $A(y) = (a^{ij}(y))_{i,j=1,2,\dots,d}$ be a measurable, bounded, symmetric and uniformly elliptic (d, d)-matrix valued function on $\mathbf{R}^d$. Suppose that each component of $A(y)$ is a periodic function with period 1 in each coordinate y^i $(1 \leqq i \leqq d)$. Consider the symmetric diffusion processes $\{Y_t^\varepsilon\}$ $(\varepsilon > 0)$ on $\mathbf{R}^d$ whose Dirichlet forms are

$$\mathscr{E}^{(\varepsilon)}(u, v) = \sum_{i,j=1}^{d} \int_{\mathbf{R}^d} \frac{\partial u}{\partial y^i}(y) \frac{\partial v}{\partial y^j}(y) a^{ij}\left(\frac{y}{\varepsilon}\right) dy.$$

Fukushima [7] discussed the homogenization problem for $\{Y^\varepsilon\}$ and showed that $\{Y^\varepsilon\}$ is convergent weakly in $C([0, \infty) \to \mathbf{R}^d)$ as $\varepsilon \to 0$.

Kusuoka formulated the above homogenization problem as a limit theorem of the stochastic line integral $\varepsilon \int_{X[0, t/\varepsilon^2]} \alpha$ of the 1-form α. Here X_t is a symmetric diffusion process on a torus $\mathbf{T}^d=[0,1)^d$ whose (extended recurrent) Dirichlet space $(\mathscr{E},\mathscr{F})$ is given by

$$\mathscr{E}(u,v)=\sum_{i,j=1}^{d}\int_{\mathbf{T}^d}\frac{\partial u}{\partial x^i}(x)\frac{\partial v}{\partial x^j}(x)a^{ij}(x)dx$$

and

$$\begin{aligned}\mathscr{F}=\{u\in H^1(\mathbf{T}^d);\ &\int_{\mathbf{T}^d}u(x)dx=0,\ u(x^1,\ldots,x^{i-1},0+,x^{i+1},\ldots,x^d)\\ &=u(x^1,\ldots,x^{i-1},1-,x^{i+1},\ldots,x^d)\\ &\quad\text{for a.e. }(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^d)\ (1\leqq i\leqq d)\}.\end{aligned}$$

In particular we can see that $Y_t^\varepsilon-Y_0^\varepsilon$ is equivalent to $(\varepsilon\int_{X[0,t/\varepsilon^2]}dx^i)_{1\leqq i\leqq d}$. For $1\leqq i\leqq d$ there exists the unique bounded function $\chi^i\in\mathscr{F}$ such that

$$\mathbb{E}(dx^i-d\chi^i,du)=0\qquad\text{for any }u\in\mathscr{F}\tag{5.3}$$

(see [7]). It holds that for $1\leqq i\leqq d$

$$\varepsilon\int_{X[0,t/\varepsilon^2]}dx^i=\varepsilon\int_{X[0,t/\varepsilon^2]}(dx^i-d\chi^i)+\varepsilon\chi^i(X_{t/\varepsilon^2})-\varepsilon\chi^i(X_0)$$

and the above stochastic line integrals are well defined (Remark 5.1). Since the main part of the above expression is $(\varepsilon\int_{X[0,t/\varepsilon^2]}(dx^i-d\chi^i))_{1\leqq i\leqq d}$, the covariance function of the limiting Wiener process is determined by $\mathbb{E}(dx^i-d\chi^i,dx^j-d\chi^j)$, the harmonic part of the 1-forms dx^i (in the sense of (5.3)), $i,j=1,2,\ldots,$ d.

Acknowledgement. I would like to thank Professor Shigeo Kusuoka for allowing me to refer to his unpublished result.

References

1. Benveniste, A., Jacod, J.: Systèmes de Lévy des processus de Markov. Inventiones Math. **21**, 183–198 (1973)
2. Dellacherie, C., Meyer, P.A.: Probabilités et potentiel Theorie des martingales. Paris: Hermann 1980
3. Föllmer, H.: Calcul d'Itô sans probabilités. Séminaire de Probabilités XV. Lecture Notes in Math. **850**. pp. 143–150. Berlin-Heidelberg-New York: Springer 1981
4. Fukushima, M.: On additive functionals admitting exceptional sets. J. Math. Kyoto Univ. **19**, 191–202 (1979)
5. Fukushima, M.: A decomposition of additive functionals of finite energy. Nagoya Math. J. **74**, 137–168 (1979)
6. Fukushima, M.: Dirichlet Forms and Markov Processes. Amsterdam-Oxford-New York: North-Holland, Tokyo: Kodansha 1980
7. Fukushima, M.: A generalized stochastic calculus in homogenization. Proceeding of the Symposium "Quantum Fields-Algebras, Processes" edited by L. Streit. Wien-New York: Springer 1980
8. Fukushima, M.: Capacitary maximal inequalities and an ergodic theorem. Probability Theory and Mathematical Statistics. Lecture Notes in Math. **1021**. pp. 130–136. Berlin-Heidelberg-New York: Springer 1983

9. Fukushima, M., Takeda, M.: A transformation of a symmetric Markov process and the Donsker-Varadhan theory. Osaka J. Math. **21**, 311–326 (1984)
10. Ikeda, N., Manabe, S.: Integral of differential forms along the path of diffusion processes. Publ. RIMS, Kyoto Univ. **15**, 827–852 (1979)
11. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. Amsterdam-Oxford-New York: North-Holland, Tokyo: Kodansha 1981
12. LeJan, Y.: Mesures associées a une forme de Dirichlet. Applications. Bull. Soc. Math. France **106**, 61–112 (1978)
13. Meyer, P.A.: Martingales locales functionnelles additives (I). Séminaire de Probabilités XII. Lecture Notes in Math. **649**. pp. 775–785. Berlin-Heidelberg-New York: Springer 1978
14. Nakao, S.: On representation of continuous additive functionals of zero energy. (to appear).
15. Osada, H.: Homogenization of diffusion processes with random stationary coefficients. Probability Theory and Mathematical Statistics. Lecture Notes in Math. **1021**, pp. 507–517. Berlin-Heidelberg-New York: Springer 1983
16. Ôshima, Y., Yamada, T.: On some representations of continuous additive functionals locally of zero energy. J. Math. Soc. Japan **36**, 315–339 (1984)
17. Stroock, D.W., Varadhan, S.R.S.: Multidimensional Diffusion Processes. Berlin-Heidelberg-New York: Springer 1979

Received December 6, 1983; in final form June 1, 1984