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The k-record Processes are i.i.d.

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Abstract. Let $X_1, X_2, \ldots$ be i.i.d. positive random variables, and let ρ_n be the initial rank of X_n (that is, the rank of X_n among $X_1, \ldots, X_n$). Those observations whose initial rank is k are collected into a point process N^k on $\mathbb{R}^+$, called the k-record process. The fact that $\{N^k; k=1,2,\ldots\}$ are *independent and identically distributed* point processes is the main result of the paper. The proof, based on martingales, is very rapid. We also show that given $N^1, \ldots, N^k$, the "lifetimes" in rank k of all observations of initial rank at most k are independent geometric random variables.

These results are generalised to continuous time, where the analogue of the i.i.d. sequence is a "time-space" Poisson process. Initially, we think of this Poisson process as having values in $\mathbb{R}^+$, but subsequently we extend to Poisson processes with values in more general Polish spaces (for example, Brownian excursion space) where ranking is performed using real-valued attributes.

1. Introduction

Let $X_1, X_2, \ldots$ be i.i.d. (independent, identically distributed) positive random variables with distribution function F; we assume that $F(x)<1$ for all x, and $F(0)=0$. Define the *initial rank* ρ_n of X_n by

$$\rho_n \equiv \text{no. of } k \leqq n \text{ such that } X_k \geqq X_n,$$

and define for each $j=1,2,\ldots$ the stopping times

$$T_0^j \equiv 0, \qquad T_{n+1}^j \equiv \inf\{k\colon k>T_n^j, \rho_k=j\} \qquad (n\in\mathbb{Z}^+).$$

The times $T_1^j, T_2^j, \ldots$ are the times at which observations with initial rank j occur, and the values of those observations are

$$R_n^j \equiv X_{T_n^j} \qquad (n=1,2,\ldots).$$

It is clear that $R_1^j < R_2^j < \ldots$, and we may think of the points $\{R_n^j \colon n \in \mathbb{N}\}$ as a point process in $\mathbb{R}^+$. For the present, this and other point processes will be most conveniently formulated by the *counting process*, here given by

$$N_x^j \equiv \sum_{n=1}^{\infty} 1\{R_n^j \leqq x\} \qquad (x \geqq 0).$$

We call this point process the *j-record process*. Thus the 1-record process is just the ordinary record sequence. Collectively, these processes are *partial record processes*.

If the law of the X_i is exponential with mean 1 then it is not hard to see that N^1 is a unit Poisson process, that is, $N_x^1 - x$ is a martingale. However it is far from obvious that

$$\{N^j; j \geqq 1\} \textit{ are i.i.d. unit Poisson processes;} \tag{1.1}$$

this is a special case of our main result, Theorem 1.

Next we consider lifetimes of observations in various ranks. If $T_n^j = t$, then the rank of $X_t \equiv R_n^j$ among $X_1, \ldots, X_t$ is j, but as more observations are taken, the rank of X_t among the larger sample $X_1, \ldots, X_m$ increases as $m \geqq t$ increases. We define the *lifetime* $L_n^{j,k}$ of the observation R_n^j in rank $k (k \geqq j)$ to be the number of integers m such that X_t is ranked k^{th} among $X_1, \ldots, X_m$.

Lifetimes depend in a complicated way on the partial record processes. Where this dependence becomes transparent is in that

conditionally on $\{N^j; j = 1, 2, \ldots, k\}$ *the lifetimes* $L_n^{j,k}$ *in rank* k *are independent, for* $j = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots,$ *with laws*

$$P(L_n^{j,k} = l \mid N^i;\, i = 1, \ldots, k) = (1 - F(R_n^j))\, F(R_n^j)^{l-1} \qquad (l \in \mathbb{N}). \tag{1.2}$$

The reader will agree, we feel, that this result is intuitively 'right' even without the result (1.1).

Our proof of (1.1) is straightforward and short, but, being largely computational, it does not give a very clear *explanation* of the amazing fact that the N^j are i.i.d.. We outline now an argument that makes (1.1) plausible, even though it would not be easy to formulate rigorously. For simplicity, $F(x) = 1 - e^{-x}$.

Fix some $k > 1$ and consider the j-record processes for $j \leqq k$. Discard every observation of initial rank bigger than k, leaving a subsequence $Y_j \equiv X_{n_j}$. We shall consider the first point $Y \equiv R_1^k$ of the k-record process and show that, conditional on $\mathscr{G}_k$, the σ-field generated by $N^1, \ldots, N^{k-1}$, Y has density e^{-x}. Obviously, none of the first $k-1$ observations can be in the k-record process, although the k^{th} might be. Figure 1 illustrates the situation after the first $k-1$ observations have been made (taking $k=5$):

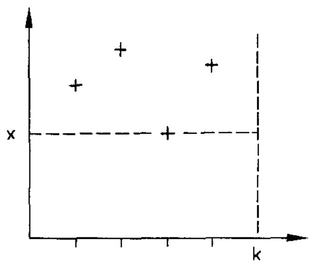

Fig. 1

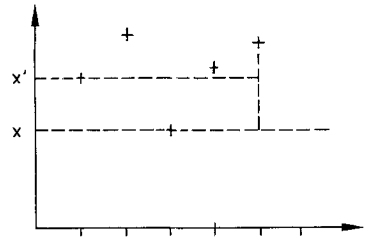

Fig. 2

Let x be the smallest of the first $k-1$ observations. At the k^{th} observation, one of two things may happen:

(i) $X_k \leqq x$, in which case $X_k = Y$;

(ii) $X_k > x$, in which case $Y > x$.

In the first case,

$$P(Y \in dy | \mathscr{G}_k) = e^{-y} dy \qquad \text{for } 0 \leqq y \leqq x.$$

In the second case, Y must be greater than x, so any subsequent observation in $[0, x]$ can be ignored. So the situation is now that illustrated in Fig. 2 where x' is the second smallest of the first k observations. But if we ignore observations in $[0, x]$, the observations in (x, ∞) are i.i.d. exponential random variables, so the picture in Fig. 2 is essentially the same as in Fig. 1; we are waiting for the first rank k observation. By the argument in the first case then,

$$P(Y - x \in dy | \mathscr{G}_k, Y > x) = e^{-y} dy \qquad \text{for } 0 \leqq y \leqq x' - x$$

But $P(Y > x | \mathscr{G}_k) = e^{-x}$, so that $P(Y \in dy | \mathscr{G}_k) = e^{-y} dy$ for $0 \leqq y \leqq x'$ and the conditional density of Y is negative exponential, not just on the (stochastic) interval $[0, x]$, but now also on the larger stochastic interval $[0, x']$. Repeating the argument, we extend the conditional density to the whole of $\mathbb{R}^+$.

The instant T_1^k of the first observation of initial rank k is a renewal epoch, so the whole procedure starts afresh, and the random variables $R_{n+1}^k - R_n^k$ are independent negative exponentials. This makes it at least (and at most!) plausible that N^k is independent of $N^1, \ldots, N^{k-1}$.

The statement (1.1), or equivalently the F-continuous case of our Theorem 1, is known. It seems to have first been stated by Ignatov [8], but there the reader is referred for proof to Ignatov [7], a paper which despite its date has not yet appeared (in any form). Motivated by Ignatov's assertion, A.J. Stam and the first author independently found proofs, different from each other and from that below, in 1982. See Stam [15] and Goldie [5]. Meanwhile P. Deheuvels, knowing nothing of the above references, had also formulated and proved (1.1), and his work has appeared as Deheuvels [4]. For convenience's sake, and in keeping with an old tradition, we shall refer to (1.1) as 'Ignatov's theorem'.

Conditional independence of lifetimes (1.2) was also found independently by Stam and the first author, motivated again by a result in Ignatov [8] giving the $k=1$ version of (1.2). Actually the $k=1$ case is a rediscovery of a result of Shorrock [14] Theorem 2.1. To Shorrock [12, 13] is also due the structure of the record process for general F, a structure which we show carries over to the partial record processes. So it is appropriate that we use the name "Shorrock processes" for the point processes which arise in this way.

The plan of the rest of the paper is as follows. The point processes N^j are Poisson only when F is an exponential law. For general F the larger class of Shorrock processes is needed; it is set up and characterised in § 2. In § 3, the characterisation is used to prove Theorem 1, by identifying a suitable family of martingales. §4 is devoted to (1.2). § 5 extends everything into continuous time. As befits a generalisation section, there we shall only sketch the relevant proofs. The extensions turn out fruitfully: we shall find ourselves splitting a 'time-space Poisson process' by initial rank into i.i.d. 'space' point processes. That has immediate application to the jumps of Lévy processes. On more general spaces, for instance excursion space, one can rank values by some chosen real-valued 'attribute' (§ 6). Different attributes will give different results, and we close with two selected examples based on Brownian motion.

2. Shorrock Processes

Let N_t be a counting process (a $\mathbb{Z}^+$-valued increasing càdlàg process on $[0, \infty)$, with unit jumps). Let $\mathscr{F}_t^N \equiv \sigma(N_s;\ s \leqq t)$ be the canonical filtration of N. Let F be a distribution function, $F(0)=0$, $F(t)<1$ for all t; then we say N_t is a *Shorrock process* (*derived from* F) if

$$M_t \equiv N_t - \int_{(0,t]} \frac{dF}{1-F_{s-}} \quad \text{is an } (\mathscr{F}_t^N)\text{-martingale.} \tag{2.1}$$

Remarks. (i) The case $F(x)=1-e^{-x}$ furnishes us with the familiar Poisson martingale $N_t - t$.

(ii) Let $\tau_1 < \tau_2 < \dots$ be the jump times of N;

$$\tau_{n+1} \equiv \inf\{t > \tau_n;\ N_t \neq N_{t-}\} \qquad (n \geqq 0)$$

where τ_0 is defined to be zero. Then the law of τ_1 is F.

(iii) If $c \equiv \sup\{t; F(t)<1\}$ is finite, the definition of M has to be reinterpreted in (c, ∞). We have assumed $c=+\infty$ because in the records application, if $c<+\infty$ one of two cases arises

(a) $F(c-)=1$, so that F is concentrated on $[0, c)$. In this case we can identify $[0, c)$ with $\mathbb{R}_+$ by a strictly increasing continuous map and reduce to the case $c=+\infty$.

(b) $F(c-)<1$, in which case F has an atom at c, and there will be infinitely many X_n taking that value. This is not very interesting from the records point of view; we could always replace F by

$$\tilde{F}(x) := \begin{cases} F(x) & (x<c) \\ F(c-) + \{1-e^{-(x-c)}\}\,\Delta F_c & (x \geqq c) \end{cases}$$

and we could deduce all the information about records for the law F from the records for $\tilde{F}$.

(iv) Defining

$$A_t \equiv \int_{(0,t]} \frac{dF_s}{1-F_{s-}} \qquad (t \geqq 0),$$

it is plain that A can be any right continuous function on $[0,\infty)$ with the properties.

$$A \text{ non-decreasing}, \quad A(0)=0, \quad A(\infty)=\infty, \quad \Delta A_t < 1 \,\forall t. \tag{2.2}$$

Here, $\Delta A_t \equiv A_t - A_{t-}$. Removing the jumps, we get the continuous part, A^c. In terms of A, (2.1) can be rephrased in the form: *a Shorrock process on* $[0,\infty)$ *is a counting process with deterministic compensator A satisfying* (2.2).

Counting processes with deterministic compensator have been characterised by Brémaud [3], and Kabanov, Lipster and Shiryayev (see Lipster and Shiryayev [10] Theorem 18.9). Paraphrasing their characterisation, and combining with the above, we find that *a Shorrock process is a counting process N having independent increments and with increment law given by*

$$E e^{i\lambda(N_t - N_s)} = \left\{ \prod_{s<u\leqq t} (1-(1-e^{i\lambda})\Delta A_u) \right\} \exp\left(-\int_s^t (1-e^{i\lambda})\, dA_u^c\right),$$
$$(0 \leqq s < t < \infty,\ \lambda \in \mathbb{R}),$$

where A satisfies (2.2). In other words, it is a Poisson process of intensity measure dA_t^c together with independent $0-1$ valued random variables at the jumps of A, and as a Shorrock process, in particular, the $0-1$ variables must each have positive probability of being zero, and the cumulative mean $A_t = EN_t$ must tend to ∞ as $t \to \infty$.

We shall use the characterisation in generating-functional form. Let $\phi\colon \mathbb{R}^+ \to (0,1]$ be measurable; then

$$E\left(\prod_{n\geqq 1} \phi(\tau_n)\right) = \left\{\prod_{s>0} (1-(1-\phi_s)\Delta A_s)\right\} \exp\left(-\int_0^\infty (1-\phi_s)\, dA_s^c\right), \tag{2.3}$$

where the τ_n are the jump-times of N. To prove this from the characterisation, just approximate $1-\phi$ by step-functions. Conversely, (2.3) is essentially the Laplace functional of the process (replace ϕ by $e^{-\phi}$), hence determines its law.

In particular, it is easy to see that if F has no atoms, then the Shorrock process N_t is a *Poisson* process, with intensity measure $dA_x \equiv F[x,\infty)^{-1} F(dx)$.

3. The k-record Processes are i.i.d.

We turn now to the main result.

Theorem 1. *The k-record processes* N^k $(k=1,2,\ldots)$ *are i.i.d. Shorrock processes derived from F.*

Remarks. As already stated, the proof is essentially computational, but first we set up some notation, and explain the central idea. Let $X_n^1 \geqq X_n^2 \geqq \ldots \geqq X_n^k$ be the first k order-statistics of $X_1, \dots, X_n$, with the convention $X_n^j = 0$ if $j > n$. Let the ϕ_j: $\mathbb{R}_+ \to (0, 1]$ be measurable and, as will be adequate for characterisation, equal to 1 outside some compact set. Let

$$U_j \equiv \prod_{n \geqq 1} \phi_j(R_n^j);$$

then if we can prove

$$E\left(\prod_{j=1}^{k} U_j\right) = \prod_{j=1}^{k} \left\{ \exp\left(-\int_0^\infty (1-\phi_j(s))\, dA_s^c\right) \prod_{s>0} (1-(1-\phi_j(s))\, \Delta A_s) \right\}, \tag{3.1}$$

the theorem follows from (2.3).

Let $\mathscr{F}_n$ be the σ-field generated by $X_1, \dots, X_n$, and let $U \equiv \prod_1^k U_j$. Then U is a bounded non-negative random variable and $E(U|\mathscr{F}_n)$ is a bounded non-negative martingale. Some of the factors in U_j are known at time n; explicitly, if

$$V_n^j \equiv \prod_{\{m:\, T_m^j \leqq n\}} \phi_j(X_{T_m^j})$$

then $E(U_j|\mathscr{F}_n) = V_n^j E(U_j/V_n^j|\mathscr{F}_n)$. Now U_j/V_n^j is the product of $\phi_j(R_r^j)$ over those r such that $R_r^j > X_n^j$, so *if* the theorem is true, then

$$E(U_j/V_n^j) = f_j(X_n^j)$$

where

$$f_j(x) \equiv \exp\left(-\int_x^\infty (1-\phi_j(s))\, dA_s^c\right) \prod_{s>x} (1-(1-\phi_j(s))\, \Delta A_s).$$

We shall indeed prove that

$$M_n \equiv \prod_{j=1}^{k} V_n^j f_j(X_n^j) \qquad \textit{is a bounded martingale.} \tag{3.2}$$

Then $M_0 = \prod_1^k f_j(0)$, and $M_\infty = U$ because $f_j(x) = 1$ for large enough x, so by the optional sampling theorem

$$EM_\infty = EU = \prod_1^k f_j(0) = EM_0,$$

which is, of course, (3.1).

Proof. In view of the preceding remarks, all we need do is to prove that M_n defined by (3.2) is a martingale, and for that, the following rather surprising formula covers a key calculation.

Proposition 3.1. *For* $0 \leqq a \leqq b$,

$$\int_{(a,b]} \phi_j(s)\, f_j(s)\, dF_s = f_j(a)\, \bar{F}(a) - f_j(b)\, \bar{F}(b),$$

where $\bar{F} \equiv 1 - F$.

Proof. For ease of notation, we omit the subscript j. Writing $\psi \equiv 1-\phi$, the defining formula for f becomes

$$f_x = \exp\left(-\int_x^\infty \psi_s dA_s^c\right) \prod_{s>x} (1-\psi_s \Delta A_s)$$

whence

$$\begin{aligned} df_t &= f_t \psi_t dA_t^c + \Delta f_t = f_t \psi_t dA_t^c + f_t - (1-\psi_t \Delta A_t) f_t \\ &= f_t \psi_t dA_t = f_t \psi_t dF_t / \bar{F}_{t-} \end{aligned}$$

Now differentiating $f_t \bar{F}_t$ by parts,

$$\begin{aligned} d(f_t \bar{F}_t) &= \bar{F}_{t-} df_t + f_t d\bar{F}_t \\ &= f_t \psi_t dF_t - f_t dF_t \\ &= -f_t \phi_t dF_t \end{aligned}$$

as required.

We now assemble the pieces. To prove that M_n is a martingale, note that under the convention $X_n^0 \equiv +\infty$, we can write

$$\begin{aligned} E(M_{n+1}|\mathscr{F}_n) &= E(M_{n+1};\, X_{n+1} \leqq X_n^k|\mathscr{F}_n) + \sum_{j=1}^k E(M_{n+1};\, X_n^j < X_{n+1} \leqq X_n^{j-1}|\mathscr{F}_n) \\ &= M_n P(X_{n+1} \leqq X_n^k|\mathscr{F}_n) + \sum_{j=1}^k E(M_{n+1};\, X_n^j < X_{n+1} \leqq X_n^{j-1}|\mathscr{F}_n). \end{aligned} \tag{3.3}$$

Now if $X_n^j < X_{n+1} \leqq X_n^{j-1}$, then $X_{n+1}^r = X_n^r$ for $r<j$, $X_{n+1}^j = X_{n+1}$, and $X_{n+1}^r = X_n^{r-1}$ for $j<r\leqq k$. Thus

$$\begin{aligned} &E(M_{n+1};\, X_n^j < X_{n+1} \leqq X_n^{j-1}|\mathscr{F}_n) \\ &\quad = \prod_{l=1}^k V_n^l \prod_{r=1}^{j-1} f_r(X_n^r) \prod_{s=j+1}^k f_s(X_n^{s-1}) \int_{(X_n^j, X_n^{j-1}]} f_j(x)\phi_j(x) F(dx) \\ &\quad = \prod_{l=1}^k V_n^l \prod_{r=1}^{j-1} f_r(X_n^r) \prod_{s=j+1}^k f_s(X_n^{s-1}) \{f_j(X_n^j)\bar{F}(X_n^j) - f_j(X_n^{j-1})\bar{F}(X_n^{j-1})\}, \end{aligned}$$

using Proposition 3.1.

Thus the sum in (3.3) telescopes, leaving

$$E(M_{n+1}|\mathscr{F}_n) = M_n F(X_n^k) + \prod_{l=1}^k V_n^l \prod_{r=1}^k f_r(X_n^r)\bar{F}(X_n^k) = M_n,$$

completing the proof of Theorem 1.

4. Lifetimes

So far we have thought of the i.i.d. sequence $\{X_n\}$ as given, and have derived the counting processes $\{N^j\}$ from it. But Theorem 1 tells us that the probabilistic structure of $\{N^j\}$ is every bit as nice as that of $\{X_n\}$, so one might ask whether one can recover $\{X_n\}_{n\geqq 1}$ from knowledge of the j-record processes $\{N^j\}_{j\geqq 1}$. Clearly we can find the *values* of the X_n, but what about the *order*? A little thought shows that this too can be deduced from the N^j inductively, for if we know the order in which the observations $X^j_{T_n}\equiv R^j_n$ $(n=1,2,\ldots,j=1,2,\ldots,k-1)$ appear, we can insert in that order the observations R^k_n $(n=1,2,\ldots)$. Indeed, the first k-record value, R^k_1, is inserted when there have appeared $k-1$ of the j-record values $(j=1,\ldots,k-1)$ that are larger than R^k_1. The subsequent values R^k_n are inserted similarly.

However, this inductive method of recovering the X_n from the N^j is unsatisfactory in that the original sequence X_n is obtained only as a limit; after we have ordered the observations in $N^1,\ldots,N^k$, we still do not know the time-gaps between those observations. But the conditional law of the gaps is actually very simple, as the next result shows.

Fix k, and define $W_0\equiv 0$, $W_{n+1}\equiv\inf\{m: m>W_n,\ \rho_m\leqq k\}$. Thus the times $L_n\equiv W_n-W_{n-1}$ are the gaps between observations of initial rank at most k. Let $Y_n\equiv X_{W_n}$, as in §1, and abbreviate $X^k_{W_n}$ to S_n.

Theorem 2. *Conditional on* $\{Y_1,Y_2,\ldots\}$, *or equivalently, conditional on* $\{N^1,N^2,\ldots,N^k\}$, *the* L_n *are independent, with distributions*

$$P(L_n=l)=F(S_{n-1})^{l-1}\{1-F(S_{n-1})\}\qquad (l=1,2,\ldots).$$

(Recall that $S_n\equiv X^k_{W_n}=0$ *for* $n<k$.*)*

Proof. As noted, the σ-fields generated by $\{Y_1,Y_2,\ldots\}$ and $\{N^1,\ldots,N^k\}$ coincide, so it is sufficient to prove that

$$E\left(\psi(Y_1,\ldots,Y_N)\prod_{n=1}^{N} z_n^{L_n}\right)=E\left(\psi(Y_1,\ldots,Y_N)\prod_{n=1}^{N}\frac{z_n\bar F(S_{n-1})}{1-z_nF(S_{n-1})}\right)$$

for each $z_n\in(0,1]$, $\psi\colon \mathbb{R}^N\to[0,1]$.

This is true if $N=1$, and if we suppose true for $N\leqq M$, by conditioning on $\mathscr{F}_{W_M}$ firstly,

$$\begin{aligned}
&E\left(\psi(Y_1,\ldots,Y_{M+1})\prod_{n=1}^{M+1} z_n^{L_n}\right)\\
&=E\left(\prod_{n=1}^{M} z_n^{L_n}\cdot\sum_{j=1}^{\infty}F(S_M)^{j-1}z_{M+1}^{j}\int_{(S_M,\infty)}\psi(Y_1,\ldots,Y_M,x)\,F(dx)\right)\\
&=E\left(\prod_{n=1}^{M} z_n^{L_n}\cdot\tilde\psi(Y_1,\ldots,Y_M;\,S_M)\frac{z_{M+1}\bar F(S_M)}{1-z_{M+1}F(S_M)}\right)
\end{aligned}$$

where $\tilde\psi(y_1,\ldots,y_M;s)\equiv\int_{(s,\infty)}\psi(y_1.\ldots.y_M,x)\,F(dx)/\bar F(s)$;

$$=E\left(\tilde\psi(Y_1,\ldots,Y_M;\,S_M)\frac{z_{M+1}\bar F(S_M)}{1-z_{M+1}F(S_M)}\prod_{n=1}^{M}\frac{z_n\bar F(S_{n-1})}{1-z_nF(S_{n-1})}\right)$$

by the inductive hypothesis, since S_M is a function of $Y_1,\ldots,Y_M$;

$$=E\left(\psi(Y_1,\ldots,Y_{M+1})\prod_{n=1}^{M+1}\frac{z_n\bar{F}(S_{n-1})}{1-z_nF(S_{n-1})}\right)$$

since $\tilde{\psi}(Y_1,\ldots,Y_M;S_M)=E(\psi(Y_1,\ldots,Y_{M+1})|\mathscr{F}_{W_M})$, and the product is $\mathscr{F}_{W_M}$-measurable. This completes the proof.

The second result stated in the introduction is a simple corollary; indeed, the lifetime $L_n^{j,k}$ of the observation R_n^j in rank k is L_r for some r, and as Theorem 2 tells us the conditional law of L_r given $N^1,\ldots,N^k$, the result (1.2) follows.

5. Continuous Time

Suppose that the observations $X_1, X_2, \ldots$ occur not at the integer time points $1,2,\ldots$ but at times $T_1, T_2, \ldots$ where T_1, T_2-T_1, $T_3-T_2, \ldots$ are i.i.d. exponentially distributed random variables independent of $\{X_n\}$, of rate μ. Then the points $\{(T_n, X_n)\}_{n\in\mathbb{N}}$ form a point process ξ in $\mathbb{R}_+^2$ which it is easy to see is a Poisson process of intensity $\mu dt\times F(dx)$. Writing λ for Lebesgue measure and defining $v(dx)\equiv\mu F(dx)$, the intensity may be rewritten as $\lambda\times v$. The process ξ is a 'time-space Poisson process'. It forms a plot in a 'time-space diagram' of the jumps of the compound Poisson process

$$X(t)\equiv\sum_{n=1}^{\infty}X_n 1\{T_n\leqq t\}\qquad(t\geqq 0).$$

Partial records of the jump heights X_n are, as we have seen, i.i.d. Shorrock processes, and conditional laws of lifetimes, instead of being independent geometric distributions, are now independent geometric compounds of exponential distributions. For these we can use the elementary fact that if $Y_1, Y_2, \ldots$ are i.i.d. exponential of rate μ, and L is independent with law $P(L=l)=p(1-p)^{l-1}(l\in\mathbb{N})$, then $\sum_1^L Y_j$ has an exponential law of rate $p\mu$; this implies that conditional on the first k of the partial record processes, the lifetime in rank k of an observation R_n^j of initial rank $j\leqq k$ is an exponential random variable with rate $\mu F(R^j{}_n,\infty)\equiv v(R_n^j,\infty)$, and that the lifetimes in rank k of the different observations in the first k partial records processes are independent.

Thus if the characteristic measure v of the time-space Poisson process ξ is *finite*, the situation is essentially that considered in the first part of the paper. But if v were to be *σ-finite* (as happens in most cases of interest), there are some slight changes. Indeed, the initial rank $r(t,x)$ of a point (t,x) of ξ, defined by

$$r(t,x)\equiv\xi((0,t]\times[x,\infty)),$$

will be finite a.s. iff $v(x,\infty)<\infty$ for all $x>0$. As in §2, remark (iii), there are minor complications if the support of v is bounded above, so we shall assume

$$0<v(x,\infty)<\infty\qquad\text{for all } x\in(0,\infty).\tag{5.1}$$

The only difficulty now is that the partial records processes cannot be thought of as counting processes; for each $\varepsilon > 0$, there will be infinitely many k-record values in $(0, \varepsilon)$. Thus we think of the k-record process as an integer-valued random measure ψ_k; explicitly, for all Borel subsets B of $(0, \infty)$,

$$\psi_k(B) \equiv \int_{\mathbb{R}^+ \times \mathbb{R}^+} I_B(x) I_{\{r(t,x)=k\}} \xi(dt, dx). \tag{5.2}$$

To specify the distribution of the ψ_k, we say an integer-valued random measure ψ on $(0, \infty)$ is a *Shorrock process* (*derived from* ν) if for each measurable ϕ: $(0, \infty) \to (0, 1]$ which is equal to 1 outside some compact set,

$$E \prod_{s \in D(\psi)} \phi(s) = \exp\left(-\int_0^\infty (1-\phi_s)\, dA_s^c\right) \prod \{1 - (1-\phi_s)\, \Delta A_s\}.$$

Here, $dA_s \equiv \nu[s, \infty)^{-1} \nu(ds)$, and $D(\psi) \equiv \{s; \psi(\{s\}) \neq 0\}$.

As point processes, Shorrock processes can be characterised by the properties:

(i) ψ has no multiple points;

(ii) ψ has no sure points (that is, $P(\psi(\{s\})=1)<1$ for all s)

(iii) for disjoint Borel $A_1, \ldots, A_n$, $\psi(A_1), \ldots, \psi(A_n)$ are independent;

(iv) $\psi((0, \infty)) = +\infty$ a.s.

Note that, from (2.3), this definition coincides with the earlier one in the case where ν is a probability measure. The analogue of Theorem 1 is the following.

Corollary 5.1. *The partial record processes $\psi_1, \psi_2, \ldots$ are i.i.d. Shorrock processes derived from ν.*

Proof. To check that the ψ_i are independent Shorrock processes, it is enough to consider the restriction of the ψ_i to (ε, ∞), for arbitrary positive ε. However, it is an elementary observation that the restriction of the ψ_i to (ε, ∞) can equally well be obtained by firstly restricting ξ to $\mathbb{R}^+ \times (\varepsilon, \infty)$ and then deriving the partial records processes from that. But the restriction of ξ to $\mathbb{R}^+ \times (\varepsilon, \infty)$ is a time-space Poisson process with a *finite* characteristic measure, and for this the partial records processes are i.i.d. Shorrock processes, as we have already seen.

If (t, x) is a point of ξ of initial rank j, then for each $k \geq j$, the point has a *time of entry into rank k*,

$$\tau_k \equiv \inf\{u > t;\ \xi((t, u] \times (x, \infty)) = k - j\},$$

and its lifetime in rank k is then $\tau_{k+1} - \tau_k$. We associate this lifetime with the point x of ψ_j and denote it $L^{j,k}(x)$. We than have the following analogue of Theorem 2.

Corollary 5.2. *Conditional on $\psi_1, \ldots, \psi_k$, the random variables $L^{j,k}(x)$ ($j = 1, 2, \ldots, k$, $x \in D(\psi_j)$) are independent exponential random variables, with law*

$$P(L^{j,k}(x) > t \mid \psi_1, \ldots, \psi_k) = \exp(-t\, \nu(x, \infty)).$$

Proof. Again, it is sufficient to consider the restrictions of ψ_i to (ε, ∞) for arbitrary $\varepsilon > 0$, and for this case the result follows immediately from the discussion at the beginning of the section.

From knowledge of the j-record processes $\{\psi_j\}$ one can recover the space coordinates x of all the points of ξ, *in the time-order of occurrence*, by a two-level induction. If we choose some sequence $c_n \downarrow 0$, then all points (t, x) of ξ that have $x > c_1$ can be reconstructed in time-order by the inductive algorithm described in §4. Similarly, all points with $x > c_2$ can be reconstructed in their time order, adding a new 'layer' to the reconstruction of the time space diagram, and so on.

Remarkably, when $\nu(\mathbb{R}_+) = \infty$ the process ξ itself can be reconstructed on a probability-1 event from knowledge of $\{\psi_j\}_{j \in \mathbb{N}}$ alone. For let $\{x_1^i, x_2^i, \dots\}$ be the sequence of x-values exceeding c_i, in their order of occurrence, as obtained above. These sequences, for $i = 1, 2, \dots$ together make up a *nested array* in the sense of Greenwood & Pitman [6]. The condition of Theorem 2.2 of that paper is easy to check, and the theorem gives an explicit construction.

The most obvious example of a σ-finite time-space Poisson process comes from the upward jumps of a Lévy process; the results of this section say that the upward jumps can be split into i.i.d. Shorrock processes according to initial rank.

6. Attributes

In this section, we consider a time-space Poisson process Ξ on $\mathbb{R}^+ \times S$, where S is a Polish space, and the intensity measure is $\lambda \times m$, where λ is Lebesgue measure, and m is the (σ-finite) characteristic measure of Ξ. We suppose given a measurable $q\colon S \to (0, \infty)$ such that

$$0 < (m \circ q^{-1})(x, \infty) < \infty \qquad \text{for all } x \in (0, \infty). \tag{6.1}$$

We define in a natural way a Poisson process ξ on $\mathbb{R}^+ \times \mathbb{R}^+$ by

$$\xi(A \times C) \equiv \Xi(A \times q^{-1}(C))$$

for A, C Borel subsets of $\mathbb{R}^+$, corresponding to mapping the point (t, s) of Ξ to $(t, q(s))$. The intensity measure of ξ is simply $\lambda \times \nu$, where $\nu \equiv m \circ q^{-1}$. To avoid certain complications, we assume that ν *has no atoms* (see Goldie [5] for a full account).

We use the real-valued attribute q to rank the points of Ξ; we define the *initial q-rank* of the point (t, s) of Ξ to be

$$r(t, s) \equiv \xi((0, t] \times [q(s), \infty)).$$

Condition (6.1) is equivalent to (5.1), ensuring that all the initial q-ranks are finite a.s. Having ranked the points of Ξ using q, we split Ξ into its partial q-record processes Ψ_i, point processes on S defined by

$$\Psi_k(B) = \int_0^\infty \int_S I_B(s)\, I_{\{r(t,s) = k\}}\, \Xi(dt, ds) \tag{6.2}$$

for Borel $B \subseteq S$, just as before (5.2). Similarly we define the lifetime in rank k, $L^{j,k}(s)$, of a point (t,s) of Ξ of initial q-rank $j \leq k$ to be the lifetime in rank k of $(t, q(s))$. We have the following result.

Proposition 6.1. *Assuming* (6.1) *and that* v *has no atoms, the partial* q*-record processes* $\Psi_1, \Psi_2, \ldots$ *are i.i.d. Poisson processes with intensity measure* n, *where* $n \ll m$ *and*

$$\frac{dn}{dm}(s) = v(q(s), \infty)^{-1}.$$

Conditional on $\{\Psi_1, \ldots, \Psi_k\}$ *the lifetimes* $L^{j,k}(s)$ $(j=1, \ldots, k,\ s \in D(\Psi_j))$ *are independent exponential random variables,*

$$P(L^{j,k}(s) > u \mid \Psi_1, \ldots, \Psi_k\} = \exp\{-u\, v(q(s), \infty)\}.$$

Proof. Since S is a Polish space, there exists a regular conditional distribution for m given q (that is, a kernel K: $(0,\infty) \times \mathscr{B}(S) \to [0,1]$ such that for each $A \in \mathscr{B}(S)$, the Borel σ-field of S, the random variable $K(q(s), A)$ is (a version of) the conditional expectation of A given q). For a proof of this, see, for example, Bourbaki [2] §2, No. 7, Prop. 13, or alternatively, modify the usual result (such as appears in Williams [17] II.69, for example).

The key observation is that one can obtain a realisation of the Poisson point process Ξ in two stages:

(i) Obtain a realisation of the Poisson point process ξ on $\mathbb{R}^+ \times \mathbb{R}^+$ with measure $\lambda \times v$;

(ii) to each point (t,x) of ξ, assign a point (t,s) of Ξ, where the distribution of s given ξ is $K(x,\cdot)$.

Let the point process so constructed be denoted Ξ'. To prove that it has the law of Ξ, take any measurable ϕ: $\mathbb{R}^+ \times S \to (0,1]$, and calculate:

$$E\{\prod_{(t,s)\in D(\Xi')} \phi(t,s)\} = E\{\prod_{(t,x)\in D(\xi)} \theta(t,x)\}$$

where $\theta(t,x) \equiv \int_S K(x,ds)\,\phi(t,s)$;

$$= \exp\left\{-\int_0^\infty dt \int_0^\infty v(dx)(1-\theta(t,x))\right\}$$

$$= \exp\left\{-\int_0^\infty dt \int_0^\infty v(dx) \int K(x,ds)(1-\phi(t,s))\right\}$$

$$= \exp\left\{-\int_0^\infty dt \int m(ds)(1-\phi(t,s))\right\}$$

by definition of K;

$$= E\{\prod_{(t,s)\in D(\Xi)} \phi(t,s)\}.$$

In view of this construction, the result is obvious; the partial records processes $\psi_1, \psi_2, \ldots$ of ξ are i.i.d. Shorrock processes derived from v (and thus are Poisson processes with intensity $\nu[x,\infty)^{-1}\nu(dx)$), and to obtain Ψ_k from ψ_k, at each point x of ψ_k, we place a point s in S according to the law $K(x,\cdot)$, independently of all the other Ψ_j. The reader may formulate this with greater precision if desired. The statement about the laws of the lifetimes needs no proof, as it is merely a statement about the process ξ.

A good example of the application of Proposition 6.1 is to *excursions of Brownian motion* from 0. Let S be the Polish space $C([0,\infty),\mathbb{R})$ with the usual topology of uniform convergence on compact sets. Letting

$$\sigma(s)\equiv\inf\{t>0:\ s(t)=0\} \qquad (s\in S),$$

the excursions lie in the subset

$$S_e\equiv\{s\in S:\ s(0)=0,\,0<\sigma(s)<\infty,\,s(t)=0\ \forall t\geqq\sigma(s)\}$$

of S. The *excursion process* is a time-space process Ξ, namely a Poisson process on $\mathbb{R}_+\times S$ of intensity $\lambda\times m$, where the measure m on S is supported by S_e and is called the *excursion law*. This formulation is equivalent to the original "Poisson point process" formulation of Itô [9]. The time variable t is actually the local time of the Brownian motion B at 0. The path of B determines the realisation of Ξ and may be reconstructed from it. Now there are many ways to formulate the excursion law m, and for us it is best to relate the formulation to the attribute to be used for ranking.

Example 1. Let us rank by duration: $q(s)\equiv\sigma(s)$. (On $S\backslash S_e$ define q arbitrarily.) For local time we use the normalisation implied by

$$L_t\equiv\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int_0^t 1\{0\leqq B_u<\varepsilon\}\,du \qquad (t\in\mathbb{R}_+),$$

where off a null event the limit exists for all t. The inverse process $L^{-1}(x)\equiv\inf\{t:\ L(t)>x\}$ is then the driftless stable subordinator having Levy measure $(2\pi t^3)^{-\frac{1}{2}}dt$, and since q represents duration the measure $\nu=m\circ q^{-1}$ may be identified with this Lévy measure. To find m we use a product-space representation of S_e (see, for example, Balkema [1]). The *sign* of $s\in S_e$ is $\operatorname{sgn}(s)=+1$ or -1 according as $s(t)>0$ or <0 for all $t\in(0,\sigma(s))$. (One or other alternative must occur.) The *normalised excursion* is $\bar{s}$ defined by

$$\bar{s}(t)\equiv\sigma^{-\frac{1}{2}}|s(t\sigma)|, \qquad 0\leqq t\leqq 1,$$

where $\sigma\equiv\sigma(s)$. It belongs to

$$\bar{S}_e\equiv\{s\in C[0,1]:\ s(0)=0=s(1),\ s(t)>0\,\forall t\in(0,1)\}.$$

We identify S_e with the product space $(0,\infty)\times\{-1,1\}\times\bar{S}_e$ by means of the map $s\leftrightarrow(\sigma(s),\operatorname{sgn}(s),\bar{s})$. As a measure on this product space, m decomposes into the product measure,

$$m=\nu\times\beta\times\pi,$$

where $\nu(dt)=(2\pi t^3)^{-\frac{1}{2}}dt$ as noted above, $\beta\{-1\}=\frac{1}{2}=\beta\{1\}$, and π is a probability measure on $\bar{S}_e$. Let R be a $BES(3)$ process i.e. the radial part of standard

Brownian motion in $\mathbb{R}^3$. Then π turns out to be the probability law of the *BES*(3) bridge;

$$Z_t = \begin{cases} (1-t)R(t/(1-t)) & (0 \leqq t < 1) \\ 0 & (t=1) \end{cases}$$

We calculate

$$\nu(\sigma(s), \infty) = (\tfrac{1}{2}\pi\,\sigma(s))^{-\frac{1}{2}},$$

hence

$$n(ds) = \frac{\nu(d\sigma)\,\beta(\mathrm{sgn})\,\pi(d\tilde{s})}{\nu(\sigma, \infty)} = \frac{d\sigma}{2\sigma}\,\frac{1}{2}\,\pi(d\tilde{s}).$$

On the product space $(0, \infty) \times \{-1, 1\} \times \tilde{S}_e$ this is the measure $\theta \equiv \eta \times \beta \times \pi$, where $\eta(d\sigma) = (2\sigma)^{-1} d\sigma$. Proposition 6.1 shows that the excursions having initial duration rank j, constituting the "points" of the processes Ψ_j, $j = 1, 2, \ldots$, give i.i.d. Poisson processes on S_e, each of intensity θ. "Lifetimes" of excursions are the amounts of *local* time they spend in the various ranks. The second statement of Prop. 6.1 expresses these lifetimes as having conditionally independent exponential laws.

Example 2. Here we rank by peak height. On S_e define now $q(s) \equiv \sup_{t \geqq 0} |s(t)|$. Let

$$\tilde{S}_e \equiv \{s\colon s \in S_e,\ \mathrm{sgn}(s) = 1,\ q(s) = 1\}.$$

The normalised excursion is now $\tilde{s} \in \tilde{S}_e$ defined by

$$\tilde{s}(t) = (1/x)\,|s(x^2 t)| \qquad (t \in [0, \infty))$$

where $x \equiv q(s)$. This time we identify S_e with the product space $(0, \infty) \times \{-1, 1\} \times \tilde{S}_e$ by means of the map $s \leftrightarrow (x, \mathrm{sgn}, \tilde{s})$, where $x \equiv q(s)$ and $\mathrm{sgn} \equiv \mathrm{sgn}(s)$. A small extension of results of Williams [16, 17] II.67, and Rogers [11], shows that the excursion law m becomes the product measure $\nu \times \beta \times \mathrm{p}$ on the product space, where $\nu(dx) = x^{-2} dx$, $\beta\{-1\} = \frac{1}{2} = \beta\{1\}$, and p on $\tilde{S}_e$ is defined as follows. Let r_1 and r_2 be independent *BES*(3) processes (started at 0). Let $\sigma_i \equiv \inf\{t\colon r_i(t) = 1\}$. Then p is the law of the random element F of $\tilde{S}_e$ defined by

$$F(t) \equiv \begin{cases} r_1(t) & (0 \leqq t \leqq \sigma_1) \\ 1 - r_2(t - \sigma_1) & (\sigma_1 \leqq t \leqq \sigma_1 + \sigma_2) \\ 0 & (\sigma_1 + \sigma_2 \leqq t < \infty). \end{cases}$$

In this example we have $\nu(x, \infty) = 1/x$, hence the intensity measure θ of the partial records Poisson processes becomes $\eta \times \beta \times p$, where $\eta(dx) = x^{-1} dx$. Proposition 6.1 applies as before.

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