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Feynman-Kac Functionals and Positive Solutions of $\frac{1}{2}\Delta u + qu = 0$

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Summary. Let D be a bounded C^2 domain in $\mathbb{R}^d$ and let q be a bounded Borel function in D. For $x \in D$ and $z \in \partial D$ suppose (X_t) under the law $P^{x;z}$ is Brownian motion in D starting at x and conditioned to converge to z. Let τ be the lifetime of (X_t). We show that if the quantity $E^{x;z}\left\{\exp\left[\int\limits_0^\tau q(X_s)\,ds\right]\right\}$ is finite for one $x \in D$ and one $z \in \partial D$, then this quantity remains bounded as x varies over D and z varies over ∂D. This may be considered one quantitative expression of the qualitative statement that no matter where Brownian motion in D eventually hits ∂D, it goes all over D before it gets there. We apply this result to show that if the equation $\frac{1}{2}\Delta u + qu = 0$ admits a non-negative solution in D, which is strictly positive on a subset of ∂D of positive harmonic measure, then for any non-negative bounded Borel function f on ∂D it admits a unique bounded solution u satisfying $u = f$ on ∂D, and this solution u is non-negative.

1. Introduction

Let D be a bounded domain in $\mathbb{R}^d$ and let $q: D \to \mathbb{R}$ be a bounded Borel function. Let $((X_t); P^x, x \in \mathbb{R}^d)$ be Brownian motion in $\mathbb{R}^d$ and let $\tau = \inf\{t > 0: X_t \notin D\}$. Recall that $\tau < \infty$ a.s. For each Borel function $f: \partial D \to \mathbb{R}$ let

$$u_f(x) = E^x\left\{\exp\left[\int\limits_0^\tau q(X_s)\,ds\right] f(X_\tau)\right\} \tag{1.1}$$

for each $x \in D$ for which the expectation makes sense. It was shown by Chung and Rao [4] that if u_f is defined and finite at one point of D, then it is locally bounded in D; moreover, they showed that in this case if f is bounded, then so is u_f. The function u_1 obtained by setting $f = 1$ in (1.1) shall be called the

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gauge of D. If the gauge of D is finite and f is bounded, then $u=u_f$ is the unique bounded solution of the boundary value problem:

$$\tfrac{1}{2}\Delta u+qu=0 \quad \text{in } D, \tag{1.2a}$$

$$u=f \quad \text{on } \partial D. \tag{1.2b}$$

(See Sect. 3 and the references cited there for further discussion of this.) We point out that the gauge of D can be identically $+\infty$ in spite of the boundedness of D and q. This is not pathological and is related to the possibility of non-existence of positive solutions of (1.2a). (See [8, 4], and Sect. 3.)

Now Williams [13] has shown that if ∂D is of class C^2 and if for some non-empty relatively open subset A of ∂D we have u_{1_A} finite, then the gauge of D is finite. This result may be regarded as one quantitative expression of the qualitative statement that no matter where Brownian motion in D eventually hits ∂D, it goes all over D before it gets there. To see this, suppose q takes on only the values 0 and 1 and A and B are non-empty relatively open subsets of D which border, respectively, on the part of D where $q=0$ and on the part of D where $q=1$. Since a Brownian path can reach A without entering the part of D where $q=1$, while to reach B it must traverse this part of D, it might seem that u_{1_B} could be infinite with u_{1_A} still finite. But Williams' result tells us that this is impossible.

In this paper we strengthen this result of Williams in the following way. For $x\in D$ and $z\in\partial D$ let $P^{x;z}$ be the probability law for Brownian motion in D starting at x and conditioned to converge to z. (See discussion and references below.) We show that if ∂D is of class C^2 and if for some $x_0\in D$ and some $z_0\in\partial D$ we have

$$E^{x_0;z_0}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\}<\infty,$$

then in fact

$$\sup_{x\in D}\sup_{z\in\partial D} E^{x;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\}<\infty.$$

As corollaries we recover a result of Chung [4] and obtain a variation on a result of Aizenman and Simon [1]. In Sect. 3 we discuss the boundary value problem (1.2) and then show how our results relate to the existence of positive solutions of (1.2).

The rest of this introduction is devoted to establishing some notation and to recalling some facts that we shall be needing. Let $p_t(x, y)$ be the transition density for Brownian motion killed on exit from D (see [10], pp. 33–41). For $\alpha\geqq 0$ let

$$G_\alpha(x, y)=\int_0^\infty e^{-\alpha t}p_t(x, y)\,dt \qquad (x, y\in D)$$

and for each Borel function $f\colon \partial D\to\mathbb{R}$ let

$$H_f^\alpha(x)=E^x\{e^{-\alpha\tau}f(X_\tau)\}$$

for each $x\in D$ for which the expectation makes sense. If $\alpha=0$, we drop it from the notation and just write $G(x, y)$, $H_f(x)$. For the analogous objects defined with respect to some open set W different from D we write G_α^W, etc. G is called the Green function of D and we call G_α the α-Green function of D. If $L: D\times D\to[0, \infty]$ and $\varphi: D\to[0, \infty]$ are Borel, we write $L\varphi$ for the function defined by

$$(L\varphi)(x)=\int_D L(x, y)\,\varphi(y)\,dy \qquad (x\in D).$$

In particular this explains what $p_t\varphi$ and $G_\alpha\varphi$ mean. $G_\alpha\varphi$ is called the α-potential of φ (relative to D). Fix a reference point $p_*\in D$. It will be convenient for us to take ∂D to be the Martin boundary of D in case this differs from the usual boundary of D. Then X_τ should be redefined to be $\lim_{t\uparrow\tau} X_t$ where the limit is taken in the Martin topology. Let $K: D\times\partial D\to(0, \infty)$ be the Martin kernel relative to the reference point p_*. For $z\in\partial D$ we shall sometimes write K^z for the function $K(\cdot, z)$. If z is a minimal point of the Martin boundary, then Doob [6] has shown that it is meaningful to speak of Brownian motion in D conditioned to converge to z and that this process has transition density $p_t^z(x, y)$ given by

$$p_t^z(x, y)=\frac{1}{K(x, z)}\,p_t(x, y)\,K(y, z) \qquad (x, y\in D)$$

and hence has α-potential density G_α^z given by

$$G_\alpha^z(x, y)=\frac{1}{K(x, z)}\,G_\alpha(x, y)\,K(y, z) \qquad (x, y\in D).$$

Hunt and Wheeden [7] have shown that if the usual boundary of D is Lipschitz, then the Martin boundary of D may be identified with the usual boundary of D and all points $z\in\partial D$ are minimal. We shall state most of our results for the case where the usual boundary of D is of class C^2, but we shall prove them under the weaker assumption that D is what we call *Green-smooth* (Definition 2.7). We do not know whether the Martin boundary of a Green-smooth domain may be identified with its usual boundary. For $x\in D$, μ_x will denote the harmonic measure for D relative to x; i.e., the measure on ∂D defined by $\mu_x(dz)=P^x\{X_\tau\in dz\}$. As is well known (Harnack's inequality), if x_1 and x_2 are any two points of D, then each of μ_{x_1} and μ_{x_2} is bounded by a constant times the other. We shall write μ for μ_{p_*} and $L^1(\partial D)$ for $L^1(\mu)$. We mention in passing that if the boundary of D is of class $C^{1+\varepsilon}$ and if σ is $d-1$ dimensional Lebesgue measure on ∂D, then there is a constant $C_1\in(1, \infty)$ such that $C_1^{-1}\mu\leqq\sigma\leqq C_1\mu$. This follows easily from results of Widman [12]. (Widman states and proves his results only in dimension $d\geqq 3$, but presumably they are valid for $d=2$ as well.)

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2. Results

We state our results for the case where the boundary of D is of class C^2, but we shall prove them for more general domains D which we call Green-smooth (see (2.7) below for the definition).

Theorem 2.1. *Suppose the boundary of D is of class C^2 and that for some $x_0 \in D$ and some $z_0 \in \partial D$ we have*

$$E^{x_0; z_0}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\} < \infty. \tag{2.1}$$

Then there exist constants $A_1, A_2 \in (0, \infty)$ such that for all $x \in D$ and all $z \in \partial D$ we have

$$A_1 \leqq E^{x; z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\} \leqq A_2. \tag{2.2}$$

Corollary 2.2. *Suppose the boundary of D is of class C^2 and that for some Borel set $A \subseteq \partial D$ with $\mu(A) > 0$ we have u_{1_A} finite. Then the gauge of D is finite.*

Corollary 2.3. *Suppose the boundary of D is of class C^2 and gauge of D is finite. Then for any compact set $\Gamma \subseteq D$ there is a constant $C < \infty$ such that for all $f \in L^1(\partial D)$ we have u_f well defined throughout D and*

$$\sup_{x \in \Gamma} |u_f(x)| \leqq C \int_{\partial D} |f|\,d\mu.$$

Remark. In the special case where D is a sufficiently small open ball, this result was proved, by a different method, by Aizenman and Simon in [1]. (They also considered certain unbounded q's.)

Corollary 2.4. *Suppose the boundary of D is of class C^2 and the gauge of D is finite. Then there is a constant $C < \infty$ such that for all $f \in L^1(\partial D)$ we have*

$$\int_D |u_f(x)|\,dx \leqq C \int_{\partial D} |f(z)|\,\mu(dz).$$

Remark. This result was proved by a different method by Chung in [2].

Corollary 2.5. *Suppose the boundary of D is of class C^2 and f is a non-negative Borel function on ∂D such that $u_f(x) < \infty$ for some $x \in D$. Then $f \in L^1(\partial D)$.*

Proof of Corollary 2.2. For each $x \in D$ we have

$$E^x\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\middle| X_\tau\right\} = E^{x; X_\tau}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\} \quad P^x \text{ a.s.}$$

Hence

$$u_{1_A}(p_*) = \int_{\partial D} 1_A(z)\, E^{p_*; z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\} \mu(dz),$$

where $p_* \in D$ is the reference point fixed in the introduction. Since $\mu(A)>0$ and $u_{1_A}(p_*)<\infty$, there exists $z_0 \in A$ for which (2.1) holds with $x_0=p_*$. Letting A_2 be as in (2.2), we find that the gauge u_1 of D satisfies

$$u_1(x)=\int_{\partial D} E^{x;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\}\mu_x(dz)\leqq A_2<\infty$$

for all $x\in D$. □

Proof of Corollary 2.3. It is an immediate consequence of Harnack's inequality for harmonic functions that there is a constant $C_1<\infty$ such that for all $x\in\Gamma$ we have $\mu_x\leqq C_1\,\mu$ so

$$|u_f(x)|=\left|\int_{\partial D} f(z)E^{x;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\}\mu_x(dz)\right|\leqq C\int_D |f(z)|\,\mu(dz)$$

where $C=C_1A_2$ and A_2 is as in (2.2). □

We defer the proof of Corollary 2.4.

Proof of Corollary 2.5. In view of Theorem 2.1, either $f=0$ μ_x - a.e. or

$$u_f(x)=\int_{\partial D} f(z)\,E^{x;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\}\mu_x(dz)\geqq A_1\int_{\partial D} f(z)\,\mu_x(dz)$$

where $A_1\in(0,\infty)$ is as in (2.2). Now from Harnack's inequality, there is a constant $\varepsilon\in(0,\infty)$ such that $\mu_x\geqq\varepsilon\mu$. Then

$$\int_{\partial D} f\,d\mu\leqq\frac{1}{A_1\varepsilon}u_f(x)$$

in the second case, while $f=0$ μ-a.e. in the first case. Thus $f\in L^1(\partial D)$ in either case. □

Lemma 2.6. *Suppose the boundary of D is of class C^2. Let φ be a bounded non-negative Borel function in D such that $\varphi>0$ on a set of strictly positive Lebesgue measure in D. Then there are constants C_1, $C_2\in(0,\infty)$ such that for all $x\in D$,*

$$C_1\,\mathrm{dist}(x,\mathbb{R}^d\setminus D)\leqq G\varphi(x)\leqq C_2\,\mathrm{dist}(x,\mathbb{R}^d\setminus D).$$

Proof. Since ∂D is of class C^2 and compact, the radius of curvature of ∂D is bounded away from 0. Hence there exists $r\in(0,\infty)$ such that a ball of radius r can be rolled all over the inside of ∂D and another ball of radius r can be rolled all over the outside of ∂D. Or, to state what we have in mind more precisely though perhaps less vividly, if we let $B=\{y\in\mathbb{R}^d\colon \|y\|<r\}$, then:

(a) For each $x\in D$ with $\mathrm{dist}(x,\mathbb{R}^d\setminus D)\leqq r$, there exists $p\in D$ such that $x\in B+p\subseteq D$ and $\mathrm{dist}(x,\mathbb{R}^d\setminus(B+p))=\mathrm{dist}(x,\mathbb{R}^d\setminus D)$;

(b) For each $x\in D$, there exists $p\in\mathbb{R}^d\setminus D$ such that $\overline{B+p}\subseteq\mathbb{R}^d\setminus D$ and $\mathrm{dist}(x,\overline{B+p})=\mathrm{dist}(x,\mathbb{R}^d\setminus D)$.

Now let $K=\{y\in D: \operatorname{dist}(y,\mathbb{R}^d\setminus D)\geqq r/2\}$. Then K is a compact subset of D, $G\varphi$ is bounded away from 0 on K, and $G1_K$ is bounded so there exists $\varepsilon>0$ such that $G\varphi\geqq\varepsilon G1_K$ on K. Then $G\varphi\geqq\varepsilon G1_K$ throughout D by the domination principle (see [9], p. 138 or [10], p. 175). Also $G\varphi\leqq sG1$ where $s=\sup\varphi$. Thus it suffices to show that there are constants $\gamma_1, \gamma_2\in(0,\infty)$ such that for all $x\in D$,

$$G1_K(x)\geqq\gamma_1 \operatorname{dist}(x,\mathbb{R}^d\setminus D), \tag{2.3}$$

$$G1(x)\leqq\gamma_2 \operatorname{dist}(x,\mathbb{R}^d\setminus D). \tag{2.4}$$

Let $V=\{y\in\mathbb{R}^d: \|y\|<r/2\}$ and let $A=\{y\in\mathbb{R}^d: r<\|y\|<r+\operatorname{diam}D\}$. Since $G1_K$ is bounded away from 0 on each compact subset of D, we need only check (2.3) for x near ∂D. For such x, (2.3) follows from (a) by comparison with G^B1_V. The estimate (2.4) follows from (b) by comparison with G^A1. This completes the proof of the lemma. □

Definition 2.7. We shall say D is Green-smooth iff whenever φ and ψ are bounded non-negative Borel functions in D with $\psi>0$ on a set of strictly positive Lebesgue measure in D, there exists a constant $C<\infty$ such that $G\varphi\leqq CG\psi$.

Lemma 2.8. *Suppose the boundary of D is of class C^2. Then D is Green-smooth.*

Proof. Clearly this follows from Lemma 2.6. □

Example 2.9. Even if D has a Lipschitz boundary, D may fail to be Green-smooth.

Proof. Let the dimension $d=2$. Under the conformal map T: $z\to z^2$, the rose-leaf

$$D:\ -\frac{\pi}{4}\leqq\theta\leqq\frac{\pi}{4},\qquad 0<r<2\cos 2\theta$$

is transformed into the disc

$$W:\ 0<r<2\cos\theta.$$

The variables r and θ are polar coordinates, of course. The function

$$g(z)=\frac{1-|z-1|^2}{|z|^2}=\frac{2\cos\theta-r}{r}$$

is positive and harmonic in W. (It is a multiple of the Poisson kernel for W with pole at 0.) Hence $h=g\circ T$ is positive and harmonic in D. Now

$$\begin{aligned}\int_D h &\geqq \int_0^1\int_{-\pi/8}^{\pi/8}\frac{2\cos 2\theta-r^2}{r^2}\,r\,d\theta\,dr\\ &\geqq\int_0^1\frac{\sqrt{2}\pi}{4r}\,dr=+\infty.\end{aligned}$$

As is shown below (Lemma 2.11), if D were Green-smooth, then every positive harmonic function in D would be integrable. Thus D cannot be Green-smooth. Clearly ∂D is Lipschitz. This completes the proof of the example. □

Lemma 2.10. *Suppose D is Green-smooth and φ, ψ are bounded non-negative Borel functions in D with $\psi>0$ on a set of strictly positive Lebesgue measure in D. Let $\alpha\in[0,\infty)$. Then there is a constant $C<\infty$ such that $G_\alpha\varphi\leqq CG_\alpha\psi$.*

Proof. If $\alpha=0$ this is just the definition of Green-smoothness. Suppose $\alpha>0$. Let K be a compact subset of D of strictly positive Lebesgue measure. Using the domination principle, it is easy to see that it suffices to show that

$$G_\alpha 1\leqq\gamma G_\alpha 1_K \tag{2.5}$$

for a suitable constant $\gamma<\infty$. Let

$$T=\inf\{t>0: X_t\in K\}$$

and let

$$v=P^{\cdot}\{T<\tau\},$$

$$v_\alpha=E^{\cdot}\{e^{-\alpha T}; T<\tau\}$$

be the capacitary potential and the α-capacitary potential, respectively, of K relative to D. Let $\varepsilon_1=\inf_K G_\alpha 1_K$. Then $\varepsilon_1>0$. Let $\gamma_1=1/\varepsilon_1$. Then

$$v_\alpha\leqq\gamma_1 G_\alpha 1_K. \tag{2.6}$$

Now by Jensen's inequality applied to the function $e^{-\alpha t}$,

$$\exp\{-\alpha E^{\cdot}[T; T<\tau]/v\}\leqq v_\alpha/v. \tag{2.7}$$

From (2.7) we obtain

$$v\leqq v_\alpha\exp\{\alpha E^{\cdot}(\tau)/v\}=v_\alpha\exp\{\alpha G1/v\}. \tag{2.8}$$

Since D is Green-smooth,

$$G1\leqq\gamma_2 G1_K \tag{2.9}$$

for a suitable constant $\gamma_2<\infty$. Using the domination principle, it is easy to see that

$$G1_K\leqq\gamma_3 v \tag{2.10}$$

for a suitable constant $\gamma_3<\infty$. From (2.8) through (2.10) we have

$$v\leqq v_\alpha\exp(\alpha\gamma_2\gamma_3). \tag{2.11}$$

Finally from (2.9) through (2.11) together with (2.6) we have

$$G1\leqq\gamma_2\gamma_3\exp(\alpha\gamma_2\gamma_3)\gamma_1 G_\alpha 1_K. \tag{2.12}$$

This completes the proof of the lemma. □

Lemma 2.11. *Suppose D is Green-smooth. Let $x_0 \in D$. Then there is a constant $C<\infty$ such that for any non-negative superharmonic function h in D,*

$$\int_D h(x)\,dx \leqq C h(x_0).$$

Proof. Let B be an open ball centred at x_0 and satisfying $\bar{B} \subseteq D$. Let $\varphi=(1/b)1_B$ where b is the Lebesgue measure of B. Then $\int_D h(x)\varphi(x)\,dx \leqq h(x_0)$. Thus it suffices to find a constant C, independent of h, such that $\int_D h(x)\,dx \leqq C\int_D h(x)\varphi(x)\,dx$. Since D is Green-smooth, $\exists\, C<\infty$ such that $G1 \leqq CG\varphi$. If h is of the form $G\psi$ for some non-negative Borel function ψ in D, then

$$\begin{aligned}\int_D h(x)\,dx &= \int_D G\psi(x)\,dx\\ &= \int_D G1(x)\psi(x)\,dx\\ &\leqq C\int_D G\varphi(x)\psi(x)\,dx\\ &= C\int_D G\psi(x)\varphi(x)\,dx\\ &= C\int_D h(x)\varphi(x)\,dx.\end{aligned}$$

But any non-negative superharmonic function h in D is the limit of an increasing sequence of such potentials $G\psi$. Thus we may complete the proof of the lemma by applying the monotone convergence theorem. □

Lemma 2.12. *Suppose D is Green-smooth. Then there is a constant $C<\infty$ such that for all $x \in D$ and all $z \in \partial D$,*

$$E^{x;z}\{\tau\} \leqq C. \tag{2.13}$$

Proof. We have $E^{x;z}\{\tau\}=G^z 1(x)=(1/K^z(x))GK^z(x)$ so what we must show is that

$$GK^z \leqq CK^z. \tag{2.14}$$

Now

$$\begin{aligned}GK^z &= \int_0^\infty p_t K^z\,dt\\ &= \int_0^1 p_t K^z\,dt + \int_1^\infty p_t K^z\,dt\\ &\leqq K^z + Gp_1 K^z\end{aligned}$$

so it suffices to show that

$$Gp_1 K^z \leqq C_1 K^z, \tag{2.15}$$

for then we can take $C=1+C_1$. Now $K^z(p_*)=1$ so by Lemma 2.11,

$$\int_D K^z(x)\,dx \leqq C_2. \tag{2.16}$$

Hence

$$p_1 K^z \leqq C_3 \tag{2.17}$$

where $C_3=(2\pi)^{-d/2}C_2$. Let H be a compact subset of D of strictly positive Lebesgue measure. By Harnack's inequality,

$$1_H \leqq C_4 K^z. \tag{2.18}$$

Consequently, on H we have

$$G1_H \leqq C_5 K^z. \tag{2.19}$$

But then by the domination principle, (2.19) holds throughout D. Since D is Green-smooth,

$$G1 \leqq C_6 G1_H. \tag{2.20}$$

Now from (2.17), (2.20), and (2.19) we see that (2.15) holds with $C_1=C_3C_6C_5$. This completes the proof of the lemma. □

Remark 2.13. Whether or not D is Green-smooth, there exists a constant $C_1<\infty$ such that

$$E^x\{\tau\} \leqq C_1 \tag{2.21}$$

for all $x\in D$. Indeed (2.21) holds whenever D has finite Lebesgue measure. See [10], p. 123, (9). K.L. Chung has observed that this immediately implies that for any $\alpha\in[0,\infty)$ we have

$$\inf_{x\in D} E^x\{e^{-\alpha\tau}\}>0, \tag{2.22}$$

since $E^x\{e^{-\alpha\tau}\}\geqq e^{-\alpha E^x\{\tau\}}$ by Jensen's inequality applied to the convex function $e^{-\alpha t}$. In view of Lemma 2.12, if D is *Green-smooth*, then

$$\inf_{x\in D,\, z\in\partial D} E^{x;z}\{e^{-\alpha\tau}\}>0 \tag{2.23}$$

for any $\alpha\in[0,\infty)$ by the same argument.

Remark 2.14. Let us now recall Young's inequality for convolutions: If p, q, $r\in[1,\infty]$ (this q is not our function q) with $r^{-1}=p^{-1}+q^{-1}-1$ and if $g\in L^p(\mathbb{R}^d)$, $h\in L^q(\mathbb{R}^d)$, then $g*h\in L^r(\mathbb{R}^d)$ and $\|g*h\|_r\leqq\|g\|_p\|h\|_q$, where $*$ denotes convolution. (For a proof, see [14], pp. 37–38.) Using this it is easy to show that if n is an integer $>d/2$ and M is the operator of multiplication by a bounded Borel function in D, then $(G_\alpha M)^n$ is bounded as a linear operator from $L^1(D)$ into $L^\infty(D)$.

Proof of Theorem 2.1. In the proof of this theorem we shall replace the hypothesis that the boundary of D is of class C^2 by the weaker hypothesis that D is Green-smooth. (The same replacement may be made in Corollaries 2.2 through 2.5.) Choose $\alpha\in[0,\infty)$ such that $\rho=q+\alpha>0$ in D and let M be the operator of multiplication by the bounded Borel function ρ. For $z\in\partial D$, let

$v^z = E^{\cdot;z}\{e^{-\alpha\tau}\}$. We have

$$E^{\cdot;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\} = E^{\cdot;z}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left[\int_0^\tau \rho(X_s)\,ds\right]^k e^{-\alpha\tau}\right\}$$

$$= \sum_{k=0}^{\infty} E^{\cdot;z}\{\int_{0<s_1<\ldots<s_k<\tau} \rho(X_{s_1})\ldots\rho(X_{s_k})\,ds_k\ldots ds_1\, e^{-\alpha\tau}\}$$

$$= \sum_{k=0}^{\infty}(G_\alpha^z M)^k v^z \tag{2.24}$$

where the last step follows from the Markov property. Fix an integer $n > d/2$. From (2.24) we have

$$E^{\cdot;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\}$$
$$= v^z + \frac{1}{K^z}\left\{\sum_{k=1}^{n}(G_\alpha M)^k K^z v^z + \sum_{l=0}^{\infty}(G_\alpha M)^l (G_\alpha M)[(G_\alpha M)^n K^z v^z]\right\}. \tag{2.25}$$

Now let H be a fixed compact subset of D of strictly positive Lebesgue measure. C_1, C_2, ... will denote finite constants $\geqq 1$. From Lemma 2.12 (see (2.14)) it is clear that

$$v^z + \frac{1}{K^z}\sum_{k=1}^{n}(G_\alpha M)^k K^z v^z \leqq C_1. \tag{2.26}$$

From (2.23) we have

$$v^z \geqq C_2^{-1}. \tag{2.27}$$

We can therefore take A_1 to be C_2^{-1}. Since $K^z(p_*) = 1$, Harnack's inequality implies that

$$K^z \geqq C_3^{-1} 1_H. \tag{2.28}$$

Also, by Lemma 2.11, we have

$$\|K^z\|_{L^1(D)} \leqq C_4. \tag{2.29}$$

From (2.27) through (2.29) we obtain

$$C_5^{-1}(G_\alpha M)^n 1_H \leqq (G_\alpha M)^n K^z v^z \leqq C_6. \tag{2.30}$$

Here $C_5 = C_2 C_3$ and the right-hand inequality holds because $n > d/2$. Lemma 2.10 and (2.30) imply

$$C_7^{-1}(G_\alpha M)1 \leqq (G_\alpha M)[(G_\alpha M)^n K^z v^z] \leqq C_8 (G_\alpha M)1. \tag{2.31}$$

From (2.1), (2.25), and the left-hand inequality in (2.31) we obtain

$$\sum_{l=0}^{\infty}[(G_\alpha M)^l (G_\alpha M)1](x_0) < \infty. \tag{2.32}$$

But by the method used to prove (2.24) we have

$$\begin{aligned} u_1 &= \sum_{k=0}^{\infty} (G_\alpha M)^k H_1^\alpha \\ &\leqq 1+\sum_{l=0}^{\infty} (G_\alpha M)^l (G_\alpha M) 1. \end{aligned} \tag{2.33}$$

From (2.32) and (2.33) we find that $u_1(x_0)<\infty$. But then by Theorem 1.2 of Chung and Rao [4],

$$u_1 \leqq C_9. \tag{2.34}$$

Now $H_1^\alpha \geqq C_2^{-1}$ (see (2.27)) so from (2.33) and (2.34) we obtain

$$\sum_{k=0}^{\infty} (G_\alpha M)^k 1 \leqq C_{10}. \tag{2.35}$$

As we saw in the proof of Lemma 2.12 (see (2.20) and (2.19)), we have

$$G1 \leqq C_{11} K^z. \tag{2.36}$$

Therefore

$$\begin{aligned} \frac{1}{K^z}\sum_{l=0}^{\infty}(G_\alpha M)^l (G_\alpha M)[(G_\alpha M)^n K^z v^z] &\leqq C_8 \frac{1}{K^z}\sum_{l=0}^{\infty}(G_\alpha M)^l (G_\alpha M)1 && \text{(from (2.31))} \\ &= C_8 \frac{1}{K^z}(G_\alpha M)\sum_{k=0}^{\infty}(G_\alpha M)^k 1 \\ &\leqq C_{12}\frac{1}{K^z}(G_\alpha M)1 && \text{(from (2.35))} \\ &\leqq C_{13}\frac{1}{K^z} G1 \\ &\leqq C_{14} && \text{(from (2.36)).} \end{aligned}$$

Thus we may take $A_2=C_1+C_{14}$. This completes the proof of the theorem. □

Remark 2.16. It follows immediately from Theorem 2.1 (in its version for Green-smooth D) that if D is Green-smooth and the gauge of D is finite, then there are constants A_1, $A_2 \in (0,\infty)$ such that for all non-negative Borel functions f on ∂D,

$$A_1 H_f \leqq u_f \leqq A_2 H_f. \tag{2.37}$$

Proof of Corollary 2.4. Since $\mu_x(dz)=K(x,z)\,\mu(dz)$,

$$\int_D |u_f(x)|\,dx = \int_D \left| \int_{\partial D} f(z) E^{x;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\} K(x,z)\,\mu(dz)\right| dx \leqq C\int_{\partial D}|f|\,d\mu$$

where $C=A_2 C_4$, A_2 is as in (2.2), and C_4 is as in (2.29). □

3. Connections with PDE Theory

We begin this section with a discussion of when and in what sense u_f solves the boundary value problem (1.2). These questions have been considered by other authors (Khas'minskii [8], Chung and Rao [4], Aizenman and Simon [1]) so we shall be brief. Our main purpose will be to indicate the sense in which (1.2b) may be interpreted when f is not continuous. Following that we shall give an application of Corollary 2.2 to PDE theory. Let us mention that in Propositions 3.1 and 3.2 below our blanket assumption that D is bounded can be replaced by the assumption that D has finite Lebesgue measure.

We begin our discussion by pointing out that if u is any locally L^1 function in D which satisfies (1.2a) in the sense of distribution theory, then u may be modified on a set of Lebesgue measure 0 in D so as to become continuous. This was shown by Aizenman and Simon [1], Theorem 1.5. In our case where q is bounded, a very simple proof can be given using Weyl's lemma and the smoothing properties of Green-potential operators in relatively compact open subsets of D. We leave it to the reader to work this out if he so desires.

Proposition 3.1. *Suppose the gauge of D is finite and $f\colon \partial D \to \mathbb{R}$ is Borel and bounded. Let $u = u_f$. Then u is bounded and continuous in D, satisfies $(1/2)\Delta u + qu = 0$ in D in the sense of distributions, and satisfies $u = f$ on ∂D in the sense that for each $x \in D$, $\lim_{t \uparrow \tau} u(X_t) = f(X_\tau)$ P^x-a.s.*

Proof. That u is bounded in D was proved by Chung and Rao [4], Theorem 1.2. Applying the Markov property to the identity

$$\exp\left[\int_0^\tau q(X_s)\,ds\right] = 1 + \int_0^\tau q(X_s) \exp\left[\int_s^\tau q(X_t)\,dt\right] ds$$

after multiplying by $f(X_\tau)$ and taking expectations, yields

$$u = H_f + G(qu). \tag{3.1}$$

In doing this an interchange in order of integration is performed whose validity follows from the finiteness of $G(|q|\,u_{|f|})$, which in turn follows from the boundedness of $u_{|f|}$. As is well known, $\Delta H_f = 0$ while $\Delta G(qu) = -2qu$ in D so $(1/2)\Delta u + qu = 0$ in D (in the sense of distributions). Finally for each $x \in D$ we have $\lim_{t \uparrow \tau} H_f(X_t) = f(X_\tau)$ P^x-a.s. (see [5]) and $\lim_{t \uparrow \tau} G(qu)(X_t) = 0$ P^x-a.s. (see [10], Theorem 5.4.5 and Proposition 6.1.2). This completes the proof of the proposition. $\square$

Now let us fix an increasing sequence (D_n) of open connected relatively compact subsets of D satisfying $\bigcup_n D_n = D$. For each n, let $\tau_n = \inf\{t > 0 : X_t \notin D_n\}$. Let us also fix a point $x_0 \in D_0$. If v and f are Borel functions in D and on ∂D respectively, then the event $F = \{\lim_{t \uparrow \tau} v(X_t)$ exists and is equal to $f(X_\tau)\}$ satisfies $\theta_{\tau_n}^{-1}[F] = F$ for each n, where θ_{τ_n} is the usual translation operator, so $P^{\cdot}\{F\}$ is harmonic in D. Thus if $\lim_{t \uparrow \tau} v(X_t) = f(X_\tau)$ P^x-a.s. for some $x \in D$, then it is so for all $x\in D$. Similarly, if $\lim_{n\to\infty} v(X_{\tau_n})=f(X_\tau)$ P^x-a.s. for some $x\in D$, then it is so for all $x\in D$. Finally, it is clear that if $\lim_{t\uparrow\tau} v(X_t)=f(X_\tau)$ P^x-a.s., then $\lim_{n\to\infty} v(X_{\tau_n}) =f(X_\tau)$ P^x-a.s.

Proposition 3.2. *Suppose the gauge of D is finite and $f\colon \partial D\to\mathbb{R}$ is Borel and bounded. Assume v is bounded and continuous in D, satisfies $(1/2)\Delta v+qv=0$ in D in the sense of distributions and satisfies $v=f$ on ∂D in the sense that $\lim_{n\to\infty} v(X_{\tau_n})=f(X_\tau)$ P^{x_0}-a.s. Then $v=u_f$.*

Proof. Let $w=G(qv)$. Then $\Delta v=-2qv=\Delta w$ so by Weyl's lemma (see [11], p. 5.21, Lemma 1) there is a harmonic function h in D such that $v=h+w$ a.e. in D. By continuity, $v=h+w$ throughout D. It is easy to see that h is bounded and then that $h=H_f$. Thus

$$v=H_f+G(qv). \tag{3.2}$$

Using the Markov property, it is easy to check that

$$H_f=H_f^\alpha+\alpha G_\alpha H_f. \tag{3.3}$$

From (3.2), (3.3), and the resolvent equation, we deduce

$$v=H_f^\alpha+G_\alpha(\rho v). \tag{3.4}$$

Here $\alpha\in[0,\infty)$ is chosen so that $\rho=q+\alpha$ is non-negative. Let M be the operator of multiplication by ρ. By successive substitution from (3.4) we find

$$v=\left[\sum_{k=0}^{n}(G_\alpha M)^k H_f^\alpha\right]+(G_\alpha M)^{n+1}v. \tag{3.5}$$

Now by hypothesis, $u_1<\infty$. But

$$u_1=\sum_{k=0}^{\infty}(G_\alpha M)^k H_1^\alpha \tag{3.6}$$

and by (2.21), $\inf_D H_1^\alpha>0$. Therefore $(G_\alpha M)^{n+1}v\to 0$ as $n\to\infty$. Thus

$$v=\sum_{k=0}^{\infty}(G_\alpha M)^k H_f^\alpha. \tag{3.7}$$

That is, $v=u_f$. This completes the proof of the proposition. □

We now turn to an application of Corollary 2.2 to PDE theory. Observe that a solution of the boundary value problem (1.2) need not be non-negative even if f is non-negative. (For example, if $d=1$, $D=(\pi/4, 7\pi/4)$, $q\equiv 1/2$, and $f\equiv 1$, then (1.2) has a unique solution, namely $u(x)\equiv 2^{1/2}\cos x$; but then $u(\pi)<0$. Of course in this example the gauge of D must be infinite for otherwise we would have a contradiction of Proposition 3.2.) Our application of Corollary 2.2 concerns the existence of non-negative solutions of (1.2). We state it for ∂D of class C^2 but it is true if D is just Green-smooth, provided ∂D is taken to be the Martin boundary of D if this differs from the Euclidean boundary (in which case X_τ should be taken to be $\lim_{t \uparrow \tau} X_t$, with the limit being taken in the Martin topology).

Proposition 3.3. *Assume the boundary of D is of class C^2. Suppose v is non-negative, continuous, and satisfies $(1/2)\Delta v + qv = 0$ in D in the sense of distributions. Then there is a non-negative Borel function g on ∂D such that $\lim_{t \uparrow \tau} v(X_t) = g(X_\tau)$ P^{x_0}-a.s. If $P^{x_0}\{g(X_\tau) > 0\} \neq 0$, then for each bounded non-negative Borel function f on ∂D there is a unique bounded continuous function u in D which satisfies $(1/2)\Delta u + qu = 0$ in D in the sense of distributions and satisfies $u = f$ on ∂D in the sense that for each $x \in D$, $\lim_{t \uparrow \tau} u(X_t) = f(X_\tau)$ P^x-a.s.; moreover, u is non-negative.*

Proof. For $0 \leqq t < \tau$, let

$$Y_t = \exp\left[\int_0^t q(X_s)\, ds\right] v(X_t). \tag{3.8}$$

By the repeated substitution method of the proof of Proposition 3.2 it is easy to show that for each n

$$v \geqq E^{\cdot}\{Y_{\tau_n}\} \quad \text{in } D_n. \tag{3.9}$$

The non-negativity of v is crucial here. If v vanishes at some point of D_n, then $E^{\cdot}\{Y_{\tau_n}\}$ also vanishes there and hence vanishes throughout D_n. Thus either $v \equiv 0$ in D or $v > 0$ throughout D. In the former case there is nothing more to prove. Therefore let us assume that $v > 0$ throughout D. Then $\inf_{\partial D_n} v > 0$ so from (3.9) the gauge of D_n is finite so in fact

$$v = E^{\cdot}\{Y_{\tau_n}\} \quad \text{in } D_n \tag{3.10}$$

by Proposition 3.2 applied to D_n. From (3.10) and the strong Markov property, for each $x \in D$, (Y_t) is a continuous non-negative local martingale under P^x on the stochastic time interval $[0, \tau)$; it is a uniformly integrable martingale under P^x on $[0, \tau_n)$ for each n. Hence there is a random variable Z such that $\lim_{t \uparrow \tau} Y_t = Z$ P^{x_0}-a.s. Let $V = \exp\left[-\int_0^\tau q(X_s)\, ds\right] Z$. Then $\lim_{t \uparrow \tau} v(X_t) = V$ P^{x_0}-a.s. and hence P^x-a.s. for each $x \in D$ (see the discussion preceding Proposition 3.2). We may assume V is measurable relative to the σ-field

$$\mathscr{G} = \bigcap_n \sigma(X_{\tau_k} : k \geqq n). \tag{3.11}$$

Let $h_n = E^{\cdot}\{V \wedge n\}$ in D. Then $0 \leqq h_n \leqq n$ and h_n is harmonic in D. As ∂D is Lipschitz, the Martin boundary of D may be identified with ∂D according to Hunt and Wheeden [7]. Hence by Theorem 9.1 of Doob [6] (with the h there equal to 1), there is a bounded non-negative Borel function g_n on ∂D such that $\lim_{t \uparrow \tau} h_n(X_t) = g_n(X_\tau)$ P^x-a.s. for $x \in D$. But by the martingale convergence theorem, $\lim_{k \to \infty} h_n(X_{\tau_k}) = V \wedge n$ P^x-a.s. for $x \in D$. Let $g = \sup_n g_n$. Then $V = g(X_\tau)$ P^x-a.s. for

$x \in D$. Now from (3.10) and Fatou's lemma we have

$$v \geqq u \qquad \text{in } D. \tag{3.12}$$

If $P^{x_0}\{g(X_\tau)>0\} \neq 0$, then by Corollary 2.2 the gauge of D is finite. The remaining assertions then follow from Propositions 3.1 and 3.2. □

4. Concluding Remarks

In closing, let us mention some open problems. First, can one relax our requirement that q be bounded, say along the lines considered by Aizenman and Simon [1]? Second, can one relax significantly our requirement that ∂D be of class C^2? In connection with this, it would be desirable to find a weaker form of "Green-smoothness" which would still be adequate to establish our main results.

Since this paper was written, Zhongxin Zhao has shown that (2.2) holds for q of the class considered by Aizenman and Simon, provided D is a *ball* with finite gauge. He is thus very close to providing an affirmative answer to the first open question stated in the preceding paragraph.

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