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Conditional Limit Theorems for Asymptotically Stable Random Walks

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§1

Let $\{S(m), m \geqq 0\}$ be a random walk (r.w.) with $S(0) \equiv 0$, $S(m) = \sum_{r=1}^{m} X_r$ for $m \geqq 1$, where the X_r are i.i.d. Assume that with probability one $\limsup S_n = +\infty$, $\liminf S_n = -\infty$, so that the r.w. is oscillatory. Then for each $k \geqq 1$ the k^{th} strict descending ladder index N_k is proper, where $N_k = \min\{m \geqq N_{k-1}$ s.t. $S(N_{k-1} + m) < S(N_{k-1})\}$ and $N_0 \equiv 0$. By "the k^{th} excursion of the r.w." we will mean the shifted and stopped section of its path, $\{S(\{N_{k-1}+m\}_\wedge N_k) - S(N_{k-1}),\ m \geqq 0\}$. Suppose also that norming constants c_n exist such that $W_n \Rightarrow W$, where W_n is the normed process defined by $W_n(t) = c_n^{-1} S([nt])$, W is a stable process of index $0 < \alpha \leqq 2$, and $\Rightarrow$ stands for weak convergence on the space $\mathscr{D} = D([0, \infty))$. Define the excursions of W_n in an analogous way, so that the first excursion of W_n coincides with the stopped process $\dot{W}_n$ which has $\dot{W}_n(t) = W_n(t_\wedge T_n)$, with $T_n = n^{-1} N_1 = \Delta(W_n)$, where for $z \in \mathscr{D}$ $\Delta(z)$ denotes $\inf\{t \geqq 0 : z(t) < 0\}$.

In this paper we are concerned with the weak convergence of a conditioned process $W_n^{(A)}$ to a limiting process $W^{(A)}$. The conditioning is to involve the behaviour of the r.w. only up to time N_1, so we require $W_n^{(A)}$ to be equivalent to $\{W_n | \dot{W}_n \in A\}$, where A is some subset of $\mathscr{D}$ with $P\{W_n \in A\} > 0$ for all large n. The basic example is when

$$A = \{z \in \mathscr{D} : \Delta(z) > 1\} \qquad \text{so that } \dot{W}_n \in A \Leftrightarrow S(m) \geqq 0 \text{ for } 0 \leqq m \leqq n,$$

and we are dealing with "r.w.'s conditioned to stay non-negative". In this context Bolthausen [3] showed that $W_n^{(A)}$ is equivalent to $W_n(K_n + \cdot) - W_n(K_n)$, where K_n is the time at which the first excursion of W_n whose length exceeds 1 begins. In the case $\alpha = 2$ he also established that $W_n(K_n + \cdot) - W_n(K_n) \Rightarrow W(K + \cdot) - W(K)$, where K is defined analogously for Brownian Motion W. Again in the case $\alpha = 2$, Shimura [19] showed that the same method works for a large class of subsets of $\mathscr{D}$, and in this paper we show that the same is true for $0 < \alpha \leqq 2$. Technically, the main difficulty in extending the argument stems from the fact that, for $\alpha \neq 2$, the stable process W has discontinuous paths.

Crucially, however, it turns out that, a.s., the paths of W are continuous at the beginning of each "excursion" and do not have an upwards jump at the end of each "excursion". (See Lemma 2 in § 2.)

The "excursions" of W coincide with the "excursions away from zero" of the process Y, where $Y(t)=W(t)-\inf_{0\leqq s\leqq t} W(s)$. In case $\alpha=2$ Y is, of course, equivalent to Reflected Brownian Motion $|W|$, so that identification of the limit process $W^{(A)}$ seems possible, and has indeed been achieved in the case $A=\{z: \Delta(z)>1\}$. For $\alpha \neq 2$, equivalence between Y and $|W|$ fails, and identification of $W^{(A)}$ seems out of the question. (See, however, Remark 4 in § 4.) Nevertheless the functional conditional central limit theorem which we state and prove in § 2 can be used, for example, to compute the asymptotic behaviour of the tail probabilities of a variety of functionals of the first excursion of the r.w. This is demonstrated in § 3, and § 4 contains some remarks about related work.

§ 2

Throughout the paper we will be making the following assumption about X, a typical step of the r.w. ($X \in D(\alpha, \beta)$ means that X is in the domain of attraction of a stable law of index $\alpha<2$, with symmetry parameter $\beta \in [-1,+1]$, and $X \in D(2)$ means that X is in the domain of attraction of the Normal law.)

Assumption 2.1. *One of the four following hold:*

$$X \in D(2) \quad \text{and} \quad E(X)=0; \tag{2.1a}$$

$$X \in D(\alpha,\beta) \quad \text{with} \quad 1<\alpha<2 \quad \text{and} \quad -1\leqq\beta\leqq+1 \quad \text{and} \quad E(X)=0; \tag{2.1b}$$

$$X \in D(1,0) \quad \text{and} \quad E(X)=0; \tag{2.1c}$$

$$X \in D(\alpha,\beta) \quad \text{with} \quad 0<\alpha<1 \quad \text{and} \quad |\beta|<1. \tag{2.1d}$$

Under this assumption, norming constants c_n exist such that $W_n \Rightarrow W$, where W is a stable process with the corresponding parameters, and WLOG we can assume that $W(1)$ has the standard stable distribution, so that, e.g. in case $\alpha=2$ W is standard Brownian Motion. Observe also that under (2.1) $P\{W(1)\geqq 0\} \in (0,1)$, so that both $(S(m), m\geqq 0)$ and $\{W(t), t\geqq 0\}$ are oscillatory.

As in § 1, we will write $\Delta(z)=\inf\{t\geqq 0: z(t)<0\}$ for $z \in \mathscr{D}$, and introduce "excursion space" $\mathscr{E}=\{z\in\mathscr{D}: z(0)=0$ and $\Delta(z)<\infty\}$. On $\mathscr{D}$ we will use the metric d which induces the J_1 topology but on $\mathscr{E}$ we will introduce another metric $\dot{d}$ given by $\dot{d}(z_1,z_2)=|\Delta(z_1)-\Delta(z_2)|+d(z_1(\cdot \wedge \Delta(z_1)), z_2(\cdot \wedge \Delta(z_2)))$.

Assumption 2.2. *The subset A of $\mathscr{E}$ is such that with respect to the $\dot{d}$ metric, $0 \notin \bar{A}$.*

For $z \in \mathscr{D}$, let $m(t)=\inf_{0\leqq s\leqq t} z(s)$, $y(t)=z(t)-m(t)$ and introduce the ladder-point set $L(z)$ which is defined to be the closure of the set $\{t: y(t)=0$ and $\exists$ no $\delta>0$ with $y(s)=0$ for all $s\in(t-\delta,t]\}$. The excursion intervals of z are the maximal finite open intervals contained in $(0,\infty)\setminus L(z)$, and $E(z)$ denotes the totality of all such intervals. For $I=(\tau,v)\in E(z)$ write $\Delta_I(z)=v-\tau$, $\theta_I(z)$ for the function $z(\cdot+\tau)-z(\tau)$, and $\alpha_I(z)$ for the function $\theta_I(z)(\cdot \wedge \Delta_I(z))$. Finally for $A\subset\mathscr{E}$ set $E_A(z)=\{I\in E(z): \alpha_I(z)\in A\}$.

Assumption 2.3. *If ∂A denotes the boundary of $A \subset \mathscr{E}$ in the metric $\dot{d}$,*

$$P\{E_{\partial A}(W)=\phi, E_A(W) \neq \phi\}=1.$$

Notice that whenever (2.2) holds, for each $z \in \mathscr{D}$ $E_A(z)$ has no finite limit points so we may denote by $\hat{I}_A(z)$ the first member of $E_A(z)$ when $E_A(z) \neq \phi$, and put $\hat{I}_A(z)=(\infty, \infty)$ when $E_A(z)=\phi$.

Theorem. *If* (2.1) *holds and A is a fixed measurable subset of $\mathscr{E}$ satisfying* (2.2) *and* (2.3) *we have $W_n^{(A)} \Rightarrow W^{(A)}$ on $\mathscr{D}$ where $W^{(A)}=\theta_{\hat{I}_A}(W)$.*

A basic ingredient in our proof is the following identity, first observed in a special case by Bolthausen [3].

Lemma 1. *For measurable $A \subset \mathscr{E}$, all n* s.t. *$P\{\dot{W}_n \in A\}>0$, and measurable $B \subset \mathscr{D}$*

$$P\{W_n \in B \mid \dot{W}_n \in A\}=P\{\theta_{\hat{I}_A}(W_n) \in B\}. \tag{2.4}$$

Proof. The definition of the ladder point set has been framed so that, a.s., $\hat{I}_A(W_n)$ coincides with $(n^{-1} N_{K-1}, n^{-1} N_K)$, where $K=\min\{k \geqq 1$ s.t. $W_n(n^{-1} N_{k-1} + \cdot \wedge n^{-1}(N_k - N_{k-1})) \in A\}$. Thus (2.4) extends Lemma 3.1 of Bolthausen [3] by allowing A to be any measurable subset of $\mathscr{E}$ (he has $A=\{z: \Delta(z)>1\}$) and by considering $\theta_{\hat{I}_A}(W_n)$, which involves S_m for all $m \geqq N_{K-1}$, rather than $\alpha_{\hat{I}_A}(W_n)$, which involves S_m for $N_K \geqq m \geqq N_{K-1}$. However the proof is essentially the same, and is omitted.

For an arbitrary $z \in \mathscr{D}$, the analysis of $E(z)$ is quite tricky; however the sample functions of W belong, a.s., to a subset of $\mathscr{D}$ with some convenient properties:

Lemma 2. *$P\{W \in \mathscr{D}^*=1\}$, where $\mathscr{D}^*=\bigcap_{i=1}^{6} \mathscr{D}_i$ and $\mathscr{D}_i$ is the subset of $\mathscr{D}$ which has property (i) below.*

(1) *$(0, v) \notin E(z)$ for any $v>0$;*

(2) *$L(z)$ coincides with the closure of $\{t: y(t)=0\}$;*

(3) *$E(z) \cap [t, \infty) \neq \phi$ for every $t<\infty$;*

(4) *$z(\cdot)$ is continuous at all local extrema;*

(5) *there exists no $0 \leqq t_0 < t_1 < t_2 < \infty$ with $m(t_0)=m(t_1)=m(t_2)$ and $y(t_0) = y(t_1)=0$;*

(6) *$(\tau, v) \in E(z) \Rightarrow z$ is continuous at τ and either $z(v-)=z(v)=z(\tau)$ or $z(v-) > z(\tau) \geqq z(v)$.*

Proof.

(1) $(0, v) \in E(W) \Rightarrow W(t) \geqq W(0)=0$ for $0<t<v$, which has probability zero since a.s. there exists $t_n \downarrow 0$ with $W(t_n) \in (-\infty, 0)$; in other words 0 is regular for $(-\infty, 0)$.

(2) This follows from the fact that, a.s., $W(\cdot)$ is not monotone in any interval of positive length.

(3) This just says that the ladder point set is recurrent.

(4) Millar [15] has established this result for any process with independent increments for which 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$; it is essentially a consequence of the fact that jump times are Markov times.

(5) If $z \notin \mathscr{D}_5$ then there are rationals $0<r_1<r_2<\infty$ with $m(r_1)=\inf_{r_1 \leqq s \leqq r_2} z(s)$. But for fixed r_1 and r_2, $P\{\inf_{0 \leqq s \leqq r_1} W(s)=\inf_{r_1 \leqq s \leqq r_2} W(s)\}=0$, and hence $P\{W \notin \mathscr{D}_5\}=0$.

(6) Note first that $(\tau, v) \in E(z) \Rightarrow \tau$ is a local minimum of z, so that if $z \in \mathscr{D}_4$, z must be continuous at τ. Next if also $z \in \mathscr{D}_2$, $z(v-)<z(v) \Rightarrow y(v)>0$ so we can find $v_n \to v$ with $y(v_n)=0$; but $\tau<v_n<v$ is impossible ($\because v_n \in L(z)$) and $v_n>v$, $v_n \downarrow v \Rightarrow y(v)=0$ by right continuity. Since $m(v-)=z(\tau)$, it follows that $z(v) \leqq z(\tau) \leqq z(v-)$. If $z(v-)=z(\tau)>z(v)$ then we have that for $t_0>v$, the reversed path $\bar{z}(t)=z(t_0)-z((t_0-t)-)$, $t \leqq t_0$, has $\bar{z}(t) \leqq \bar{z}(t_0-v)<\bar{z}((t_0-v)-)$ for $t \in (t_0-v, t_0-\tau)$. Using duality and the fact that jump times are Markov times, it follows that $P\{W \in \mathscr{D}_6\}=1$.

We will also need some facts about convergence in the metric d on $\mathscr{D}$.

Lemma 3. (i) *Let*

$$L_c^{(k)}(z)=\sup_{\substack{0 \leqq t-c<t_1<t<t_2 \leqq t+c;\\ 0 \leqq t \leqq k}} \{\min\{|z(t_1)-z(t)|, |z(t_2)-z(t)|\}+\sup_{0 \leqq h \leqq c} \{|z(h)-z(0)|\}.$$

Then $z_n \to z$ *on* $\mathscr{D}$ iff $z_n(t) \to z(t)$ *at some set of points* t *which is everywhere dense in* $(0, \infty)$ *and for each* k $\lim_{c \downarrow 0} \limsup_{n \to \infty} L_c^{(k)}(z_n)=0$.

(ii) *Suppose* $z_n \to z$ *on* $\mathscr{D}$, $t_n \to t$ *and* $l=\lim_{n \to \infty} z_n(t_n)$ *exists. Then either* $l=z(t)$ *or* $l=z(t-)$.

(iii) *Suppose* $z_n \to z$ *on* $\mathscr{D}$, $t_n \to t$ *and* z *is continuous at* t. *Then* $z_n(t_n) \to z(t)$ *and* $z_n(t_n-) \to z(t)$.

Proof. (i) That this is the appropriate extension to $\mathscr{D}$ of a result for $D[0,1]$ in Skorokhod [20, Th. 2, p. 200] follows from Theorem 3 of Lindvall [14].

(ii) It is easily seen that we can find $u_n<t_n<v_n$ with $u_n \to t$ $v_n \to t$ such that $z_n(u_n) \to z(t-)$, $z_n(v_n) \to z(t)$. But if $l \neq z(t)$, $l \neq z(t-)$, $\min\{|z_n(u_n)-z_n(t_n)|, |z_n(v_n)-z_n(t_n)|\} \nrightarrow 0$, which contradicts $\lim_{c \downarrow 0} \limsup_{n \to \infty} L_c^{(k)}(z_n)=0$ for any fixed $k>t$.

(iii) Any subsequence of $\{z_n(t_n)\}$ contains a convergent subsequence, and by (ii) its limit must be $z(t)$. This shows that $z_n(t_n) \to z(t)$. But if some subsequence existed with $z_n(t_n-) \to l \neq z(t)$, we could find $s_n \to t$ with $z_n(s_n) \to l$, so this is impossible, and $z_n(t_n-) \to z(t)$.

The main part of the proof of the Theorem is contained in the next two lemmas.

Lemma 4. *Suppose* $z_n \in \mathscr{D}$, $z_n \to z \in \mathscr{D}^*$, $I_n=(\tau_n, v_n) \in E(z_n)$ *and* $\tau_n \to \tau$, $v_n \to v$ *where* $0 \leqq \tau<v<\infty$. *Then*

(i) $I=(\tau, v) \in E(z)$;

(ii) $\theta_{I_n}(z_n) \xrightarrow{d} \theta_I(z)$;

(iii) $\alpha_{I_n}(z_n) \xrightarrow{d} \alpha_I(z)$.

Proof. (i) Writing $\tilde{z}(t)=\min(z(t), z(t-))$ for $t>0$, $\tilde{z}(0)=z(0)$, it follows from (ii) of Lemma 3 that $\liminf_{t_n \to t} \tilde{z}_n(t_n) \geqq \tilde{z}(t)$ for $0 \leqq t<\infty$. It follows easily from $I_n \in E(z_n)$

that $z_n(t) \geqq \tilde{z}_n(\tau_n)$ for $0 \leqq t < v_n$ and hence that $z(t) \geqq \tilde{z}(\tau)$ for every $t \in [0, v)$ which is a continuity point of $z(\cdot)$. By right-continuity, this inequality is valid for arbitrary $t \in [0, v)$, and hence τ is a local minimum of $z(\cdot)$. Since $z \in \mathscr{D}_4$, z is continuous at τ and $z(\tau) = \tilde{z}(\tau) = m(\tau)$, so that $y(\tau) = 0$. We know $\tilde{z}(t) \geqq z(\tau)$ for $t \in (\tau, v)$; if $\tilde{z}(t_1) = z(\tau)$ for some $t_1 \in (\tau, v)$ then t_1 is a local minimum of z and for any $t_2 \in (t_1, v)$ $m(\tau) = m(t_1) = m(t_2)$ and $y(\tau) = y(t_1) = 0$. Since $z \in \mathscr{D}_5$ this is impossible so $\tilde{y}(t) > 0$ for $t \in (\tau, v)$. Finally another simple consequence of $I_n \in E(z_n)$ is that $z_n(v_n) \leqq \tilde{z}_n(\tau_n)$; since z is continuous at τ, (iii) of Lemma 3 gives $\tilde{z}_n(\tau_n) \to z(\tau)$, and it follows that $\tilde{z}(v) \leqq \liminf \tilde{z}_n(v_n) \leqq z(\tau) = m(v-)$. Since $z(v-) \geqq m(v-)$ this gives $z(v) \leqq z(\tau)$ so that $m(v) = z(v)$ and $y(v) = 0$. From $y(\tau) = y(v) = 0$, $\tilde{y}(t) > 0$ for $t \in (\tau, v)$ and $z \in \mathscr{D}_2$ it is immediate that $I \in E(z)$.

(ii) Since $z_n(\tau_n) \to z(\tau)$, it suffices to prove that $z_n^* \to z^*$, where $z_n^*(\cdot) = z_n(\tau_n + \cdot)$, $z^*(\cdot) = z(\tau + \cdot)$. But $z_n^*(t) \to z^*(t)$ at each point t such that $\tau + t$ is a continuity point of $z(\cdot)$, and this set is everywhere dense on $[0, \infty)$. Also if $k^* = k + \tau + \gamma$, where $\gamma > 0$, then for all large enough n $L_c^{(k)}(z_n^*) \leqq L_c^{(k^*)}(z_n) + \sup_{0 \leqq h \leqq c} |z_n(\tau_n + h) - z_n(\tau_n)|$, and since z is continuous at τ it follows easily that

$$\lim_{c \downarrow 0} \limsup_{n \to \infty} L_c^{(k)}(z_n^*) \leqq \lim_{c \downarrow 0} \limsup_{n \to \infty} L_c^{(k^*)}(z_n)$$
$$+ \lim_{c \downarrow 0} \{ \sup_{\tau \leqq t \leqq \tau + c} |z(t) - z(t-)| + \sup_{0 \leqq h \leqq c} |z(\tau + h) - z(\tau)| \},$$

and the result follows by (i) of Lemma 3.

(iii) Write $x_n(\cdot) = \theta_{I_n}(z_n(\cdot)) = z_n(\tau_n + \cdot) - z_n(\tau_n)$, $x(\cdot) = \theta_I(z(\cdot)) = z(\tau + \cdot) - z(\tau)$, so that $x_n \to x$ and we need to show that $\hat{x}_n \to \hat{x}$, where $\hat{x}_n(\cdot) = x_n(\delta_{n \wedge} \cdot) = \alpha_{I_n}(z_n(\cdot))$, $\hat{x}(\cdot) = x(\delta_\wedge \cdot) = \alpha_I(z(\cdot))$ and $\delta_n = v_n - \tau_n \to \delta = v - \tau > 0$. Note first that $\hat{x}_n(t) \to \hat{x}(t)$ at all $t \in [0, \delta)$ which are continuity points of $\hat{x}(\cdot)$, and at all $t > \delta$, provided $x_n(\delta_n) \to x(\delta)$. If x is continuous at δ, this follows by (ii) of Lemma 3. If x is not continuous at δ, then since $z \in \mathscr{D}_6$ we have $x(\delta-) > 0$ and $x(\delta) \leqq 0$. Thus if $x_n(\delta_n) \nrightarrow x(\delta)$ by (ii) of Lemma 3 there is some subsequence along which $x_n(\delta_n) \to x(\delta-) > 0$; however $x_n(\delta_n) = z_n(v_n) - z_n(\tau_n) \leqq z_n(v_n) - \tilde{z}_n(\tau_n) \leqq 0$, which leads to a contradiction. Suppose now that for some k $\lim_{c \downarrow 0} \limsup_{n \to \infty} L_c^{(k)}(\hat{x}_n) > 0$. Then either there is $h_n \downarrow 0$ with $\lim |\hat{x}_n(h_n) - \hat{x}_n(0)| > 0$, or $u_n < t_n < v_n$ with $|t_n - u_n| \to 0$, $|t_n - v_n| \to 0$ and $\hat{\varepsilon}_n = \hat{\varepsilon}_n(u_n, t_n, v_n) \nrightarrow 0$, where $\hat{\varepsilon}_n = \min\{|\hat{x}_n(u_n) - \hat{x}_n(t_n)|, |\hat{x}_n(v_n) - \hat{x}_n(t_n)|\}$. Since $\delta > 0$, $\lim |\hat{x}_n(h_n) - \hat{x}_n(0)| = \lim |x_n(h_n) - x_n(0)|$ so the first case is incompatible with $x_n \to x$. In the second case $\hat{x}_n(v_n) = \hat{x}_n(t_n) = x_n(\delta_n)$ for $t_n \geqq \delta_n$, so we may take $t_n < \delta_n$ for all n and assume, WLOG, that $t_n \to t \leqq \delta$. However if $t < \delta$ then for all sufficiently large n $\hat{\varepsilon}_n$ coincides with ε_n, the corresponding quantity for x_n, so that $\hat{\varepsilon}_n \to 0$. We may therefore take $t = \delta$. But then $u_n < t_n < v_n' = \min(v_n, \delta_n)$ and $|v_n' - t_n| \to 0$, so that $\varepsilon_n' = \varepsilon_n(u_n, t_n, v_n') \to 0$ and $\lim_{c \downarrow 0} \limsup_{n \to \infty} L_c^{(k)}(\hat{x}_n) = 0$.

We now introduce $J_n = \left\{ z \in \mathscr{D} \text{ s.t. for } k = 0, 1, \ldots, z \text{ is constant on } \left[\frac{k}{n}, \frac{k+1}{n}\right) \right\}_\cap \{E(z)_\cap (t, \infty) \neq \phi, \text{ all } t \geqq 0\}$ and remark that for each $n \geqq 1$, $P\{W_n \in J_n\} = 1$, so that essentially we are only concerned with the situation where $z_n \to z$, $z_n \in J_n$ and $z \in \mathscr{D}^*$.

Lemma 5. *Suppose $A \subset \mathscr{E}$ and $z \in \mathscr{D}^*$ are such that $0 \notin \bar{A}$, $E_{\partial A}(z) = \phi$ and $E_A(z) \neq \phi$. Suppose also that $z_n \in J_n$ for $n \geqq 1$, $z_n \to z \in \mathscr{D}^*$, and $\hat{I}_A(z) = (\hat{\tau}, \hat{v})$, $\hat{I}_A(z_n) = (\hat{\tau}_n, \hat{v}_n)$. Then $\hat{\tau}_n \to \hat{\tau}$ and $\hat{v}_n \to \hat{v}$.*

Proof. Since $E_A(z) \neq \phi$ and $z \in \mathscr{D}_1$, $0 < \hat{\tau} < \hat{v} < \infty$. Since $z_n \in J_n$ $L(z_n)$ is discrete and for all large enough n we may define $I_n = (\tau_n, v_n)$ where $\tau_n = \max\{\tau \leqq \hat{\tau} + \rho$ s.t. $(\tau, v) \in E(z_n)$ for some $v\}$, and $\rho = \frac{1}{2}(\hat{v} - \hat{\tau})$, noting that $v_n > \hat{\tau} + \rho$. Then there is some subsequence along which $\tau_n \to \tau'$, $v_n \to v'$ where $\tau' \leqq \hat{\tau} + \rho$ and $v' \geqq \hat{\tau} + \rho$. It is easily seen that $v' = \infty$ contradicts $z \in \mathscr{D}_2 \cap \mathscr{D}_5$, and if $\tau' > \hat{\tau}$ we have $z_n(\tau_n) < z_n(\hat{\tau})$ for all large enough n. Since z is continuous at $\hat{\tau}$, it follows that $\tilde{z}(\tau') \leqq z(\hat{\tau})$, which contradicts $\tilde{z}(t) > z(\hat{\tau})$ for $t \in (\hat{\tau}, \hat{v})$. Thus $\tau' \leqq \hat{\tau} < \hat{\tau} + \rho \leqq v' < \infty$ and Lemma 4(i) applies, giving $I' = (\tau', v') \in E(z)$. But $\hat{I} = (\hat{\tau}, \hat{v}) \in E(z)$ and $\hat{I} \cap I' \neq \phi$; from the maximal nature of $E(z)$ it follows that $I' = \hat{I}$, so that $\tau_n \to \hat{\tau}$, $v_n \to \hat{v}$. Now if $\hat{\tau}_n = \tau_n$ for all sufficiently large n, the lemma is proved, so assume the contrary. Then either $\alpha_{I_n}(z_n) \notin A$ occurs i.o. or $\alpha_{I_n}(z_n) \in A$ and $\hat{\tau}_n < \tau_n$ occurs i.o. But Lemma 4 gives $\alpha_{I_n}(z_n) \xrightarrow{d} \alpha_{\hat{I}}(z)$ and $\alpha_{\hat{I}}(z) \in \operatorname{int}(A)$ since $E_{\partial A}(z) = \phi$, so the first case is impossible. In the second case, $\hat{\tau}_n < \tau_n$ implies that $\hat{v}_n \leqq \tau_n \to \hat{\tau}$, so there is some subsequence along which $\hat{\tau}_n \to \tau_0$, $\hat{v}_n \to v_0$, where $0 \leqq \tau_0 \leqq v_0 \leqq \hat{\tau}$. Now $\tau_0 = v_0$ means that $0 \in \bar{A}$, so $\tau_0 < v_0$ and Lemma 4 gives $I_0 = (\tau_0, v_0) \in E(z)$. But since each $\alpha_{\hat{I}_n}(z_n) \in \operatorname{int}(A)$ it follows that $\alpha_{I_0}(z) \in A$ and $\hat{I}$ is not the first member of $E_A(z)$. This contradiction establishes the result.

Proof of Theorem. First we note that $P\{\dot{W}_n \in A\} > 0$ for all sufficiently large n; for $P\{\dot{W}_n \in A\} = 0$ implies $P\{E_A(W_n) = \phi\} = 1$, and hence $P\{\hat{\tau}_A(W_n) = \infty\} = 1$. But Lemma 5 and the facts that $W_n \Rightarrow W$, $P\{W_n \in J_n\} = 1$, $P\{W \in \mathscr{D}^*\} = 1$ show that $\hat{\tau}_A(W_n) \xrightarrow{D} \hat{\tau}_A(W)$, and $\hat{\tau}_A(W) < \infty$ a.s. by assumption 2.3. Thus Lemma 1 applies and we need to show $\theta_{\hat{I}_A}(W_n) \Rightarrow \theta_{\hat{I}_A}(W)$. But this is a consequence of $W_n \Rightarrow W$, the continuous mapping theorem, Lemma 5 and (ii) of Lemma 4.

§ 3

If ϕ is a non-negative functional defined on excursion space $\mathscr{E}$, we will denote by Φ the random variable $\phi(\dot{S})$, where $\dot{S}$ is the stopped random walk process $\{S([t] \wedge N_1), t \geqq 0\}$, so that Φ is determined by the first excursion of the random walk.

Definition. The class $\mathscr{F}$ consists of all non-negative measurable $\phi: \mathscr{E} \to [0, \infty)$ such that for each $y > 0$ the set $A_y = \{z \in \mathscr{E}$ s.t. $\phi(z) > y\}$ satisfies (2.2) and (2.3) and some norming sequence $\lambda_n \geqq 0$ has the following scaling property:

$$\phi(\dot{W}_n) = \frac{1}{\lambda_n} \phi(\dot{S}) \qquad \text{for each } n \geqq 1. \tag{3.1}$$

The following are examples of functionals satisfying this definition, together with the appropriate norming sequences and values of Φ;

Example 1. $\phi(z(\cdot))=\Delta(z)$; $\lambda_n=n$; $\Phi=N$.

Example 2. $\phi(z(\cdot))=\inf\{u: z(u)=\sup_{0\leq v\leq\Delta} z(v) \text{ or } z(u-)=\sup_{0\leq v\leq\Delta} z(v)\}$; $\lambda_n=n$;

$$\Phi=\min\{m: S(m)=\max_{0\leq j\leq N} S(j)\}.$$

Example 3. $\phi(z(\cdot))=\sup_{0\leq u\leq\Delta} z(u)$; $\lambda_n=c_n$; $\Phi=\max_{0\leq m\leq N} S(m)$

Example 4. $\phi(z(\cdot))=|z(\Delta)|$; $\lambda_n=c_n$; $\Phi=|S(N)|$.

Example 5. $\phi(z(\cdot))=\Delta^\gamma|z(\Delta)|^\delta$; $\lambda_n=n^\gamma\{c(n)\}^\delta$ $(\gamma>0,\ \delta>0)$; $\Phi=N^\gamma|S(N)|^\delta$.

Example 6. $\phi(z(\cdot))=\int_0^\Delta\{z(u)\}^\varepsilon du$; $\lambda_n=n c_n^\varepsilon$; $\Phi=\sum_1^N\{S(m)\}^\varepsilon$ $(\varepsilon>0)$.

The following result follows immediately from our Theorem:

Corollary 1. *Suppose that for $i=1,2$, $\phi^{(i)}$ belongs to $\mathscr{F}$ with norming sequence $\lambda_n^{(i)}$. Then for each $y>0$*

$$P\{\Phi^{(2)}>y\lambda_n^{(2)}|\Phi^{(1)}>\lambda_n^{(1)}\}\to P\{\phi^{(2)}\{W^{(A_1^{(1)})}\}>y\},\tag{3.2}$$

$$P\{\Phi^{(1)}>y\lambda_n^{(1)}|\Phi^{(2)}>\lambda_n^{(2)}\}\to P\{\phi^{(1)}\{W^{(A_1^{(2)})}\}>y\}.\tag{3.3}$$

In the case that $\alpha=2$ and $\phi^{(1)}(z(\cdot))=\Delta(z)$ is the functional of Example 1, the process $W^{(A_1^{(1)})}=W^+$ is known as scaled Brownian Meander. It has been studied by various authors (see [4, 8, 12, 13]) and many of its properties are known. For example, the distribution of $\sup_{0\leq t\leq 1} W^+(t)$ is known (see, e.g. (2.3) of [8]). From this, the R.H.S. of (3.2) can be deduced when $\phi^{(2)}(z(\cdot))=\sup_{0\leq u\leq\Delta} z(u)$ is the functional of Example 3. (See [5], where an explicit version of (3.3) in this special case is also given.) For certain other choices of $\phi^{(2)}$ (e.g. that of Example 2) less explicit formulae for R.H.S. of (3.2) can be deduced from the results of Imhof ([13], §4). In other cases, particularly for $\alpha\neq 2$, there seems little hope of computing these limit distributions. Nevertheless Corollary 1 still yields some useful information. To see this, note that if both the R.H.S. of (3.2) and the R.H.S. of (3.3) are positive when $y=1$ and $A^{(1)}=\{\Delta(z)>1\}$, then $P\{N>n\}=P\{\Phi^{(1)}>\lambda_n^{(1)}\}\sim cP\{\Phi^{(2)}>\lambda_n^{(2)}\}$. ($c$ here denotes a generic finite positive constant.) In case $\alpha=2$, $\mathrm{Var}(X_1)<\infty$, it is known that $P\{N>n\}\sim cn^{-\frac{1}{2}}$, and in all other cases that $n^\rho P\{N>n\}$ is slowly varying (s.v.) as $n\to\infty$, (see, e.g. Rogozin [18]) where $\rho=P\{W(1)<0\}=\frac{1}{2}$ if $\alpha=2$, $=\frac{1}{2}-\frac{1}{\pi\alpha}\tan^{-1}\left(\beta\tan\frac{\pi\alpha}{2}\right)$ if $0<\alpha<2$.

Corollary 2. *Let W^+ denote the limiting process in our theorem when $A=\{z:\Delta(z)>1\}$ and suppose $\phi\in\mathscr{F}$ is such that $P\{\phi(W^+)>1\}>0$ and $P\{\Delta(W^{(A_1)})>1\}>0$. Then if $\alpha=2$ and $E(X_1^2)<\infty$, $n^{\frac{1}{2}}P\{\Phi>\lambda_n\}\to c$, and in all other cases $n^\rho P\{\Phi>\lambda_n\}$ is* s.v. *at* ∞.

In all examples of interest, λ_n is regularly varying (r.v.) with positive index, μ say, so it then follows from Corollary 2 that $P\{\Phi>n\}$ is r.v. with index

$-\rho/\mu$, and we can conclude that $\Phi \in D(\rho/\mu, 2)$ if $\rho/\mu < 2$. Since c_n is r.v. with index $1/\alpha$, this is the case in Examples 2–6, and it is also easy to check that the other conditions hold for these examples, except for Example 4 and Example 5 in case $\alpha=2$, when $\phi(W^+)\equiv 0$. It therefore follows, with the above exceptions, that the functionals in these examples belong to domains of attractions with indices ρ (Example 2), $\rho\alpha$ (Examples 3, 4), $\rho\alpha/(\gamma\alpha+\delta)$ (Example 5), $\rho\alpha/(\alpha+\varepsilon)$ (Example 6).

In certain cases it is possible to improve on Corollary 2 by finding the asymptotic behaviour of the s.v. function that appears there. To see this, recall that under assumption 2.1, $N \in D(\rho, 1)$ and $Z=-S(N) \in D(\alpha\rho, 1)$ if $\alpha\rho<1$, and if $\alpha\rho=1$, Z is relatively stable. Let $a(n)$, $b(n)$ denote the norming constants for N and Z respectively, which are asymptotically unique and determined by the relation $nP\{N>a(n)\}\to 1$, $nP\{Z>b(n)\}\to 1$, when $\alpha\rho \neq 1$.

Lemma. *$c(a(n)) \sim c\cdot b(n)$ as $n\to\infty$.*

Proof. This result is implicit in the proof of Corollary 3.3 of [10]. In the case $\alpha\rho\neq 1$, it can also be deduced from $P\{Z>c(n)\}\sim cP\{N>n\}$. As an example of the way this can be applied, let $M=\max\limits_{1\leq m\leq N} S(m)$ denote the Φ of Example 3.

Corollary 3. (i) *$nP\{M>b(n)\}\to c$ as $n\to\infty$ in all cases.*

(ii) *If $\alpha=2$ and $E(X_1^2)<\infty$, $nP\{M>n\}\to c$.*

(iii) *If $\alpha\leq 1$ or $1<\alpha<2$ and $\beta\neq 1$, $\alpha\rho<1$ and $M\in D(\alpha\rho, 1)$ with norming constants $b(n)$.*

(iv) *If $1<\alpha<2$ and $\beta=1$, so that $\alpha\rho=1$, M is relatively stable. Furthermore,*

$nP\{M>n\}\to c$ iff $E(Z)<\infty$ iff $\int_1^\infty x^{-1}R(x)\,dx<\infty$, *where* $R(x)=P\{X\leq -x\}/P\{X>x\}$.

Proof. (i) This follows from $P\{M>c(a(n))\}\sim cP\{N>a(n)\}\sim cn^{-1}$ and use of the lemma.

(ii) In this case $E(Z)<\infty$ and we can take $b(n)=n$.

(iii) In this case the lemma gives $b(n)$ r.v. of index $\alpha\rho$, and the result follows.

(iv) In case $\alpha\rho=1$, (i) says that $P\{M>n\}$ is r.v. with index -1, which implies that M is relatively stable. Furthermore $b(n)\sim$ c.n iff $E(Z)<\infty$, and the second equivalence follows from Corollary 3 of [6].

Note. (ii) and (iv) have been established by Pakes [16], in the special case of left-continuous random walk. A direct proof of (ii) in the general case is available in [5], where it is also shown that $c=E(Z)$.

§4

Remark 1. In the special case $\alpha=2$, our theorem should be compared with that of Shimura [19]. Although his context is slightly more general his result is essentially the same as ours, but his assumptions are somewhat different and in one respect appear to be inadequate. Specifically, instead of $0\notin\bar{A}$ he makes the weaker assumption that $P\{E_A(W)$ has no finite limit point$\}=1$. It appears that, in the argument corresponding to our Lemma 5 (which is only sketched in [19]), the possibility that $\tau_0 = \nu_0$ has been overlooked. (This has been confirmed by Shimura, in a private communication.)

Remark 2. Again in case $\alpha = 2$, Greenwood and Perkins [9] have a result (Theorem 9) which essentially contains ours. Their method is quite different, and is applied to the situation where N is replaced by "the time of first exit from a curved boundary". It should also be mentioned that they deduce a functional limit theorem for the randomly normed process $(W_N(\cdot), T_n)$, conditioned on $\dot{W}_n \in A$ (a special case of this is Theorem 2 of Hooghiemistra [11]), and exactly the same argument works in our case.

Remark 3. The fact that a weak limit for $W^{(A)}$ exists when $A = \{z: \Delta(z) > 1\}$ has been proved by Durrett [7], again using different methods.

Remark 4. In the special case of left-continuous r.w., note that N coincides with N_B, the first hitting time of the set B, when $B = \{-1\}$. Belkin ([1, 2]) has studied $(W_n | N_B > n)$ for integer-valued, aperiodic r.w. His main results specifically exclude the case of left-continuous r.w., but in § 5 of [1] he calculates the characteristic function $\Psi_\alpha(t)$ of the limit distribution of $(W_n(1) | N_B > n)$ for left-continuous r.w. satisfying (2.1b), which by our theorem must coincide with $E(e^{itW^+(1)})$ in case $1 < \alpha < 2$, $\beta = +1$. However there is an error in his calculation, which arises because he assumes that the proof of his Theorem 3.1 goes through unchanged in this case. In fact the second term in his Eq. (3.3) does not tend to zero, and the correct result is that if $E(e^{it(W)(1)}) = \phi_\alpha(t)$ then

$$\Psi_\alpha(t) = 1 - b\,|t|^\alpha \int_0^1 x^{-\frac{1}{\alpha}} \phi_\alpha(t(1-x)^{\frac{1}{\alpha}})\,dx + \mathrm{i}k\,t\,\phi_\alpha(t) \tag{4.1}$$

for some constant k. [(4.1) can also be established in a similar way to that used by Pechinkin [17] in case $\alpha = 2$, starting from his Eq. (10).] In principle (4.1) determines the distribution of $W^+(1)$ in this case, and then the finite-dimensional distributions of W^+ are also determined, as in Belkin ([2], Eq. (3.1)).

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References

1. Belkin, B.: A limit theorem for conditioned recurrent random walk attracted to a stable law. Ann. Math. Statist. **41**, 146–163 (1970)
2. Belkin, B.: An invariance principle for conditioned recurrent random walk attracted to a stable law. Z. Wahrscheinlichkeitstheor. Verw. Geb. **21**, 45–64 (1972)
3. Bolthausen, E.: On a functional central limit theorem for random walk conditioned to stay positive. Ann. Probab. **4**, 480–485 (1976)
4. Chung, K.L.: Excursions in Brownian Motion. Ark. Math. **14**, 155–177 (1976)
5. Doney, R.A.: A note on conditioned random walk. J. Appl. Probab. **20**, 409–412 (1983)
6. Doney, R.A.: On the existence of the mean ladder height for random walk. Z. Wahrscheinlichkeitstheor. Verw. Geb. **59**, 373–382 (1982)
7. Durrett, R.T.: Conditioned limit theorems for some null recurrent Markov processes. Ann. Probab. **6**, 798–828 (1978)

8. Durrett, R.T., Iglehart, D.L.: Functionals of Brownian meander and Brownian excursion. Ann. Probab. **5**, 130–135 (1977)
9. Greenwood, P.E., Perkins, E.: Limit theorems for excursions from a moving boundary. Theory Probab. Appl. **29**, 517–528 (1984)
10. Greenwood, P.E., Teugels, J.L.: Harmonic renewal measures and bivariate domains of attraction. Z. Wahrscheinlichkeitsteor. Verw. Geb. **61**, 527–539 (1982)
11. Hooghiemistra, G.: Conditioned limit theorems for waiting-time processes of the $M|G|1$ queue. J. Appl. Probab. **20**, 675–688 (1983)
12. Iglehart, D.L.: Functional central limit theorems for random walk conditioned to stay positive. Ann. Probab. **4**, 608–619 (1974)
13. Imhof, J.P.: Density factorizations for Brownian Motion, Meander and the three-dimensional Bessel Process, and applications. [To appear in J. Appl. Probab.]
14. Lindvall, T.: Weak convergence of probability measures and random functions in the function space $D[0, \infty)$. J. Appl. Probab. **10**, 109–121 (1973)
15. Millar, P.W.: Zero-one laws and the minimum of a Markov process. Trans. Am. Math. Soc. **226**, 365–391 (1977)
16. Pakes, A.G.: On the maximum and absorption time of left-continuous random walk. J. Appl. Probab. **15**, 292–299 (1978)
17. Pechinkin, A.V.: The limit distribution for a random walk with absorption. Theory Probab. Appl. **25**, 580–584 (1980)
18. Rogozin, B.A.: The distributions of the first ladder moment and height and fluctuation of a random walk. Theory Probab. Appl. **16**, 575–595 (1972)
19. Shimura, M.: A class of conditional limit theorems related to ruin problem. Ann. Probab. **11**, 40–45 (1983)
20. Skorokhod, A.V.: The theory of random processes. National Lending Library, Boston Spa. 1971

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