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Some Limit Theorems for Partial Sums of Quadratic Forms in Stationary Gaussian Variables

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Summary. Limit theorems with a non-Gaussian limiting distribution have been obtained, under appropriate conditions for partial sums of instantaneous nonlinear functions of stationary Gaussian sequences with long range dependence by a number of people. The normalization has typically been n^α, with $\frac{1}{2}<\alpha<1$ where n is the sample size. Here examples of limit theorems are given for quadratic functions with long range memory (not instantaneous) with a normalization n^α, $0<\alpha<\frac{1}{2}$.

Introduction. Let $\{X_j; j=\dots,-1,0,1,\dots\}$ be a strictly stationary sequence with mean $E(X_j)\equiv 0$ and variance $0<\sigma^2(X_j)<\infty$. A great deal of research has been devoted to determining the domain of the central limit theorem (asymptotic normality for partial sums) for such processes using measures of asymptotic independence like, for example, strong mixing (see [2]). However, one can even have asymptotic normality under special circumstances with long range dependence [4]. A clear picture of the limits of the domain of the central limit theorem is not yet available. However, a number of results have been obtained on limit theorems outside of this domain with, of course, nonnormal limiting distributions. The following class of processes has drawn special attention. Let $\{Y_j, j=\dots,-1,0,1,\dots\}$ be a stationary Gaussian sequence with $E(Y_j)\equiv 0$, $\sigma^2(Y_j)\equiv 1$. Consider a function G with $E[G^2(Y)]<\infty$ where Y is $N(0,1)$. Let

$$X_j=G(Y_j), \qquad EX_j\equiv 0, \tag{1}$$

be the process $\{X_j\}$ generated by the function G. Limiting distributions are then considered for

$$\sum_{j=1}^{n} G(Y_j) \tag{2}$$

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appropriately normalized or limiting (weak) processes for

$$\sum_{j=1}^{[nt]} G(Y_j) \tag{3}$$

appropriately normalized (see [1, 3, 5, 6–8]). Here $[x]$ denotes the greatest integer less than or equal to x. In all the cases in which non-Gaussian limiting distributions have been derived, the normalization has been of the form $n^\alpha L(n)$, $\frac{1}{2}<\alpha<1$ with $L(\cdot)$ a slowly varying function. Here, we restrict ourselves to quadratic functions of a stationary Gaussian sequence. By allowing noninstantaneous functions, we obtain limit laws with a normalization of form n^α with $0<\alpha<\frac{1}{2}$.

It should be noted that the limiting (weakly) processes $Z(t)$ obtained have the following self-similarity property if the normalization is n^α. The processes $Z(t)$ and $c^{-\alpha}Z(ct)$ have the same distribution for each $c>0$.

Limit Theorems. The first result is a proposition on the limiting behavior in distribution of covariance estimates when the spectral density has appropriate singular behavior at zero.

Proposition. *Let $\{Y_k\}$ be a stationary Gaussian sequence with $EY_k\equiv 0$ and covariances*

$$r_k = EY_0 Y_k \cong k^{-2\gamma} \tag{4}$$

as $|k|\to\infty$ with $0<\gamma<\frac{1}{4}$. Then the differences of the random quantities

$$n^{-1+2\gamma}\sum_{j=1}^{n-\alpha}(Y_j Y_{j+\alpha}-r_\alpha), \qquad \alpha=0,1,\ldots,s, \tag{5}$$

tend to zero in probability as $n\to\infty$ and the common limiting distribution has characteristic function

$$\exp\left\{\frac{1}{2}\sum_{k=2}^{\infty}(2it)^k c_k/k\right\} \tag{6}$$

with the constants

$$c_k=\int_0^1\!\!\ldots\!\int |x_1-x_2|^{-2\gamma}|x_2-x_3|^{-2\gamma}\ldots|x_k-x_1|^{-2\gamma}\,dx_1\ldots dx_k. \tag{7}$$

This implies that partial sums of a quadratic polynomial of finite range

$$\sum_{|\alpha|\leq s}(Y_j Y_{j+\alpha}-r_\alpha)\,a_\alpha$$

and its shifts $j=1,\ldots,n$ have the same limiting distribution when normalized by an appropriate scalar multiple of $n^{1-2\alpha}$ as $n\to\infty$.

The argument is basically that given in [3]. Suppose we consider the joint characteristic function of

$$n^{-1+2\gamma}\sum_{j=1}^{n-\alpha} Y_j Y_{j+\alpha}, \qquad \alpha=0,1,\dots,s.$$

The joint characteristic function can be written as

$$|I-2iRA|^{-\frac{1}{2}}$$

where R is the n by n covariance matrix of the process $\{Y_j\}$ and

$$A=n^{-1+2\gamma}\sum_{k=0}^{s} t_k J^k$$

where

$$J=\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ 0 & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Except for a constant, a typical k^{th} cross-cumulant, $k>1$, looks like

$$n^{-k+2\gamma k}\sum_{\substack{j_u=1\\ u=1,\dots,k}}^{n} r_{j_k+\alpha_k-j_1} r_{j_1+\alpha_1-j_2}\dots r_{j_{k-1}+\alpha_{k-1}-j_k}$$

where $\alpha_1,\dots,\alpha_k$ take on values $0,1,\dots,s$. As $n\to\infty$ all these k^{th} order terms have the same limit

$$c_k=\int_0^1\dots\int |x_1-x_2|^{-2\gamma}|x_2-x_3|^{-2\gamma}\dots|x_k-x_1|^{-2\gamma}\,dx_1\dots dx_k$$

and the proposition follows.

After a preliminary remark we shall give a class of stationary processes which are quadratic forms in Gaussian variables and such that normalization of partial sums by n^α, $0<\alpha<\frac{1}{2}$, yields a nontrivial non-Gaussian limit as $n\to\infty$. First note that

$$\sum_{n=1}^{\infty} n^{-\eta}\sin nx \simeq x^{\eta-1}\,\Gamma(1-\eta)\cos\tfrac{1}{2}\pi\eta \tag{8}$$

as $x\to 0+$ for $0<\eta<2$ (see p. 186, volume 1, Zygmund [9]). Let $\{Y_k\}$ be a stationary Gaussian sequence, $EY_k\equiv 0$, satisfying (4) with $r_0=1$. Set

$$X_k=Y_k^2-1$$

and let the process

$$U_m = \sum_k a_k X_{m-k} \tag{9}$$

where

$$a_k = \begin{cases} 0 & \text{if } k=0 \\ k^{-\beta-1} & \text{if } k>0 \\ -|k|^{-\beta-1} & \text{if } k<0, \quad \beta>0. \end{cases} \tag{10}$$

Now

$$\sum_{m=1}^{n} U_m = \sum_{m=1}^{n} \sum_k a_k X_{m-k}$$

$$= \sum_s \alpha_s(n) X_s \tag{11}$$

where

$$\alpha_s(n) = \sum_{k=1-s}^{n-s} a_k. \tag{12}$$

Notice that with $s=u$ a continuous variable (i) if

$$u<0, \ \alpha_s(n) \simeq \frac{1}{\beta}[(1-u)^{-\beta} - (n-u)^{-\beta}]$$

if u is large in absolute value, (ii) if

$$\frac{n}{2}>u>0, \ \alpha_s(n) \simeq \frac{1}{\beta}[(u-1)^{-\beta} - (n-u)^{-\beta}]$$

when u, $\frac{n}{2}-u$ are large, (iii) if

$$n>u>\frac{n}{2}, \ \alpha_s(n) \simeq \frac{1}{\beta}[-(u-1)^{-\beta} + (n-u)^{-\beta}]$$

when $n-u$ is large, (iv) if

$$u>n, \ \alpha_s(n) \simeq \frac{1}{\beta}[-(u-n)^{-\beta} + (u-1)^{-\beta}]$$

when $u-n$ is large. Our object is to look at the asymptotic distribution of (11) appropriately normalized. The variance of (11) is

$$2\sum \alpha_j(n)\,\alpha_k(n)\,r_{j-k}^2 \simeq 2n^{2-2\beta-4\gamma} \iint \alpha(x)\,\alpha(y)|x-y|^{-4\gamma}\,dx\,dy \tag{13}$$

as $n \to \infty$ if $2-2\beta-4\gamma>0$. Here

$$\alpha(x)=\begin{cases}\frac{1}{\beta}[|x|^{-\beta}-|1-x|^{-\beta}] & \text{if } x<0\\ \frac{1}{\beta}[x^{-\beta}-|1-x|^{-\beta}] & \text{if } 0<x<\frac{1}{2}\\ \frac{1}{\beta}[-x^{-\beta}+|1-x|^{-\beta}] & \text{if } \frac{1}{2}<x<1\\ \frac{1}{\beta}[-(x-1)^{-\beta}+x^{-\beta}] & \text{if } x>1.\end{cases} \tag{14}$$

The spectral density of $\{X_s\}$ in the neighborhood of $\lambda=0$ looks like $|\lambda|^{4\gamma-1}$ and we require $0<4\gamma<1$. On the other hand $2\beta>1-4\gamma$ or $2\beta+4\gamma>1$ is required so that the spectral density of $\{U_s\}$ near $\lambda=0$ looks like $|\lambda|^{2\beta+4\gamma-1}$. However $2>2\beta+4\gamma$ is also required so that (13) will diverge as $n\to\infty$. The normalization of (11) will be of the form n^α with $\alpha=1-\beta-2\gamma$ and $\frac{1}{2}>\alpha>0$. Notice that under the assumptions we have made the integral on the right of (13) is finite. The characteristic function of

$$n^{-\alpha}\sum_{m=1}^{n} U_m$$

is

$$|I-2itn^{-\alpha}RA|^{-\frac{1}{2}}\exp\{-itn^{-\alpha}\sum_s \alpha_s(n)\} \tag{15}$$

with R the covariance matrix of the Gaussian process $\{Y_k\}$ and A the diagonal matrix with the entries a_k. The characteristic function of the limiting distribution is (6) with

$$c_k=\int\dots\int\alpha(x_1)|x_1-x_2|^{-2\gamma}\alpha(x_2)|x_2-x_3|^{-2\gamma}\dots\alpha(x_k)|x_k-x_1|^{-2\gamma}\,dx_1\dots dx_k. \tag{16}$$

Theorem. *Consider the process $\{U_m\}$ (9) quadratic in Gaussian stationary variables $\{Y_m\}$ whose covariances satisfy (4). Then*

$$n^{-\alpha}\sum_{m=1}^{n} U_m, \tag{17}$$

$0<\alpha=1-\beta-2\gamma<\frac{1}{2}$, *has a limiting distribution as $n\to\infty$ with characteristic function* (6) *and the c_k's* (16).

The expression given by the characteristic function (6) and constants (16) is well defined since it is analytic in a neighborhood of zero. This follows from certain bounds on the cumulants which we will now obtain. If k is even, using the Schwarz inequality, we find

$$\begin{aligned}|c_k| \leqq & [\int \dots \int |\alpha(x_1)| |x_1 - x_2|^{-4\gamma} |\alpha(x_2)| |\alpha(x_3)| \\ & |x_3 - x_4|^{-4\gamma} |\alpha(x_4)| \dots |\alpha(x_{k-1})| \\ & |x_{k-1} - x_k|^{-4\gamma} |\alpha(x_k)| \, dx_1 \dots dx_k]^{\frac{1}{2}} \\ & [\int \dots \int |\alpha(x_k)| |x_k - x_1|^{-4\gamma} |\alpha(x_1)| |\alpha(x_2)| |x_2 - x_3|^{-4\gamma} |\alpha(x_3)| \\ & |\alpha(x_4)| |x_4 - x_5|^{-4\gamma} |\alpha(x_5)| \dots \\ & |\alpha(x_{k-2})| |x_{k-2} - x_{k-1}|^{-4\gamma} |\alpha(x_{k-1})| \, dx_1 \dots dx_k]^{\frac{1}{2}} \\ = & (\int |\alpha(x_1)| |x_1 - x_2|^{-4\gamma} |\alpha(x_2)| \, dx_1 \, dx_2)^{k/2} \end{aligned} \tag{18}$$

and the integral on the extreme right of (18) is finite. Consider now the case of k odd. Since

$$\frac{1}{(x_1 - x_k)(x_{k-1} - x_k)} = \frac{1}{x_{k-1} - x_1} \left\{ \frac{1}{x_1 - x_k} - \frac{1}{x_{k-1} - x_k} \right\}$$

if $x_1 \neq x_{k-1}$ it follows that

$$\begin{aligned}& \int |x_1 - x_k|^{-2\alpha} |x_{k-1} - x_k|^{-2\alpha} |\alpha(x_k)| \, dx_k \\ & \quad \leqq |x_1 - x_{k-1}|^{-2\alpha} 2^{2\alpha} \{ \int |x_1 - x_k|^{-2\alpha} |\alpha(x_k)| \, dx_k \\ & \qquad + \int |x_{k-1} - x_k|^{-2\alpha} |\alpha(x_k)| \, dx_k \}. \end{aligned} \tag{19}$$

Further

$$\int |x_1 - x_k|^{-2\alpha} |\alpha(x_k)| \, dx_k \leqq C(1 + |x_1|)^{-\beta} \tag{20}$$

for some constant C. Inequalities (19) and (20) together with an argument like that leading to (18) for k even imply that

$$c_k \leqq C' (\int |\alpha(x_1)| |x_1 - x_2|^{-4\gamma} |\alpha(x_2)| \, dx_1 \, dx_2)^{(k-1)/2} \tag{21}$$

for k odd with C' an appropriate constant. The bounds (18) and (21) imply that the characteristic function is analytic in a neighborhood of the origin.

Let

$$S_n(t) = n^{-\alpha} \sum_{m=1}^{[nt]} U_m, \quad 0 < \alpha = 1 - \beta - 2\gamma < \tfrac{1}{2}. \tag{22}$$

The following corollary describes the limiting distribution of $S_n(t_1), \dots, S_n(t_k)$ $0 \leqq t_1, \dots, t_k$, as $n \to \infty$ and the proof is basically that of the Theorem given above.

Corollary 1. *The asymptotic distribution of* $S_n(t_1), \dots, S_n(t_k)$ (see (22)) *as* $n \to \infty$ *has joint characteristic function*

$$\phi(z_1, \dots, z_k) = \exp \left\{ \frac{1}{2} \sum_{j=2}^{\infty} \frac{(2i)^j}{j} \sum_{\substack{m_1, \dots, m_k \geqq 0 \\ m_1 + \dots m_k = j}} \frac{k!}{m_1! \dots m_k!} z_1^{m_1} \dots z_k^{m_k} c(\tau^{(k)}) \right\} \tag{23}$$

where $\tau^{(k)}=(\tau_1,\tau_2,\ldots,\tau_k)$ *with the first* m_1 τ_i*'s equal to* t_1, *the next* m_2 τ_i*'s equal to* $t_2,\ldots,$ *and the last* m_k τ_i*'s equal to* t_k, *and*

$$c(\tau^{(k)})=\int\ldots\int\alpha(x_1,\tau_1)|x_1-x_2|^{-2\gamma}\alpha(x_2,\tau_2)\\ |x_2-x_3|^{-2\gamma}\ldots\alpha(x_k,\tau_k)|x_k-x_1|^{-2\gamma}dx_1\ldots dx_k \tag{24}$$

with

$$\alpha(x,\tau)=\begin{cases}\tau^{-\beta}\alpha(x/\tau) & \text{if } \tau\neq 0\\ 0 & \text{if } \tau=0.\end{cases} \tag{25}$$

The joint characteristic function $\phi(z_1,\ldots,z_k)$ *is analytic in the variables* z_i *for* $|z_i|$ *sufficiently small,* $i=1,\ldots,k$.

By applying Theorem 2.1 of Taqqu [6] we obtain the following Corollary on weak convergence of $S_n(t)$.

Corollary 2. *The sequence* $S_n(t)$, $0\leqq t\leqq 1$, *converges weakly as* $n\to\infty$ *to a process* $S(t)$, $0\leqq t\leqq 1$, *with continuous sample functions. The joint distribution of* $S(t_1),\ldots,S(t_k)$ *has characteristic function* (23).

The discussion in [8] suggests that aside from a constant $S(t)$ has the form

$$S(t)=\int\alpha(s,t)\int_{-\infty}^{s}(s-\xi_1)^{-\gamma-\frac{1}{2}}dB(\xi_1)\int_{-\infty}^{\xi_1}(s-\xi_2)^{-\gamma-\frac{1}{2}}dB(\xi_2)$$

where $B(.)$ is a Brownian motion.

Comment. The proposition and theorem can be generalized, without any essential change in the proof, to the case in which

$$r_k\cong k^{-2\gamma}L(k)$$

with $L(.)$ a slowly varying function and the normalization $n^{1-2\gamma}L(n)$. A result related to the theorem can be obtained by using Theorem 3 of the as yet still unpublished paper: Dobrushin, R.L., Major, P., Noncentral limit theorems for non-linear functionals of Gaussian fields. I thank a referee for his remarks.

References

1. Dobrushin, R.L.: Gaussian and their subordinated self-similar random generalized fields. Ann. Probability **7**, 1–28 (1979)
2. Ibragimov, I.A., Linnik, Yu.V.: Independent and Stationary Sequences of Random Variables. Gröningen: Walters-Noordhoff 1971
3. Rosenblatt, M.: Independence and dependence. Proc. 4th Berkeley Sympos. Math. Statist. Probab., 431–443. Univ. Calif. (1961)
4. Rosenblatt, M.: Fractional integrals of stationary processes and the central limit theorem. J. Appl. Probability **13**, 723–732 (1976)
5. Sun, T.C.: Some further results on central limit theorems for non-linear functions of a normal stationary process. J. Math. Mech., **14**, 71–85 (1965)

6. Taqqu, M.S.: Weak convergence to fractional Brownian motion and to the Rosenblatt process. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **31**, 287–302 (1975)
7. Taqqu, M.S.: Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **40**, 203–238 (1977)
8. Taqqu, M.S.: A representation for self-similar processes. manuscript
9. Zygmund, A.: Trigonometric Series. Cambridge: Cambridge University Press 1968

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