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# A Characterization of the Generalized Inverse Gaussian Distribution by Continued Fractions

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## 1. Introduction

We consider the two following probability distributions on $(0, +\infty)$.

$$\mu_{\lambda,a,b}(dx)=\frac{a^{\lambda/2}\,b^{-\lambda/2}}{2K_\lambda(\sqrt{ab})}\,x^{\lambda-1}\exp -\tfrac{1}{2}(ax+bx^{-1})\,\mathbb{1}_{(0,\infty)}(x)\,dx$$

where $\lambda$ is real and $a, b>0$ and

$$\gamma_{\lambda,a}(dx)=\frac{a^{-\lambda}}{\Gamma(\lambda)}\,x^{\lambda-1}\exp(-a^{-1}x)\,\mathbb{1}_{(0,\infty)}(x)\,dx,$$

where $\lambda$, $a>0$.

The generalized inverse Gaussian distribution introduced by Barndorff-Nielsen and Halgreen (1977) is $\mu_{\lambda,a,b}$. Since

$$\int_0^\infty \mu_{\lambda,a,b}(dx)=\int_0^\infty \mu_{-\lambda,a,b}(dx)=1, \tag{1}$$

denoting the distribution of the random variable $X$ by $L(X)$, and changing $x\mapsto\frac{1}{x}$ in the first integral yields

$$L(X)=\mu_{\lambda,a,b} \quad \text{if and only if} \quad L\left(\frac{1}{X}\right)=\mu_{-\lambda,b,a}, \tag{2}$$

$$K_\lambda(c)=K_{-\lambda}(c) \quad \text{for} \quad c>0. \tag{3}$$

Indeed (3) is a consequence of the known properties of the Bessel function $K_\lambda(c)$ (see G.N. Watson (1966), p. 78). Hence the Laplace transform of $\mu_{\lambda,a,b}$ defined for $s>0$ is

$$\hat{\mu}_{\lambda,a,b}(s)=\int_0^\infty \exp(-sx)\,\mu_{\lambda,a,b}(dx)=\left(1+\frac{2s}{a}\right)^{-\lambda/2}\frac{K_\lambda(\sqrt{(a+2s)\,b})}{K_\lambda(\sqrt{ab})}. \tag{4}$$

From (3) we get

$$\hat{\mu}_{-\lambda,a,b}(s)=\left(1+\frac{2s}{a}\right)^{\lambda/2}\frac{K_\lambda(\sqrt{(a+2s)b})}{K_\lambda(\sqrt{ab})}. \tag{5}$$

Comparing (4) and (5) we obtain our basic relation

$$\mu_{\lambda,a,b}=\mu_{-\lambda,a,b}*\gamma_{\lambda,\,2/a} \quad \text{for } \lambda, a, b>0 \tag{6}$$

where $*$ denotes convolution.

If we now consider two independent random variables $X$ and $Y$ such that

$$L(X)=\mu_{-\lambda,a,a} \quad \text{and} \quad L(Y)=\gamma_{\lambda,\,2/a} \quad \text{with } \lambda, a>0,$$

we obtain from (6)

$$L(Y+X)=\mu_{\lambda,a,a}$$

and from (2) we have

$$L(X)=L\left(\frac{1}{Y+X}\right). \tag{7}$$

Likewise if we consider three independent random variables $X$, $Y_1$ and $Y_2$ such that

$$L(X)=\mu_{-\lambda,a,b}, \quad L(Y_1)=\gamma_{\lambda,\,2/b} \quad \text{and} \quad L(Y_2)=\gamma_{\lambda,\,2/a},$$

with $\lambda, a, b>0$, we obtain from (6) and (2)

$$L(Y_2+X)=\mu_{\lambda,a,b}, \quad L\left(\frac{1}{Y_2+X}\right)=\mu_{-\lambda,b,a},$$

$L\left(Y_1+\frac{1}{Y_2+X}\right)=\mu_{\lambda,b,a}$ and eventually

$$L(X)=L\left(\frac{1}{Y_1+\frac{1}{Y_2+X}}\right). \tag{8}$$

## 2. Characterizations

Formulae (7) and (8) raise the question, whether these properties are characteristic of $\mu_{-\lambda,a,a}$ and of $\mu_{-\lambda,a,b}$. This question is answered in the affirmative by the following.

**Theorem 1.** (i) *Let $X$ and $Y$ be two independent random variables such that $X>0$ and $L(Y)=\gamma_{\lambda,\,2/a}$* for $\lambda$, $a>0$. *Then $L(X)=L\left(\frac{1}{Y+X}\right)$ if and only if $L(X)=\mu_{-\lambda,a,a}$.*

(ii) *Let $X$, $Y_1$ and $Y_2$ be three independent random variables such that $X>0$, $L(Y_1)=\gamma_{\lambda,\,2/b}$, $L(Y_2)=\gamma_{\lambda,\,2/a}$ for $\lambda, a, b>0$. Then $L(X)=L\left(\frac{1}{Y_1+\frac{1}{Y_2+X}}\right)$ if and only if $L(X)=\mu_{-\lambda,a,b}$.*

The proof, as can easily be guessed from (8), is based on elementary properties of continued fractions. We shall therefore adopt the following notations.

If $(y_n)_{n=1}^{\infty}$ is a sequence of positive numbers, we define inductively the sequence $([y_1, y_2, \dots, y_n])_{n=1}^{\infty}$ by $[y_1]=y_1$ and

$$[y_1, y_2, \dots, y_n]=y_1+\frac{1}{[y_2, \dots, y_n]} \quad \text{for } n\geqq 2.$$

The following facts are well known (see for instance (Olds (1963), 3.7)). If $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ are defined by

$$\begin{aligned} p_1&=y_1, & p_2&=y_1 y_2+1, & \text{and} \quad p_n&=y_n p_{n-1}+p_{n-2} & \text{for } n>2\\ q_1&=1, & q_2&=y_2, & \text{and} \quad q_n&=y_n q_{n-1}+q_{n-2} & \text{for } n>2 \end{aligned} \tag{9}$$

then

$$[y_1, y_2, \dots, y_n]=\frac{p_n}{q_n}, \tag{10}$$

$$[y_1, \dots, y_n]-[y_1, \dots, y_{n+1}]=\frac{(-1)^n}{q_n q_{n+1}}, \tag{11}$$

$$[y_1, \dots, y_n, k]=\frac{k p_n+p_{n-1}}{k q_n+q_{n-1}}. \tag{12}$$

The proof of Theorem 1 is based on the following.

**Theorem 2.** *If $d$ is an integer $>0$ and if $X_0, Y_1, Y_2, Y_3, \dots$ is a sequence of strictly positive independent random variables such that*

$$L(Y_{md+r})=L(Y_r), \quad \forall r=1, 2, \dots, d \quad \textit{and} \quad \forall m=0, 1, 2, \dots.$$

(i) *$Z=\lim_{n\to\infty}[Y_1, Y_2, \dots, Y_n]$ exists almost surely.*

(ii) *The Markov chain $(X_m)_{m=0}^{\infty}$ defined by*

$$\frac{1}{X_{m+1}}=\left[Y_{md+1}, \dots, Y_{md+2}, \dots, Y_{(m+1)d}, \frac{1}{X_m}\right]$$

*for $m\geqq 0$, is such that $L(X_m)$ converges to $L\left(\frac{1}{Z}\right)$ for any $L(X_0)$; and*

(iii) $L(X_0)=L\left(\dfrac{1}{\left[Y_1, Y_2, \dots, Y_d, \dfrac{1}{X_0}\right]}\right)$ *if and only if* $L(X_0)=L\left(\frac{1}{Z}\right)$.

*Proof of Theorem 2.* (i) Consider $p_n$ and $q_n$ defined by (9) with $Y_n$ replacing $y_n$. (11) implies that

$$\left(\frac{p_{2n+1}}{q_{2n+1}}\right)_{n=0}^{\infty} \quad \text{and} \quad \left(\frac{p_{2n}}{q_{2n}}\right)_{n=0}^{\infty}$$

are two adjacent sequences, and we have only to show that $q_n q_{n+1} \to \infty$ as $n \to \infty$ almost surely.

Now

$$q_{2n+1} = 1 + \sum_{k=1}^{n} (q_{2k+1} - q_{2k-1}) = 1 + \sum_{k=1}^{n} Y_{2k+1} q_{2k}$$
$$\geqq \sum_{k=1}^{n} Y_{2k} Y_2 \to \infty \quad \text{as } n \to \infty \text{ almost surely.}$$

Since $q_{2n} \geqq Y_2$, clearly $q_n q_{n+1} \to \infty$ as $n \to \infty$ almost surely.

(ii) We now use the fact that

$$\frac{1}{X_m} = \left[ Y_{(m-1)d+1}, Y_{(m-1)d+2}, \dots, Y_{md}, Y_{(m-2)d+1}, \dots, Y_{(m-1)d}, Y_1, \dots, Y_d, \frac{1}{X_0} \right]$$

has the same distribution as $\left[ Y_1, Y_2, \dots, Y_{md}, \frac{1}{X_0} \right]$.

From (10) and (12) we see that $\left[ Y_1, \dots, Y_{md}, \frac{1}{X_0} \right]$ belongs to the interval with end points

$$\frac{p_{md}}{q_{md}} \quad \text{and} \quad \frac{p_{md+1}}{q_{md+1}}.$$

$\left( \text{Since } a, b, c, d > 0 \text{ and } \frac{a}{b} < \frac{c}{d} \text{ imply } \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \right)$.

Hence, from, (i) $Z = \lim_m \left[ Y_1, \dots, Y_{md}, \frac{1}{X_0} \right]$ and $L(X_m) \xrightarrow[m+\infty]{} L\left(\frac{1}{Z}\right)$ which is the unique stationary distribution of the Markov chain $(X_m)_{m=0}^{\infty}$.

(iii) The "if" part is obvious from (ii).

Conversely, if $L(X_0) = L\left( \frac{1}{\left[ Y_1, \dots, Y_d, \frac{1}{X_0} \right]} \right)$, necessarily $L(X_0) = L(X_1)$ $= L(X_m)$ for all $m$. From (ii), $L(X_0) = L\left(\frac{1}{Z}\right)$.

*Proof of Theorem 1.* The "if" parts of (i) and (ii) are respectively (7) and (8). Conversely, we apply Theorem 2(iii) to the case $d=1$ and $L(Y_1) = \gamma_{\lambda, 2/a}$ to obtain (i), and to the case $d=2$, $L(Y_1) = \gamma_{\lambda, 2/b}$ and $L(Y_2) = \gamma_{\lambda, 2/a}$ to obtain (ii). Indeed, Theorem 2(iii) shows that the equation in $L(X_0)$

$$L(X_0) = L\left( \frac{1}{\left[ Y_1, \dots, Y_d, \frac{1}{X_0} \right]} \right)$$

has a unique solution, and (7) and (8) give such a solution for $d=1$ and 2.

*Remarks.* Applying Theorem 2 to any $d$, and $L(Y_r) = \gamma_{\lambda, 2/a_r}$ where $\lambda$, $a_1, \dots, a_d > 0$, we can consider:

$$Z = Y_1 + \cfrac{1}{Y_2 + \cfrac{1}{Y_3 + \cfrac{1}{Y_4 + \dots}}}$$

which makes sense because of Theorem 2(i). If we denote by

$$L\left(\frac{1}{Z}\right) = \mu_{-\lambda, a_d, a_{d-1}, \dots, a_1}, \tag{13}$$

$$L(Z) = \mu_{\lambda, a_1, a_d, \dots, a_2}, \tag{14}$$

it is then easy to see that

$$\mu_{\lambda, a_d, \dots, a_1} = \mu_{-\lambda, a_d, \dots, a_1} * \gamma_{\lambda, 2/a_d}. \tag{15}$$

Theorem 2 implies that for fixed $d$, the family $\mu_{\lambda, a_1, \dots, a_d}$ of probabilities on $(0, +\infty)$ satisfying (13), (14) and (15) is unique. But it is indeed a challenging problem to determine the distribution explicitly. The cases $d=1$ and $d=2$ described in Theorem 1 turn out to be lucky accidents.

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