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Moment Bounds for Stationary Mixing Sequences

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Summary. For an $r>2$ and a finite $K, E\left|\sum_{j=1}^{n} X_j\right|^r \leqq K n^{r/2}$ (all $n \geqq 1$) is obtained for a strictly stationary strong mixing sequence $\{X_j\}$. The convergence of rth$(r>2)$ absolute moments in the central limit theorem for stationary ϕ-mixing and strong mixing sequences is also studied.

1. Introduction

For an $r>2$ and a finite K,

$$E\left|\sum_{j=a+1}^{a+n} X_j\right|^r \leqq K n^{r/2} \quad \text{(all } a \geqq 0,\ n \geqq 1) \tag{1.1}$$

has been studied for various classes of random variables $\{X_j, j \geqq 1\}$. And it has been obtained that if either $\{X_j\}$ is

(i) a sequence of mutually independent random variables;

(ii) a stationary Markov sequence satisfying Doeblin's condition;

(iii) a strictly stationary ϕ-mixing sequence; or

(iv) a martingale difference sequence,

then (1.1) holds. Detailed discussion may be found in Brillinger [4], von Bahr [1], Doob [8] p. 225, Ibragimov [10] and Stout [15] p. 213. This type of bound has proved to be of considerable use in obtaining several types of limit laws, notably central limit theorems and strong laws: see e.g., Lemma 7.4, p. 225 of Doob [8] and Theorem 3.7.7, p. 211 of Stout [15].

The main purpose of this paper is to show that (1.1) holds for a strictly stationary strong mixing sequence. This result is stated in Theorems 1 and 2 of Sect. 3.

Ibragimov's [10] proof for ϕ-mixing is based on Doob's argument (see [8], pp. 225–227) which is difficult to extend straightforwards to the strong mixing case. This difficulty occurs from the difference between the basic inequalities

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(2.3) and (2.4) below. We shall show how Doob's argument can be adapted to our special case.

In Sect. 5, using Ibragimov's [10] Lemma 1.9 and our Theorems 1 and 2, we try to find sufficient conditions for the convergence of rth$(r>2)$ absolute moments in the central limit theorem for strictly stationary ϕ-mixing and strong mixing sequences. For sums of independent random variables Bernstein [2] (an alternative proof was given by Brown [5, 6]), and for martingales Hall [9] presented necessary and sufficient conditions for such convergence of moments.

2. Mixing Conditions

Let $\{X_j, j \geqq 1\}$ be a strictly stationary ϕ-mixing or strong mixing sequence. Thus, the condition (ϕ-mixing)

$$\sup_{A \in \mathcal{M}_1^k,\, B \in \mathcal{M}_{k+n}^\infty} \frac{1}{P(A)} |P(A \cap B) - P(A) P(B)| \leqq \phi(n) \downarrow 0 \qquad (n \to \infty) \tag{2.1}$$

or (strong mixing)

$$\sup_{A \in \mathcal{M}_1^k,\, B \in \mathcal{M}_{k+n}^\infty} |P(A \cap B) - P(A) P(B)| \leqq \alpha(n) \downarrow 0 \qquad (n \to \infty) \tag{2.2}$$

holds, where $\mathcal{M}_a^b$ denotes the σ-field generated by $X_j (a \leqq j \leqq b)$. Clearly ϕ-mixing sequence is strong mixing.

The following two basic inequalities (2.3) and (2.4) are used repeatedly; for their proofs we refer to Ibragimov [10] and Davydov [7]. Let ξ and η be measurable with respect to $\mathcal{M}_1^k$ and $\mathcal{M}_{k+n}^\infty$ respectively, then if (2.1) holds,

$$|E(\xi \eta) - E(\xi) E(\eta)| \leqq 2 \|\xi\|_p \|\eta\|_q [\phi(n)]^{1/p} \tag{2.3}$$

for all $1 \leqq p, q \leqq \infty$ with $p^{-1} + q^{-1} = 1$, and if (2.2) holds,

$$|E(\xi \eta) - E(\xi) E(\eta)| \leqq 12 \|\xi\|_p \|\eta\|_q [\alpha(n)]^{1/s} \tag{2.4}$$

for all $1 \leqq p, q, s \leqq \infty$ with $p^{-1} + q^{-1} + s^{-1} = 1$.

Assume that $EX_1 = 0$ and $EX_1^2 < \infty$. Set $S_n = X_1 + \ldots + X_n$, $\sigma_n^2 = ES_n^2$ and $\sigma^2 = EX_1^2 + 2 \sum_{j=2}^\infty EX_1 X_j$, and assume throughout $\sigma \neq 0$. Where no confusion is possible K, K_α, etc., denote generic constants.

3. Moment Bounds for Strong Mixing Sequences

Theorem 1. *Let $\{X_j\}$ be a strictly stationary strong mixing sequence with $EX_1 = 0$ and $E|X_1|^{r+\delta} < \infty$ for some $r > 2$ and $\delta > 0$. If*

$$\sum_{i=0}^\infty (i+1)^{r/2-1} [\alpha(i)]^{\delta/(r+\delta)} < \infty, \tag{3.1}$$

then there exists a constant K such that

$$E|S_n|^r \leqq K n^{r/2}, \quad n \geqq 1. \tag{3.2}$$

Theorem 2. *Let $\{X_j\}$ be a strictly stationary strong mixing sequence with $EX_1=0$ and $|X_1| \leqq C < \infty$ a.s. If*

$$\sum_{i=0}^{\infty} (i+1)^{r/2-1} \alpha(i) < \infty, \tag{3.3}$$

then (3.2) *holds.*

Remark. If $\{X_j\}$ is a strictly stationary ϕ-mixing sequence, (3.2) holds under less restrictive assumptions $EX_1=0$, $E|X_1|^r<\infty$ and $\sigma_n^2=\sigma^2 n(1+o(1))$ (see Lemma 1.9 of [10]).

The following corollaries are due to Serfling ([13], Theorem B and [14], Theorem 3.1). (See also [15], Theorems 3.7.5–3.7.7.)

Corollary 1. *Suppose that the assumptions of Theorem* 1 *or* 2 *hold. Then there exists a constant K such that*

$$E(\max_{1\leqq k\leqq n} |S_k|^r) \leqq K n^{r/2}, \quad n \geqq 1.$$

Corollary 2. *Suppose that the assumptions of Theorem* 1 *or* 2 *hold. Then, as $n\to\infty$*

$$S_n/[n^{1/2}(\log n)^{1/r}(\log\log n)^{2/r}] \to 0 \quad \text{a.s.}$$

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. This theorem will be proved in three cases;

(i) $r=2m$, $m=2,3,\ldots$

(ii) $r=2m+\varepsilon$, $m=1,2,\ldots,0<\varepsilon\leqq 1$.

(iii) $r=2m+\varepsilon$, $m=1,2,\ldots,1<\varepsilon<2$.

Proof of (i). Here we shall prove specifically that for $m \geqq 1$,

$$ES_n^{2m} \leqq K_\alpha \|X_1\|_{2m+\delta}^{2m} n^m, \quad n \geqq 1, \tag{4.1}$$

where K_α depends only on α and m. The proof is based on Lemma 3.1 of Sen [12]. Let us write

$$A_r(\alpha) = \sum_{i=0}^{\infty} (i+1)^{r/2-1} [\alpha(i)]^{\delta/(r+\delta)}.$$

Then, $A_r(\alpha)<\infty$ implies $A_q(\alpha)<\infty$ for $q\leqq r$. We denote by $\sum_{n,j}$ the summation over all $1\leqq i_1 \leqq \ldots \leqq i_j \leqq n$, and let $\sum_{n,j}^{(h)}$, $1\leqq h\leqq j$, be the components of $\sum_{n,j}$ for which $r_h=\max\{r_1,\ldots,r_j\}$, where $r_h=i_h-i_{h-1}$ and $i_0=1$. Then we have

$$\begin{aligned} ES_n^{2m} &\leqq [(2m)!]\, n \sum_{n,2m-1} |E(X_1 X_{i_1} \ldots X_{i_{2m-1}})| \\ &\leqq [(2m)!] \sum_{h=1}^{2m-1} \{n \sum_{n,2m-1}^{(h)} |E(X_1 X_{i_1} \ldots X_{i_{2m-1}})|\}. \end{aligned}$$

Using (2.4), we obtain that if $A_2(\alpha)<\infty$ then

$$n\sum_{n,1}|E(X_1X_{i_1})|\leqq 12n\|X\|_{2+\delta}^2A_2(\alpha), \tag{4.2}$$

and if $A_4(\alpha)<\infty$ then

$$n\sum_{n,2}|E(X_1X_{i_1}X_{i_2})|\leqq 24n\|X\|_{3+\delta}^3A_4(\alpha), \tag{4.3}$$

$$\begin{aligned} n\sum_{n,3}&|E(X_1X_{i_1}X_{i_2}X_{i_3})| \\ &\leqq 36n^2\|X\|_{4+\delta}^4A_4(\alpha)+288n^2\|X\|_{2+\delta}^4[A_2(\alpha)]^2 \\ &\leqq K_{\alpha,3}\|X\|_{4+\delta}^4n^2, \end{aligned} \tag{4.4}$$

where $X\equiv X_1$ (cf. [3], p. 196). In view of (4.2)–(4.4), we assume inductively that under the condition $A_{2m-2}(\alpha)<\infty$,

$$n\sum_{n,j}|E(X_1X_{i_1}\dots X_{i_j})|\leqq K_{\alpha,j}\|X\|_{j+1+\delta}^{j+1}n^{j^*}, \qquad n\geqq 1, \tag{4.5}$$

for $1\leqq j\leqq 2m-3$, where $j^*=k$ for $j=2k$ or $2k-1$. Then we shall show that (4.5) also holds for $j=2m-2$ and $2m-1$, under the condition $A_{2m}(\alpha)<\infty$. Applying (2.4) with $p=(2m+\delta)/h$ and $q=(2m+\delta)/(2m-h)$, for each h, $1\leqq h\leqq 2m-1$,

$$\begin{aligned} n\sum_{n,2m-1}^{(h)}&|E(X_1X_{i_1}\dots X_{i_{2m-1}})| \\ &\leqq n\sum_{n,2m-1}^{(h)}|E(X_1\dots X_{i_{h-1}})E(X_{i_h}\dots X_{i_{2m-1}})| \\ &\quad +12n\sum_{n,2m-1}^{(h)}\|X_1\dots X_{i_{h-1}}\|_{(2m+\delta)/h}\|X_{i_h}\dots X_{i_{2m-1}}\|_{(2m+\delta)/(2m-h)} \\ &\quad \cdot[\alpha(r_h)]^{\delta/(2m+\delta)}, \end{aligned} \tag{4.6}$$

and the second term on the right-hand side (rhs) of (4.6) is bounded by

$$\begin{aligned} 12n\|X\|_{2m+\delta}^{2m}&\sum_{r_h=0}^{n-1}(r_h+1)^{2m-2}[\alpha(r_h)]^{\delta/(2m+\delta)} \\ &\leqq 12n^m\|X\|_{2m+\delta}^{2m}A_{2m}(\alpha). \end{aligned}$$

The first term on the rhs of (4.6) vanishes for $h=1$ and $2m-1$, and for $2\leqq h\leqq 2m-2$, it follows along the same line as that of Lemma 3.1 in [12] that

$$\begin{aligned} n\sum_{n,2m-1}^{(h)}&|E(X_1\dots X_{i_{h-1}})E(X_{i_h}\dots X_{i_{2m-1}})| \\ &\leqq K'_{\alpha,h}\|X\|_{h+\delta}^h\|X\|_{2m-h+\delta}^{2m-h}n\sum_{i=1}^n i^{(h-1)^*-1}(n-i+1)^{(2m-1-h)^*-1} \\ &\leqq K''_{\alpha,h}\|X\|_{2m+\delta}^{2m}n^{(h-1)^*+(2m-1-h)^*}, \end{aligned}$$

where $(h-1)^*+(2m-1-h)^*$ equals m or $m-1$ according as h is even or odd. The case where $j=2m-2$ follows similarly, and thus we get (4.1).

Proof of (ii). For simplicity we introduce the following notation:

$$\hat{S}_n=\sum_{j=n+k+1}^{2n+k}X_j, \qquad c_n=E|S_n|^r \quad \text{and}$$

$$A_r(\alpha,k)=\sum_{i=k+1}^{\infty}(i+1)^{r/2-1}[\alpha(i)]^{\delta/(r+\delta)}.$$

We shall show that for $\varepsilon_1>0$ there exist K and k such that

$$E|S_n+\hat{S}_n|^r \leqq (2+\varepsilon_1)\,c_n+Kn^{r/2}, \qquad n\geqq 1.$$

Then the proof of (ii) follows from that of Lemma 7.4 in [8]. Because of the stationarity,

$$\begin{aligned} E|S_n+\hat{S}_n|^r &\leqq E(S_n+\hat{S}_n)^{2m}(|S_n|^\varepsilon+|\hat{S}_n|^\varepsilon) \\ &=2c_n+E\left\{\sum_{j=0}^{2m-1}\binom{2m}{j}S_n^j|S_n|^\varepsilon\hat{S}_n^{2m-j}+\sum_{j=1}^{2m}\binom{2m}{j}S_n^j\hat{S}_n^{2m-j}|\hat{S}_n|^\varepsilon\right\}, \end{aligned}$$

so it is sufficient to prove that for $0\leqq j\leqq 2m-1$,

$$|E(S_n^j|S_n|^\varepsilon\hat{S}_n^{2m-j})|\leqq \varepsilon_1 c_n+Kn^{r/2}, \qquad n\geqq 1, \tag{4.7}$$

and for $1\leqq j\leqq 2m$,

$$|E(S_n^j\hat{S}_n^{2m-j}|\hat{S}_n|^\varepsilon)|\leqq \varepsilon_1 c_n+Kn^{r/2}, \qquad n\geqq 1. \tag{4.8}$$

We only prove (4.8); expanding $\hat{S}_n^{2m-j}$, (4.7) follows similarly. We note that by the assumption (3.1), σ^2 exists and $\sigma_n^2=\sigma^2 n(1+o(1))$ (cf. [11], Theorem 18.5.3). Thus, there is n_0 such that

$$\left|\frac{\sigma_n^2}{n}-\sigma^2\right|<\tfrac{1}{2}\sigma^2$$

for all $n\geqq n_0$, and so for such n,

$$\tfrac{1}{2}\sigma^2 n\leqq \sigma_n^2\leqq c_n^{2/r}. \tag{4.9}$$

We also note that the following inequalities hold; from the proof of (i), for $2\leqq j\leqq 2m$,

$$\begin{aligned} \textstyle\sum_{n,j}|E(X_{i_1}\dots X_{i_j})| &\leqq n\textstyle\sum_{n,j-1}|E(X_1X_{i_1}\dots X_{i_{j-1}})| \\ &\leqq K_{\alpha,j-1}\|X\|_{j+\delta}^j n^{j/2}, \qquad n\geqq 1, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \textstyle\sum_{n,j-1}|E(X_{i_1}\dots X_{i_{j-1}}X_n)| &=\textstyle\sum_{n,j-1}|E(X_1X_{i_1}\dots X_{i_{j-1}})| \\ &\leqq K_{\alpha,j-1}\|X\|_{j+\delta}^j n^{j/2-1}, \qquad n\geqq 1. \end{aligned} \tag{4.11}$$

To obtain (4.8), we show that for each j, $1\leqq j\leqq 2m$, there exist $K_{(j)}$ and k_j such that

$$|E(S_n^j\hat{S}_n^{2m-j}|\hat{S}_n|^\varepsilon)|\leqq \varepsilon_1 c_n+K_{(j)}n^{r/2} \tag{4.12}$$

for all $n\geqq(n_0, k_j)$. Write $Y_n^j=\hat{S}_n^{2m-j}|\hat{S}_n|^\varepsilon$. Then, by (4.1),

$$E|Y_n^j|\leqq (ES_n^{2m})^{(r-j)/2m}\leqq K_\alpha\|X\|_{r+\delta}^{r-j}n^{(r-j)/2}, \qquad n\geqq 1. \tag{4.13}$$

(We do not use (4.13) when $j=1$, so it is also applicable to the case (iii).) We have for $1\leqq j\leqq 2m$,

$$|E(S_n^j \cdot Y_n^j)| \leqq j! \sum_{n,j} |E(X_{i_1} \dots X_{i_j} Y_n^j)|$$
$$\leqq j! \sum_{h=1}^{j} \sum_{n,j}^{(h)} |E(X_{i_1} \dots X_{i_j} Y_n^j)|, \tag{4.14}$$

where $\sum_{n,j}^{(h)}$, $1 \leqq h \leqq j$, are the components of $\sum_{n,j}$ for which $r_h = \max\{r_1, \dots, r_j\}$, where $r_h = i_{h+1} - i_h$ and $i_{j+1} = n+k+1$. Using (2.4),

$$\sum_{n,j}^{(h)} |E(X_{i_1} \dots X_{i_j} Y_n^j)| \leqq \sum_{n,j}^{(h)} |E(X_{i_1} \dots X_{i_h}) E(X_{i_{h+1}} \dots X_{i_j} Y_n^j)|$$
$$+ 12 \sum_{n,j}^{(h)} \|X_{i_1} \dots X_{i_h}\|_p \|X_{i_{h+1}} \dots X_{i_j} Y_n^j\|_q [\alpha(r_h)]^{1/s}, \tag{4.15}$$

where $p = (r+\delta)/h$, $q = r(r+\delta)/[r(r-h)+(r-j)\delta]$ and $s = r(r+\delta)/j\delta$. For $1 \leqq h \leqq j$, by Hölder's inequality, the second term on the rhs of (4.15) is bounded by

$$12 \|X\|_{r+\delta}^j c_n^{(r-j)/r} \sum_{i=k+1}^{n+k} (i+1)^{j-1} [\alpha(i)]^{j\delta/r(r+\delta)}$$
$$\leqq 12 \|X\|_{r+\delta}^j c_n^{(r-j)/r} [A_r(\alpha, k)]^{j/r} \left[\sum_{i=k+1}^{n+k} (i+1)^{-1+jr/2(r-j)} \right]^{(r-j)/r}. \tag{4.16}$$

By (4.9), the rhs of (4.16) is bounded by

$$K_j \|X\|_{r+\delta}^j c_n [A_r(\alpha, k)]^{j/r}, \tag{4.17}$$

if $n \geqq (n_0, k)$, where K_j does not depend on k. For $h=1$, the first term on the rhs of (4.15) vanishes, and for $h=j$ ($\geqq 2$), by (4.10) and (4.13), is bounded by

$$E|Y_n^j| \sum_{n,j} |E(X_{i_1} \dots X_{i_j})| \leqq K_j^{(j)} \|X\|_{r+\delta}^r n^{r/2}, \quad n \geqq 1. \tag{4.18}$$

Since $A_r(\alpha, k) \to 0$ as $k \to \infty$, choosing k_1 and k_2 so that

$$K_1 \|X\|_{r+\delta} [A_r(\alpha, k_1)]^{1/r} < \varepsilon_1, \quad 4K_2 \|X\|_{r+\delta}^2 [A_r(\alpha, k_2)]^{2/r} < \varepsilon_1,$$

(4.12) holds for $j=1$ and 2 with $K_{(1)}=0$ and $K_{(2)} = 2K_2^{(2)} \|X\|_{r+\delta}^r$. In order to prove (4.12) for general $3 \leqq j \leqq 2m$, $m \geqq 2$, we shall show that for $1 \leqq l \leqq j-2$ and all $n \geqq k$,

$$|E(X_{i_1} \dots X_{i_l} Y_n^j)| \leqq K_{l,j} \|X\|_{r+\delta}^{r-j+l} n^{(r-j+l)/2}, \tag{4.19}$$

where $K_{l,j}$ does not depend on k. For $3 \leqq j \leqq 2m$,

$$\sum_{i=0}^{\infty} (i+1)^{j/2-2} [\alpha(i)]^t < \infty, \tag{4.20}$$

where $t = (j-\varepsilon)/2m - (j-2)/(r+\delta)$. Indeed, since

$$t(r+\delta)/\delta = (j-\varepsilon)/2m + [r(j-\varepsilon) - 2m(j-2)]/2m\delta > (j-\varepsilon)/2m,$$

by Hölder's inequality,

$$\sum_{i=0}^{n} (i+1)^{j/2-2} [\alpha(i)]^t \leqq \sum_{i=0}^{n} (i+1)^{j/2-2} [\alpha(i)]^{(j-\varepsilon)\delta/2m(r+\delta)}$$
$$\leqq [A_r(\alpha)]^{(j-\varepsilon)/2m} \left[\sum_{i=1}^{n+1} i^{-s} \right]^{(r-j)/2m},$$

where $s=1-\varepsilon/2+(r-\varepsilon)/(r-j)>1$ (which is also true for $1<\varepsilon<2$), thus the series in (4.20) converges. Let $t_l=(j-\varepsilon)/2m-l/(r+\delta)$ for $1\leqq l\leqq j-2$. Then $t=t_{j-2}$. Using (2.4) with $p=r+\delta$ and $q=2m/(r-j)$, if $j\geqq 3$, by (4.10), (4.13) and (4.20),

$$\begin{aligned}
&\textstyle\sum_{n,1}|E(X_{i_1}Y_n^j)|\\
&\quad\leqq 12\|X\|_{r+\delta}(ES_n^{2m})^{(r-j)/2m}\sum_{i=k+1}^{n+k}[\alpha(i)]^{t_1}\\
&\quad\leqq 12\|X\|_{r+\delta}(ES_n^{2m})^{(r-j)/2m}n^{1/2}\sum_{i=0}^{n-1}(i+1)^{j/2-2}[\alpha(i)]^t\\
&\quad\leqq K_{1,j}\|X\|_{r+\delta}^{r-j+1}n^{(r-j+1)/2},\qquad n\geqq 1.
\end{aligned}\tag{4.21}$$

If $j\geqq 4$, using (2.4) with $p=r+\delta$, $q=2m(r+\delta)/[(r+\delta)(r-j)+2m]$,

$$\begin{aligned}
&\textstyle\sum_{n,2}^{(1)}|E(X_{i_1}X_{i_2}Y_n^j)|\\
&\quad\leqq 12\|X\|_{r+\delta}^2(ES_n^{2m})^{(r-j)/2m}\sum_{i=k+1}^{n+k}(i+1)[\alpha(i)]^{t_2}\\
&\quad\leqq 12\|X\|_{r+\delta}^2(ES_n^{2m})^{(r-j)/2m}(n+k+1)\sum_{i=k+1}^{n+k}(i+1)^{j/2-2}[\alpha(i)]^t,
\end{aligned}$$

and with $p=(r+\delta)/2$, $q=2m/(r-j)$,

$$\begin{aligned}
&\textstyle\sum_{n,2}^{(2)}|E(X_{i_1}X_{i_2}Y_n^j)|\leqq E|Y_n^j|\sum_{n,2}|E(X_{i_1}X_{i_2})|\\
&\quad+12\|X\|_{r+\delta}^2(ES_n^{2m})^{(r-j)/2m}\sum_{i=k+1}^{n+k}(i+1)[\alpha(i)]^{t_2},
\end{aligned}$$

so that we have

$$\begin{aligned}
&\textstyle\sum_{n,2}|E(X_{i_1}X_{i_2}Y_n^j)|\leqq(\sum_{n,2}^{(1)}+\sum_{n,2}^{(2)})|E(X_{i_1}X_{i_2}Y_n^j)|\\
&\quad\leqq K_{2,j}\|X\|_{r+\delta}^{r-j+2}n^{(r-j+2)/2},\qquad n\geqq k.
\end{aligned}\tag{4.22}$$

Let us now assume that for $1\leqq l\leqq j-4$, (4.19) holds. Then we shall show that (4.19) also holds for $l=j-3$ and $j-2$. We only prove the case of $l=j-2$ (the other case follows similarly). We have

$$\textstyle\sum_{n,j-2}|E(X_{i_1}\dots X_{i_{j-2}}Y_n^j)|\leqq\sum_{h=1}^{j-2}\sum_{n,j-2}^{(h)}|E(X_{i_1}\dots X_{i_{j-2}}Y_n^j)|.\tag{4.23}$$

Applying (2.4) with $p=(r+\delta)/h$ and $q=2m(r+\delta)/[(r+\delta)(r-j)+2m(j-2-h)]$,

$$\begin{aligned}
&\textstyle\sum_{n,j-2}^{(h)}|E(X_{i_1}\dots X_{i_{j-2}}Y_n^j)|\\
&\quad\leqq\textstyle\sum_{n,j-2}^{(h)}|E(X_{i_1}\dots X_{i_h})E(X_{i_{h+1}}\dots X_{i_{j-2}}Y_n^j)|\\
&\qquad+12\|X\|_{r+\delta}^{j-2}(ES_n^{2m})^{(r-j)/2m}\sum_{i=k+1}^{n+k}(i+1)^{j-3}[\alpha(i)]^t,
\end{aligned}\tag{4.24}$$

and the second term on the rhs of (4.24) is bounded by

$$\begin{aligned}
&12\|X\|_{r+\delta}^{j-2}(ES_n^{2m})^{(r-j)/2m}(n+k+1)^{j/2-1}\sum_{i=k+1}^{n+k}(i+1)^{j/2-2}[\alpha(i)]^t\\
&\quad\leqq K'_{j-2,j}\|X\|_{r+\delta}^{r-2}n^{(r-2)/2},\qquad n\geqq k.
\end{aligned}\tag{4.25}$$

For $h=1$, the first term on the rhs of (4.24) vanishes, and for $h=j-2$, is bounded by

$$E|Y_n^j|\sum_{n,j-2}|E(X_{i_1}\dots X_{i_{j-2}})|\leqq K''_{j-2,j}\|X\|_{r+\delta}^{r-2}n^{(r-2)/2},\qquad n\geqq 1. \tag{4.26}$$

For $2\leqq h\leqq j-3$, $1\leqq j-2-h\leqq j-4$, and by (4.11) and the assumption made,

$$\begin{aligned}&\textstyle\sum_{n,j-2}^{(h)}|E(X_{i_1}\dots X_{i_h})E(X_{i_{h+1}}\dots X_{i_{j-2}}Y_n^j)|\\ &\leqq \sum_{i_h=1}^{n}\{\textstyle\sum_{i_h,h-1}|E(X_{i_1}\dots X_{i_h})|\}\{\textstyle\sum_{n,j-2-h}|E(X_{i_1}\dots X_{i_{j-2-h}}Y_n^j)|\}\\ &\leqq K\|X\|_{r+\delta}^{r-2}n^{(r-2-h)/2}\sum_{i=1}^{n}i^{h/2-1}\qquad (K=K_{\alpha,h-1}\cdot K_{j-2-h,j})\\ &\leqq K_{j-2,j}^{(h)}\|X\|_{r+\delta}^{r-2}n^{(r-2)/2},\qquad n\geqq k.\end{aligned} \tag{4.27}$$

From (4.23) through (4.27), we have thus proved that (4.19) holds for $l=j-2$. Using then (4.21) and (4.22), the proof of (4.19) follows by the method of induction.

We return to the proof of (4.12). For $2\leqq h\leqq j-1$, $1\leqq j-h\leqq j-2$, and by (4.11) and (4.19),

$$\begin{aligned}&\textstyle\sum_{n,j}^{(h)}|E(X_{i_1}\dots X_{i_h})E(X_{i_{h+1}}\dots X_{i_j}Y_n^j)|\\ &\leqq \sum_{i_h=1}^{n}\{\textstyle\sum_{i_h,h-1}|E(X_{i_1}\dots X_{i_h})|\}\{\textstyle\sum_{n,j-h}|E(X_{i_1}\dots X_{i_{j-h}}Y_n^j)|\}\\ &\leqq K\|X\|_{r+\delta}^{r}n^{(r-h)/2}\sum_{i=1}^{n}i^{h/2-1}\qquad (K=K_{\alpha,h-1}\cdot K_{j-h,j})\\ &\leqq K_j^{(h)}\|X\|_{r+\delta}^{r}n^{r/2},\qquad n\geqq k.\end{aligned} \tag{4.28}$$

Combining (4.14)–(4.18) and (4.28), we obtain for $1\leqq j\leqq 2m$,

$$|E(S_n^j\cdot Y_n^j)|\leqq j!\,\{jK_j\|X\|_{r+\delta}^{j}c_n[A_r(\alpha,k)]^{j/r}+\sum_{h=2}^{j}K_j^{(h)}\|X\|_{r+\delta}^{r}n^{r/2}\}, \tag{4.29}$$

if $n\geqq(n_0,k)$. Thus, (4.12) holds by properly choosing $K_{(j)}$ and k_j as this has been already made for $j=1$ and 2. Let $K\geqq\max\{K_{(2)},\dots,K_{(2m)}\}$ and $k=\max\{k_1,\dots,k_{2m}\}$. Then (4.8) holds for $n\geqq(n_0,k)$. But we can choose K so that (4.8) holds also for $n<(n_0,k)$, thus (4.8) is proved.

Proof of (iii). Since $1<\varepsilon<2$,

$$\begin{aligned}E|S_n+\hat{S}_n|^r&\leqq 2^{\varepsilon-1}E(S_n+\hat{S}_n)^{2m}(|S_n|^\varepsilon+|\hat{S}_n|^\varepsilon)\\ &=2^\varepsilon c_n+2^{\varepsilon-1}E\left\{\sum_{j=0}^{2m-1}\binom{2m}{j}S_n^j|S_n|^\varepsilon\hat{S}_n^{2m-j}+\sum_{j=1}^{2m}\binom{2m}{j}S_n^j\hat{S}_n^{2m-j}|\hat{S}_n|^\varepsilon\right\}.\end{aligned}$$

It follows similarly to the proof of (ii) that for $\varepsilon_1>0$, there exist K and k such that

$$E|S_n+\hat{S}_n|^r\leqq(2^\varepsilon+\varepsilon_1)c_n+Kn^{r/2},\qquad n\geqq 1.$$

For $m \geqq 1$, $2^{\varepsilon} < 2^{(2m+\varepsilon)/2}$ (see (7.12) in [8], p. 227), so that the proof also follows along the same line as in Lemma 7.4 in [8]. Thus the proof of Theorem 1 is complete.

Proof of Theorem 2. Theorem 2 can be proved using the arguments used in the proof of Theorem 1 with a few changes. Using (2.4) with $p=q=\infty$, the proof for $r=2m$, $m=1,2,\ldots$, is similar to that of (4.1). Note that, under the assumptions of Theorem 2, σ^2 exists and $\sigma_n^2=\sigma^2 n(1+o(1))$, and thus (4.9) holds (cf. [11], Theorem 18.5.4). Choosing $p=\infty$, $q=r/(r-j)$ and $s=r/j$, the second term on the rhs of (4.15) is bounded by

$$K c_n^{(r-j)/r} \sum_{i=k+1}^{n+k} (i+1)^{j-1} [\alpha(i)]^{j/r}$$
$$\leqq K c_n^{(r-j)/r} \left[\sum_{i=k+1}^{\infty} (i+1)^{r/2-1} \alpha(i)\right]^{j/r} \left[\sum_{i=k+1}^{n+k} (i+1)^{-1+jr/2(r-j)}\right]^{(r-j)/r},$$

and so (4.17) holds. Let $t=(j-\varepsilon)/2m$, $3 \leqq j \leqq 2m$. Then the series in (4.20) converges, so that (4.19) is also obtained by using (2.4) with $p=\infty$ and $q=2m/(r-j)$. The remaining changes should be obvious.

5. Convergence of Moments in the Central Limit Theorem

Theorem 3. *Let $\{X_j\}$ be a strictly stationary ϕ-mixing sequence with $EX_1=0$ and $E|X_1|^r<\infty$ for some $r>2$. If*

$$\sum_{i=1}^{\infty} [\phi(i)]^{1/2} < \infty, \tag{5.1}$$

then as $n \to \infty$

$$E|S_n/\sigma n^{1/2}|^r \to \beta_r, \tag{5.2}$$

where β_r is the rth absolute moment of $\mathcal{N}(0,1)$.

Proof. Under the assumptions of Theorem 3 the central limit theorem

$$S_n/\sigma n^{1/2} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \tag{5.3}$$

holds (cf. [11], Theorem 18.5.2), and thus it is sufficient to prove that $\{|S_n/n^{1/2}|^r, n \geqq 1\}$ is uniformly integrable. Let $f_N(x)=x$ if $|x| \leqq N$; $=0$ if $|x|>N$, and $g_N(x)=x-f_N(x)$, and put $\bar{f}_N(x)=f_N(x)-E(f_N(X_1))$, $\bar{g}_N(x)=g_N(x)-E(g_N(X_1))$. Then, both $\{\bar{f}_N(X_j)\}$ and $\{\bar{g}_N(X_j)\}$ are ϕ-mixing with mixing coefficients $\leqq \phi(n)$. Let

$$U_{Nn}=\sum_{j=1}^{n} \bar{f}_N(X_j), \qquad V_{Nn}=\sum_{j=1}^{n} \bar{g}_N(X_j),$$

then $S_n=U_{Nn}+V_{Nn}$. Denote by $E_a(X)$ the integral of X over the set $\{X \geqq a\}$. Since $|S_n|^r \leqq 2^{r-1}(|U_{Nn}|^r+|V_{Nn}|^r)$ and $E_a(U+V) \leqq 2\{E_{a/2}(U)+E(V)\}$, we have

$$E_a|S_n/n^{1/2}|^r \leqq 2^r\{E_{a/2^r}|U_{Nn}/n^{1/2}|^r+E|V_{Nn}/n^{1/2}|^r\}. \tag{5.4}$$

Let

$$\sigma^2(N) = E(\bar{f}_N(X_1))^2 + 2 \sum_{j=2}^{\infty} E(\bar{f}_N(X_1)\,\bar{f}_N(X_j)).$$

Then $\sigma^2(N) \to \sigma^2$ as $N \to \infty$, and thus for N sufficiently large ($N \geqq N_1$, say),

$$\sigma^2(N) \geqq \tfrac{1}{2}\sigma^2 > 0.$$

For such N, since $\bar{f}_N(X_j)$ is bounded, by the remark following Theorem 2,

$$\begin{aligned} E_{a/2^r} |U_{Nn}/n^{1/2}|^r &\leqq 2^r E|U_{Nn}/n^{1/2}|^{2r}/a \\ &\leqq K_N/a, \quad n \geqq 1. \end{aligned} \tag{5.5}$$

To complete the proof we show the following

Lemma. *For* $t > 0$, *there is* N_0 *for which*

$$E|V_{Nn}|^r \leqq t n^{r/2} \tag{5.6}$$

for all $n \geqq 1$ *and* $N \geqq N_0$.

Proof of Lemma. By (2.3),

$$E V_{Nn}^2 \leqq n \left\{ 1 + 4 \sum_{i=1}^{\infty} [\phi(i)]^{1/2} \right\} E(g_N(X_1))^2.$$

Since $E(g_N(X_1))^2 \to 0$ as $N \to \infty$, the lemma is true for $r = 2$. So it is sufficient to assume that the lemma is true if r is an integer $m \geqq 2$ and prove that it is then true if $r = m + \varepsilon$, where $0 < \varepsilon \leqq 1$. Let us write

$$\hat{V}_{Nn} = \sum_{j=n+k+1}^{2n+k} \bar{g}_N(X_j), \qquad c_{Nn} = E|V_{Nn}|^r.$$

Using (2.3), and arguing as in [8], pp. 225–226, we obtain that for $\varepsilon_1 > 0$, there exist K and k such that

$$E|V_{Nn} + \hat{V}_{Nn}|^r \leqq (2 + \varepsilon_1) c_{Nn} + K t n^{r/2}, \tag{5.7}$$

for all $n \geqq 1$ and $N \geqq N_0$, where K depends on r alone. Also we obtain that for $\varepsilon_2 > 0$,

$$c_{N,2n} \leqq (2 + \varepsilon_2) c_{Nn} + 2K t n^{r/2} \tag{5.8}$$

for all $n \geqq 1$ and $N \geqq N_0$. Indeed, we have

$$\begin{aligned} c_{N,2n} &\leqq \{[(2+\varepsilon_1) c_{Nn} + K t n^{r/2}]^{1/r} + 2k c_{N1}^{1/r}\}^r \\ &= (1+\varepsilon_3)^r [(2+\varepsilon_1) c_{Nn} + K t n^{r/2}], \end{aligned}$$

(see [8], p. 226), where since $c_{N1} \leqq 2^r E|X_1|^r$,

$$\begin{aligned} \varepsilon_3 &= 2k c_{N1}^{1/r} / [(2+\varepsilon_1) c_{Nn} + K t n^{r/2}]^{1/r} \\ &\leqq 4k \|X\|_r / (K t n^{r/2})^{1/r} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus, choosing ε_1 sufficiently small and n_0 not depending on N sufficiently large, for $n \geqq n_0$,

$$(1+\varepsilon_3)^r(2+\varepsilon_1) \leqq 2+\varepsilon_2, \qquad K(1+\varepsilon_3)^r < 2K,$$

then (5.8) holds for $n \geqq n_0$. For each n, $c_{Nn} \to 0$ as $N \to \infty$, and so for some redefined (if necessary) N_0, we have

$$c_{N,2n} \leqq 2Ktn^{r/2}$$

for all $n < n_0$ and $N \geqq N_0$, thus establishing (5.8). Applying (5.8) and the fact that $c_{N1} \to 0$ as $N \to \infty$ it follows as in the proof of Lemma 7.4 in [8] that there is K_1 not depending on N and t such that

$$E|V_{Nn}|^r \leqq K_1 tn^{r/2}$$

for all $n \geqq 1$ and $N \geqq N_0$ (increase N_0, if necessary), which is (5.6) except for the constant K_1.

We return to the proof of Theorem 3. For any $\varepsilon_4 > 0$, if we choose t in (5.6) so that $t < \varepsilon_4/2^{r+1}$, then for N large enough, $2^r E|V_{Nn}/n^{1/2}|^r < \varepsilon_4/2$. For such $N (\geqq N_1)$, choose a so that $K_N/a < \varepsilon_4/2^{r+1}$, then from (5.4) and (5.5), for all $n \geqq 1$,

$$E_a|S_n/n^{1/2}|^r < \varepsilon_4,$$

which asserts that $\{|S_n/n^{1/2}|^r\}$ is uniformly integrable. This completes the proof of Theorem 3.

Theorem 4. *Let $\{X_j\}$ be a strong mixing sequence. If* (i) *the assumptions of Theorem 1 are satisfied; or* (ii) $EX_1 = 0$, $|X_1| \leqq C < \infty$ a.s. *and*

$$\sum_{i=0}^{\infty} (i+1)^{r'/2-1} \alpha(i) < \infty \qquad (r' > r > 2),$$

then (5.2) *holds.*

Proof. We use the same notation as defined in the previous proofs. Note that the central limit theorem (5.3) holds under the assumptions of Theorem 4 (cf. [11], Theorems 18.5.3-4). We first assume (i). Let $s = 2 + (r-2)(r+\delta)/\delta$ $(>r)$. If (3.1) holds, then

$$\sum_{i=0}^{\infty} (i+1)^{s/2-1} \alpha(i) = \sum_{i=0}^{\infty} \{(i+1)^{r/2-1} [\alpha(i)]^{\delta/(r+\delta)}\}^{(r+\delta)/\delta} < \infty,$$

so from Theorem 2, for all N sufficiently large,

$$\begin{aligned} E_{a/2^r}|U_{Nn}/n^{1/2}|^r &\leqq E|U_{Nn}/n^{1/2}|^s/(a/2^r)^{(s-r)/r} \\ &\leqq K_N/a^{(s-r)/r}, \qquad n \geqq 1. \end{aligned}$$

In view of the proof of Theorem 3, it is sufficient to prove that (5.6) holds under the assumption (i). When $r = 2m$, by (4.1),

$$EV_{Nn}^{2m} \leqq K_\alpha (E|\bar{g}_N(X_1)|^{2m+\delta})^{2m/(2m+\delta)} n^m, \qquad n \geqq 1.$$

Thus since $E|\bar{g}_N(X_1)|^{2m+\delta} \to 0$ as $N \to \infty$ (5.6) holds.

Now we assume that $r=2m+\varepsilon$, $0<\varepsilon<2$. To prove (5.6) for such r, we show that there exist K, not depending on N, and N_0 such that

$$c_{Nn}\leqq Kn^{r/2} \tag{5.9}$$

for all $n\geqq 1$ and $N\geqq N_0$. Let $\sigma_n^2(N)=EU_{Nn}^2$. Then since $E|\bar{f}_N(X)|^p\leqq 2^pE|X|^p$ $(p\geqq 1)$,

$$\begin{aligned}
&\left|\frac{\sigma_n^2(N)}{n}-\sigma^2(N)\right| \\
&\quad=2\left|\frac{1}{n}\sum_{j=2}^{n}(j-1)E(\bar{f}_N(X_1)\bar{f}_N(X_j))+\sum_{j=n+1}^{\infty}E(\bar{f}_N(X_1)\bar{f}_N(X_j))\right| \\
&\quad\leqq 96\|X\|_{2+\delta}^2\left\{\frac{1}{n}\sum_{j=1}^{n-1}j[\alpha(j)]^{\delta/(2+\delta)}+\sum_{j=n}^{\infty}[\alpha(j)]^{\delta/(2+\delta)}\right\}\to 0
\end{aligned}$$

as $n\to\infty$ (cf. [11], p. 348). So there is n_0 not depending on N such that

$$\left|\frac{\sigma_n^2(N)}{n}-\sigma^2(N)\right|<\tfrac{1}{4}\sigma^2 \tag{5.10}$$

for all $n\geqq n_0$ and $N\geqq 0$. On the other hand, there is N_0 such that

$$|\sigma^2(N)-\sigma^2|<\tfrac{1}{4}\sigma^2 \tag{5.11}$$

for all $N\geqq N_0$. Combining (5.10) and (5.11), we have

$$\left|\frac{\sigma_n^2(N)}{n}-\sigma^2\right|<\tfrac{1}{2}\sigma^2$$

for all $n\geqq n_0$ and $N\geqq N_0$, and so for such n and N,

$$\tfrac{1}{2}\sigma^2 n\leqq\sigma_n^2(N)\leqq d_{Nn}^{2/r} \tag{5.12}$$

where $d_{Nn}=E|U_{Nn}|^r$. Applying (5.12) it is easy to obtain as in the proof of Theorem 1 that for $\varepsilon_1>0$, there exist K and k, both not depending on N, such that

$$E|U_{Nn}+\hat{U}_{Nn}|^r\leqq(2^*+\varepsilon_1)d_{Nn}+Kn^{r/2}$$

for all $n\geqq 1$ and $N\geqq N_0$, where $2^*=2$ if $0<\varepsilon\leqq 1$; $=2^\varepsilon$ if $1<\varepsilon<2$. Hence, it follows similarly to the proof of the lemma that there exist K, not depending on N, and N_0 such that

$$d_{Nn}\leqq Kn^{r/2} \tag{5.13}$$

for all $n\geqq 1$ and $N\geqq N_0$. From Theorem 1, (5.13) and the inequality $c_{Nn}\leqq 2^{r-1}(c_n+d_{Nn})$, we get (5.9). Combining (5.9) and the fact that $E|\bar{g}_N(X_1)|^{r+\delta}\to 0$ as $N\to\infty$ with (4.16) and (4.29) (setting $k=0$), we obtain that for $t>0$, there is N_0 such that

$$c_{N,2n}\leqq 2^*c_{Nn}+tn^{r/2}$$

for all $n \geqq 1$ and $N \geqq N_0$. So the proof of (5.6) also follows from Lemma 7.4 in [8]. When the assumption (ii) holds, using Theorem 2, the uniform integrability of $\{|S_n/n^{1/2}|^r\}$ follows immediately from

$$\begin{aligned} E_a |S_n/n^{1/2}|^r &\leqq E|S_n/n^{1/2}|^{r'}/a^{(r'-r)/r} \\ &\leqq K/a^{(r'-r)/r}, \qquad n \geqq 1. \end{aligned}$$

Thus the proof of Theorem 4 is complete.

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