Z. Wahrscheinlichkeitstheorie verw. Gebiete
50, 333–355 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete


# Asymptotic Expansions for Bivariate von Mises Functionals

F. Götze

Mathematisches Institut der Universität Köln, Weyertal 86, D-5000 Köln 41

Herrn Professor Dr. Leopold Schmetterer
zu Ehren seines sechzigsten Geburtstages gewidmet

**Summary.** A Berry-Essen result and asymptotic expansions are derived for the distribution of bivariate von Mises functionals under moment and smoothness conditions.

The results apply to the Cramér-von Mises $\omega^2$ - statistic as well as to the Central Limit Theorem in Hilbert space, yielding a convergence rate $O(n^{-1+\varepsilon})$ for every $\varepsilon>0$ on centered ellipsoids.

## 1. Introduction and Notations

Let $(X,\mathscr{A},P)$ be a probability space. For a symmetric function $h: X^2\to\mathbb{R}$ and $(x_1,\dots,x_n)\in X^n$ let

(1.1) $$w_n=n^{-1}\sum_{i,j=1}^{n} h(x_i,x_j)$$

denote a bivariate von Mises functional.

In this paper we investigate the asymptotic distribution of $w_n$ under $P^n|\mathscr{A}^n$, assuming that

(1.2) $$\int h(x,\cdot)\,dP=0 \qquad P\text{-a.e.}$$

If this condition is violated, but $\int h\,dP^2=0$, then $n^{-1/2}w_n$ is asymptotically normal. See H. Callaert and P. Janssen (1978) and H. Callaert, P. Janssen and N. Veraverbeke (1978) for a Berry-Esseen Theorem and asymptotic expansions in this case. Note that (1.1) includes the so called $U$-statistics (up to an inessential difference in standardization) if we assume $h(x,x)\equiv 0$.

Assume that for some $s\geqq 3$

(1.3) $$\beta_s=\int|h(\cdot,\cdot)|^s\,dP^2+\int|h(x,x)|^s P(dx)$$

is finite.

0044-3719/79/0050/0333/$04.60

The limiting distribution function of $w_n$ under $P^n|\mathscr{A}^n$, say $\chi(x,h)$, $x\in\mathbb{R}$, was investigated by von Mises (1947). It has the c.f.

(1.4) $$\hat{\chi}(t,h)=\exp(it\int h(x,x)\,P(dx))\prod_{k=1}^{\infty}[(1-2it\lambda_k)\exp(2it\lambda_k)]^{-1/2}$$

where $\lambda_k$, $k\in\mathbb{N}$, denote the eigenvalues of the Hilbert-Schmidt operator on $L^2(X,P)$ induced by the kernel $h(\cdot,\cdot)$. The c.f. $\hat{\chi}(t,h)$ is well defined and analytic on $\mathbb{R}$. (See N. Dunford and J.T. Schwartz (1963, XI, p. 1036, Theorem 26) and T. Carleman (1922).) If all eigenvalues $\lambda_k$, $k\in\mathbb{N}$, have multiplicity 2, then

$$\chi(x,h)=1-\sum_{\lambda_k>0}\exp[-\lambda_k^{-1}x/2]\prod_{j\neq k}(1-\lambda_j\lambda_k^{-1})^{-1} \quad \text{for } x>0.$$

See Remark (2.12).

With Theorem (2.3) we prove that for $s\geqq 4$

(1.5) $$\sup_{z\in\mathbb{R}}|P^n\{w_n<z\}-\chi(z,h)|=O(n^{-\alpha}\eta_n)$$

where $\alpha=1$ and $\eta_n=o(n^{\varepsilon})$ for every $\varepsilon>0$, provided that $h(x,y)$ is not equal to a finite sum of functions $f(x)\,g(y)$ where $f,g\in L^2(X,P)$. Furthermore, if $h(x,y)$ fulfills

(1.6) $$P\{x\in X: |\int\exp[ith(\cdot,x)]\,dP|\geqq 1-n^{-1}\varepsilon_n \text{ for } |t|\geqq\eta_n^{-1}\} =O(n^{-(s-2)/2}(\log n)^{-1})$$

with $\varepsilon_n=3(s-2)\log n$ and $\eta_n$ defined in (2.5), then we prove with Theorem (2.9) the existence of an asymptotic expansion for $P^n\{w_n<x\}$ up to $O(n^{-(s-2)/2})$, say $\chi_s^{(n)}(x,h)$, starting with $\chi(x,h)$.

For a more general but less applicable continuity condition see (2.8) and Remark (2.11). Unfortunately we are not able to give an explicit formula for $\chi_s^{(n)}$ in case of a general $h(x,y)$, but its c.f. is given by

(1.7) $$\hat{\chi}_s^{(n)}(t,h)=\hat{\chi}(t,h)\left(1+\sum_{j=1}^{[(s-3)/2]}n^{-j}P_{2j}(t,h)\right)$$

where $P_{2j}(t,h)$ are meromorphic functions in $t$ depending on the moments of $h$ and the resolvent $(\mathrm{Id}-2itH)^{-1}$ of the operator $H$ induced by the kernel $h$. See (1.10)-(1.14).

Generally the inversion of (1.7) has to be done numerically. See e.g. (G.V. Martinov (1975)).

These results apply to von Mises' $\omega^2$-statistic

(1.8) $$\omega_n^2=n\int(F_n(t)-F(t))^2\,F(dt),$$

where $F(t)$ is a continuous d.f. on $\mathbb{R}$ and $F_n(t)$ denotes the empirical d.f. of a random sample of size $n$ distributed according to $F^n$, and the related statistic of Watson (1961)

(1.9) $$U_n=\omega_n^2-n[\int(F_n(t)-F(t))\,F(dt)]^2.$$

In these cases we give in Examples (2.13) and (2.16) rapidly converging power series expansions for the distribution functions up to $O(n^{-2})$.

The limiting distribution of $\omega_n^2$ was investigated by N.V. Smirnov (1937) and T.W. Anderson and D.A. Darling (1952). The convergence rate has been investigated by several authors. N.P. Kandelaki (1965) proved (1.5) with $\alpha=0$, $\eta_n=(\log n)^{-1/4}$, followed by V.V. Sazonov (1968), (1969) with $\alpha=1/10$, resp. $\alpha=1/6$, W.A. Rosenkrantz (1969) with $\alpha=1/5$, J. Kiefer (1972) and Y.Y. Nikitin (1972) with $\alpha=1/4$, A.I. Orlov (1971) with $\alpha=1/3$ and finally Orlov (1974) proved $\alpha=1/2$ and $\eta_n$ as in (1.5). S. Csörgö (1976) proved $\alpha=1/2$ and $\eta_n=\log n$, conjectured $\alpha=1$ and gave a formal expansion for the distribution of $\omega_n^2$.

Finally, if $X$ is a real separable Hilbert space, (1.5) applies to the Central Limit Theorem in $X$ and improves the convergence rates of J. Kuelbs and T. Kurtz (1974) who proved $\alpha=1/8$ and $\eta_n=1$ and V.I. Paulauskas who increased $\alpha$ to 1/6. See Remark (2.7).

Having introduced the necessary notations below we formulate the main results in Sect. 2. Section 3 contains the lemmas. Here, Lemma (3.30) may be of independent interest. The proof of the theorems can be found in Sect. 4.

*Notations*

Define $\hat{\chi}_n(t,h)=\int \exp(itw_n)\,dP^n$.

Let $\alpha=(i_1,\dots,i_m)$ resp. $\underline{r}=(r_{jk})_{j,k=1,\dots,m}$ be a vector resp. a matrix of non-negative integers. Define

$$(1.10)\quad \gamma_\alpha(t,h)=\sum{}^* \prod_{l=1}^{m} i_l! \int \prod_{j,k=1}^{m} r_{jk}!^{-1}\left[\tfrac{1}{2}R(t,h)(x_j,x_k)\right]^{r_{jk}} P^m(d\underline{x})$$

with

$$(1.11)\quad R(t,h)(x,y)=(2it)\{h(x,y)+2it\int h(x,z)[(Id-2itH)^{-1}h(\cdot,y)](z)\,P(dz)\}$$

where $\sum^*$ denotes summation over all $m\times m$ integer matrices $(r_{jk})_{j,k=1,\dots,m}$ such that

$$(1.12)\quad \sum_{j=1}^{m}(r_{jk}+r_{kj})=i_k,\qquad k=1,\dots,m.$$

Since $h(x,\cdot)\in L^2(X,P)$ $P$-a.e., $R(t,h)(x,y)$ is well defined $P^2$-a.e. and $P*d$-a.e., where $d\colon X\to X^2$ denotes the diagonal map $x\to(x,x)$. Furthermore, $\gamma_\alpha(t,h)$ does not depend on the order of the components of the vector $\alpha$ and vanishes identically if $|\alpha|=i_1+\dots+i_m$ is odd. Replacing all monomials $\beta_{i_1}\dots\beta_{i_m}$ in the one dimensional Edgeworth polynomial of order $r$, $r\geqq 0$, say $P_r(\beta)$, with cumulants expressed in terms of moments $\beta_i$, $i\geqq 2$, (see Bikjalis (1973, p. 153/5)) by the functions $\gamma_{(i_1,\dots,i_m)}(t,h)$, we obtain functions $P_r(t,h)$, $r\geqq 0$. For example $P_0(t,h)\equiv 1$, $P_1(t,h)\equiv 0$ and $P_2(t,h)=\frac{1}{24}(\gamma_4-3\gamma_{(2,2)})(t,h)+\frac{1}{72}\gamma_{(3,3)}(t,h)$. Define the expansion of $\hat{\chi}_n(t,h)$ by

(1.13) $\hat{\chi}_s^{(n)}(t,h)=\sum_{r=0}^{s-3} n^{-r/2} P_r(t,h)\,\hat{\chi}(t,h)$

where $P_r(t,h)\equiv 0$ if $r$ is odd.

Especially we have

(1.14) $P_2(t,h)=\int[\frac{1}{24}(3r_{11}^2-3(r_{11}r_{22}+2r_{12}^2))+\frac{1}{72}(6r_{12}^3+9r_{11}r_{12}r_{22})]\,dP^2$

with $r_{jk}(x_1,x_2)=R(t,h)(x_j,x_k)$, $j,k=1,2$.

In order to avoid ambiguities let $z^{1/2}$, $z\in C$, denote the branch with positive real part (see (1.4)).

For notational convenience we shall use the letter $c$ as a generic positive constant which depends only on the length of the expansions.

## 2. Results

Assume that the eigenvalues $\lambda_k\neq 0$, $k\in\mathbb{N}$, of the Hilbert-Schmidt operator on $L^2(X,P)$ pertaining to $h$ are ordered according to their absolute values i.e. $|\lambda_{k+1}|\leqq|\lambda_k|$ for every $k\in\mathbb{N}$. Let $\gamma_{n,s}$ denote the monotone increasing sequence of real numbers determined by

(2.1) $|\hat{\chi}(\gamma_{n,s}|\lambda_{a(s)}|^{-1},h)|=n^{-3(s-2)}$

where $s\geqq 3$ and $a(s)=24(s-2)+5$ if $s\geqq 4$ and $a(3)=31$. For instance,

$$\lambda_k=k^{-\alpha},\ \alpha>\tfrac{1}{2}\ \text{implies}\ \gamma_{n,s}\leqq c(s,\alpha)\left(\frac{s}{\alpha}\log n\right)^{\alpha}$$

(2.2) $\lambda_k=\exp(-k^{\beta}),\quad \beta>0$

$$\text{implies}\ \gamma_{n,s}\leqq c(s,\beta)\exp[(s(1+\beta^{-1})\log n)^{\beta/(1+\beta)}]$$

and $|\lambda_k|>0$ for every $k\in\mathbb{N}$ with $\sum_{k=1}^{\infty}\lambda_k^2<\infty$ implies $\gamma_{n,s}=o(n^{\varepsilon})$ for every $\varepsilon>0$.

(2.3) **Theorem.** *Assume that $h(x,y)$ fulfills condition (1.2). Furthermore, assume that*

(2.4) *$\beta_s$ is finite and $\lambda_{a(s)}\neq 0$ for $s\geqq 3$.*

*Then there exist constants $c_3, c_4$ such that*

(2.5) $$\sup_{z\in\mathbb{R}}|P^n\{w_n<z\}-\chi(z,h)|\leqq\begin{cases}c_3(\beta_3|\lambda_{a(3)}|^{-3})^3 n^{-1/2} & \text{if } s=3\\ c_4(\beta_4|\lambda_{a(4)}|^{-4})^6\eta_n n^{-1} & \text{if } s\geqq 4\end{cases}$$

*where $\eta_n=(\log n)^6\gamma_{n,4}^5$.*

(2.6) *Remark.* Let $b_i(z)=\int|h(\cdot,z)|^i\,dP$, $i\in\mathbb{N}$. If there exists a constant $b>0$ such that $b_3(z)+b_2(z)^{-1}\leqq b$ $P$-a.e. and $h$ fulfills (2.4) (with $s=4$), the factor $\eta_n$ on the r.h.s. of (2.5) can be replaced by 1.

(2.7) *Remark.* Consider the case where $X$ is a real separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and $P|\mathscr{A}$ is a mean zero $p$-measure defined on the Borel $\sigma$-field $\mathscr{A}$ with covariance operator, say $C$. Let $h(x,y)=\langle Bx,y\rangle$, where $B$ is a bounded, symmetric and positive semidefinite operator on $X$ and denote with $\Phi|\mathscr{A}$ the mean zero Gaussian $p$-measure with covariance operator $C$. Furthermore, define $s_n=n^{-1/2}(x_1+\ldots+x_n)$, $x_i\in X$ and $B_r=\{x\in X\colon \langle Bx,x\rangle\leqq r\}$, $r\geqq 0$.

In this case, the eigenvalues $\lambda_k$, $k\in\mathbb{N}$, pertaining to $h$ are the eigenvalues of $BC\colon X\to X$. Since

$$w_n=\langle Bs_n,s_n\rangle \quad\text{and}\quad \chi(r,h)=\Phi(B_r),\qquad r\geqq 0.$$

Theorem (2.3) applies to the Central Limit Theorem in Hilbert space for the ellipsoids $B_r$, $r\geqq 0$.

For the special case $X=\mathbb{R}^k$ and $C=B$ being the $k\times k$ identity matrix, a result of C.G. Esseen (1945, p. 92, Theorem 1) yields $c(k)\beta_2^{3/2}n^{1/(k+1)}n^{-1}$ as an upper bound for the l.h.s. of (2.5) with $\lim_{k\to\infty} c(k)=\infty$ which is better than our bound in this case.

In order to prove asymptotic expansions we need the following smoothness condition:

There exists a constant $a>0$ such that

(2.8) $$\inf_{1\leqq 2m\leqq n} P^m\{\underline{x}\in X^m\colon |\int \exp[ith(\cdot,\underline{x})]\,dP|\geqq 1-\varepsilon_n n^{-1} \text{ for } |t|\geqq \eta_n^{-1}\} \leqq an^{-(s-2)/2}(\log n)^{-1}$$

where $\varepsilon_n=3(s-2)\log n$ and $h(\cdot,\underline{x})=\sum_{i=1}^{m} h(\cdot,x_i)$.

Obviously condition (1.6) implies condition (2.8).

(2.9) **Theorem.** *Assume that $h$ fulfills condition* (1.2), (2.8) *and condition* (2.4) *with $s\geqq 4$. Then we have with the notations of* (1.13)

(2.10) $$\sup_{z\in\mathbb{R}} |P^n\{w_n<z\}-\chi_s^{(n)}(z,h)|\leqq c_s(\beta_s|\lambda_{a(s)}|^{-s})^6 n^{-(s-2)/2}$$

*where $c_s$ depends on $s$ and the constant $a$ of* (2.8).

(2.11) *Remark.* Let $P|\mathscr{A}$ denote the uniform distribution on $X=[0,1]$. Then (2.8) holds for every $s\geqq 4$ if $b_3(x)\leqq c$ $P$-a.e. (see (2.6)) and there exists constants $\delta$, $\eta>0$ such that for $P$-a.a. $x\in[0,1]$ there exists an interval $I_x\subset[0,1]$ of length larger than $\eta$ with $|\partial/\partial y\, h(y,x)|\geqq\delta$ for every $y\in I_x$.

(2.12) *Remark.* If all eigenvalues $\lambda_k$, $k\in\mathbb{N}$, have an even multiplicity, then $\hat{\chi}_s^{(n)}(t,h)$ is a meromorphic function on $\mathbb{C}$ with poles at $t=\lambda_k^{-1}/(2i)$, $k\in\mathbb{N}$. Under the assumptions of Theorem (2.10) it is not hard to see that the calculs of residues applied to the inversion formula for the d.f. yields (with integration over the contour consisting of $\mathbb{R}$ and an appropriately chosen semicircle in the half space $\operatorname{Im} t<0$ with radius going to infinity)

$$\chi_s^{(n)}(x,h)=1+\sum_{\lambda_k>0}\exp(-\lambda_k^{-1}x/2)\Big(\sum_{0\leqq 2i\leqq s-3} n^{-i}p_{2i,k}(x)\Big) \qquad \text{for } x>0$$

where $p_{2i,k}(x)$, $k\in\mathbb{N}$, denotes a polynomial in $x$ with coefficients depending on the eigenvalues $\lambda_j$, $j\in\mathbb{N}$, their multiplicity $2m_j$, $m_j\in\mathbb{N}$, $j\in\mathbb{N}$, and moments of $h$. For the case $s=3$ more details can be found in G.V. Martinov (1975, p. 788/9).

(2.13) *Example.* Let $P|\mathscr{A}$ denote the uniform distribution on $[0,1]$. Cramér-von Mises' $\omega^2$ statistic (1.8) is a statistic of type (1.1) with $h(x,y)=\frac{1}{2}(x^2+y^2)-\max(x,y)+\frac{1}{3}$. We have $\lambda_k=(k\pi)^{-2}$, $k\in\mathbb{N}$, pertaining to eigenvectors $e_k(t)=2^{1/2}\cos(\pi k x)$, $x\in[0,1]$. Hence

$$(2.14)\quad h(x,y)=\sum_{k=1}^{\infty} e_k(x)\,e_k(y)\pi^{-2}k^{-2}.$$

See E.R. Hansen (1975, p. 266, 43.1.9). Since $h(x,y)$ fulfills the conditions of Remark (2.11), the results (2.3) and (2.9) apply to this kernel for every $s\geqq 4$.

Furthermore $R(t,h)(x,y)=1-\mu\sinh^{-1}(\mu)\cosh[\mu(1-\max(x,y))]\cosh[\mu\min\cdot(x,y)]$ where $\mu=(-2it)^{1/2}$. See E.R. Hansen (1975, p. 243, 17.3.7). The relation (1.14) together with $\hat{\chi}(t,h)=(\mu\sinh^{-1}\mu)^{1/2}$ (see T.W. Anderson and D.A. Darling (1952, p. 200, 4.26)) imply after some elementary computations

$$\hat{\chi}_6^{(n)}(t,h)=(\mu\sinh^{-1}\mu)^{1/2}[1+n^{-1}(\tfrac{1}{12}-\tfrac{1}{144}\mu^2-\tfrac{1}{36}\mu\sinh^{-1}\mu-\tfrac{1}{32}\mu^2\sinh^{-2}\mu-\tfrac{7}{288}\mu\coth\mu)].$$

Expanding $\sinh^{-k/2}\mu$ in a power series in $\exp(-2\mu)$ and using the fact that $z^{\nu/4}\exp\left[-(2z)^{1/2}\left(2k+\frac{j}{2}\right)\right]$ is the one sided Laplace Transform of

$$(2\pi)^{-1/2}2^{-\nu/4}x^{-\nu/4-1}\exp\left[-\left(2k+\frac{j}{2}\right)^2/4x\right]D_{1+\nu/2}\left[\left(2k+\frac{j}{2}\right)x^{-1/2}\right],\quad x\geqq 0,$$

where $D_\nu(x)$ denotes the parabolic cylinder function (see F. Oberhettinger and L. Badii (1973, p. 259 (5.94))) we obtain

$$(2.15)\quad \chi_6^{(n)}(x,h)=\pi^{-1/2}x^{-1/4}\sum_{k=0}^{\infty}(-1)^k\Big\{u_{k,00}(x)\binom{-1/2}{k}-n^{-1}\sum_{\mu=0}^{2}\sum_{\nu=0}^{2}p_{k,\mu\nu}(x)\,u_{k,\mu\nu}(x)\Big\}$$

where

$$u_{k,\mu\nu}(x)=D_{-1/2+\nu}(x_{k,\mu})\exp(-x_{k,\mu}^2/4),\quad x_{k,\mu}=(2k+\tfrac{1}{2}+\mu)x^{-1/2},$$

$$p_{k,00}=-\tfrac{1}{12}\binom{-1/2}{k},\qquad p_{k,01}=p_{k,21}=\tfrac{7}{288}x^{-1/2}\binom{-3/2}{k},$$

$$p_{k,11}=\tfrac{1}{18}x^{-1/2}\binom{-3/2}{k},\qquad p_{k,02}=\tfrac{1}{144}x^{-1}\binom{-1/2}{k},$$

$$p_{k,22}=\tfrac{1}{8}x^{-1}\binom{-5/2}{k}$$

and $p_{k,\mu\nu}\equiv 0$ for the remaining indices $\mu,\nu$.

Note that the $n^0$-term in (2.15) is the power series of T.W. Anderson and D.A. Darling (1952, p. 202, (4.35)).

(2.16) *Example.* Let $U_n$ denote the goodness-of-fit statistic (1.9) for a circle considered by G.S. Watson (1961). Here $h(x,y)=\frac{1}{2}(|x-y|-\frac{1}{2})^2-\frac{1}{24}$ for $x,y\in[0,1]$ and $P|\mathscr{A}$ is the uniform distribution on $[0,1]$. Furthermore, the eigenvalues $\lambda_k=(2\pi k)^{-2}$, $k\in\mathbb{N}$, have multiplicity 2 and $\hat{\chi}(t,h)=\lambda\sinh^{-1}\lambda$, where $\lambda=(-it/2)^{1/2}$. This is the c.f. of the distribution function $\theta_3(\frac{1}{2},2\pi ix)$, $x\geqq 0$, where $\theta_3(s,2\pi ix)=1+2\sum_{k=1}^{\infty}\exp(-2\pi^2k^2x)\cos(2\pi ks)$ denotes the third Theta function. See G.S. Watson (1961, p. 112, (22)). By Remark (2.11), the results (2.3) and (2.9) apply in this case for every $s\geqq 4$. Since $\sin(2\pi kx)$ and $\cos(2\pi kx)$ are both eigenfunctions pertaining to the eigenvalues $\lambda_k$ we obtain by a Fourier expansion similarly as in Example (2.13): $R(t,h)(x,y)=1-\lambda\sinh^{-1}(\lambda)\cosh[(2|x-y|-1)\lambda]$. Hence, we obtain after some computations

$$\hat{\chi}_6^{(n)}(t,h)=\hat{\chi}(t,h)\left(1+n^{-1}\left\{\tfrac{1}{12}[1-(\lambda\sinh^{-1}\lambda)^2]-\tfrac{1}{36}\lambda^2\right\}\right)$$

which yields by the method of Remark (2.12)

$$(2.17)\quad \chi_6^{(n)}(x,h)=\left\{1+n^{-1}\left[\tfrac{1}{12}(\tfrac{1}{12}-5x)\frac{\partial}{\partial x}-\tfrac{1}{6}x^2\frac{\partial^2}{\partial x^2}\right]\right\}\theta_3(\tfrac{1}{2},2\pi ix),\qquad x\geqq 0.$$

Notice that Jacobi's identity for Theta functions implies

$$\theta_3(\tfrac{1}{2},2\pi ix)=(2\pi x)^{-1/2}\,2\sum_{k=1}^{\infty}\exp[-(k+\tfrac{1}{2})^2/2x],\qquad x>0.$$

See R. Bellman (1961, p. 26, (19.2)). This is a rapidly converging power series for small values of $x$.

The following tables compare the percentage points of the expansion (2.17) with the exact values of the distribution of $U_n$ in Example (2.16) for various $n$ obtained by Monte Carlo methods. See M.A. Stephens (1963, p. 311, Table 4 and 1964, p. 394, Table 1). Unfortunately the exact percentage points in Table 1 are accurate to three digits only.

**Table 1.** Upper tail percentage points for $U_n$
Significance level (%)

| $n$ | 0.5 | 1 | 2.5 | 5 | 10 | |
|---|---|---|---|---|---|---|
| 5 | 0.262 | 0.238 | 0.205 | 0.177 | 0.148 | exact |
| | 0.267 | 0.243 | 0.208 | 0.179 | 0.148 | Appr. |
| 10 | 0.283 | 0.254 | 0.213 | 0.182 | 0.150 | exact |
| | 0.285 | 0.255 | 0.214 | 0.183 | 0.150 | Appr. |
| 20 | 0.293 | 0.261 | 0.217 | 0.185 | 0.151 | exact |
| | 0.294 | 0.261 | 0.218 | 0.185 | 0.151 | Appr. |
| $\infty$ | 0.302 | 0.267 | 0.221 | 0.187 | 0.152 | |

**Table 2.** Lower tail percentage points for $U_n$
Significance level (%)

| $n$ | 0.1 | 0.5 | 1 | 2.5 | 5 | 10 | |
|---|---|---|---|---|---|---|---|
| 5 | 0.01860 | 0.02101 | 0.02281 | 0.02638 | 0.03041 | 0.03610 | exact |
| | 0.01897 | 0.02135 | 0.02311 | 0.02647 | 0.03021 | 0.03573 | Appr. |
| 7 | 0.01667 | 0.01998 | 0.02200 | – | – | – | exact |
| | 0.01723 | 0.02009 | 0.02201 | – | – | – | Appr. |
| $\infty$ | 0.01429 | 0.01763 | 0.01971 | 0.02340 | 0.2736 | 0.03306 | |

## 3. Lemmas

Let $h_k: X^2 \to \mathbb{R}$, $k \in \mathbb{N}$, be a sequence of symmetric functions

(3.1) $h_k(x, y) = \sum_{i,j=1}^{k} b_{ij,k} 1_{A_{i,k}}(x) 1_{A_{j,k}}(y)$ where

$$b_{ij,k} \in \mathbb{R}, \quad A_{i,k} \in \mathscr{A}, \; X = \sum_{i=1}^{k} A_{i,k}$$

such that

(3.2) $h_k$ converges to $h$ for $k \to \infty$ in the strong $L^s(X^2, P^2 + P * d)$ topology, where $d(x) = (x, x) \in X^2$ and $s \geqq 3$.

Let

(3.3) $T_k(x) = (1_{A_{i,k}} - P(A_{i,k}))_{i=1,\ldots,k}$.

Furthermore, let $B_k := (b_{ij,k})_{i,j=1,\ldots,k}$ and let $\Phi_k | \mathscr{B}^k$ denote the $k$ dimensional centered normal distribution with covariance $C_k := \operatorname{cov}(P * T_k)$.

Let $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product in $\mathbb{R}^k$. Using assumption (1.2) and (3.2) we may choose $h_k$ (see the definition of $h'$ below) such that in addition to (3.1) and (3.2)

(3.4) $\int h_k(x, y) P(dy) = 0$ holds for every $x \in X$. Hence,

(3.5) $h_k(\cdot, \cdot) = \langle B_k T_k(\cdot), T_k(\cdot) \rangle$.

For a symmetric function $h$: $X^2 \to \mathbb{R}$ let $h''(x, y) = h(x, y) 1_{A_n}$, where $A_n = \{(x, y) : |h(x, y)| \leqq n^{1/2}\}$.

Define

$$h'(x, y) = h''(x, y) - \int (h''(x, \cdot) + h''(\cdot, y)) \, dP + \int h'' \, dP^2.$$

It is not hard to see that (3.1), (3.2) and (3.4) are fulfilled for $h'_k(\cdot, \cdot)$, $k \in \mathbb{N}$, and that there exists a symmetric $k \times k$ matrix $B'_k$ such that

(3.6) $h'_k(\cdot, \cdot) = \langle B'_k T_k(\cdot), T_k(\cdot) \rangle$.

Furthermore, Čebyšev's inequality together with (3.5) implies for every $(x, y) \in X^2$

(3.7) $|h(x, y) - h'(x, y)| \leqq c n^{-(r-1)/2} [|h(x, y)|^r + \|h(\cdot, x)\|_s^r + \|h(y, \cdot)\|_s^r + \beta_s^{r/s}]$

where $\|h(\cdot, x)\|_s = [\int |h(\cdot, x)|^s dP]^{1/s}$. For $\beta_s$ see (1.3).
Note that (3.7) holds for $h'_k$ as well, since

(3.8) $\int |h_k(\cdot, \cdot)|^s dP^2 + \int |h_k \circ d(\cdot)|^s dP \leqq 2\beta_s \quad$ for $k \geqq k_0$.

Finally note that

(3.9) $|h'(x, y)| \leqq 4 n^{1/2} \quad$ for every $(x, y) \in X^2$.

The same relation holds for $h'_k$.

For reasons of simplicity we shall assume by now that $\beta_s = 1$.

3.10 **Lemma.** *Let* $\underline{v} \in (\mathbb{R}^k)^m$, $\underline{\varepsilon} \in \mathbb{R}^m$ *and* $\underline{\varepsilon} \cdot \underline{v} := \sum_{i=1}^{m} \varepsilon_i v_i$. *Then*

(i)

(3.11) $$\int \exp[it\langle B_k(u + \underline{\varepsilon} \cdot \underline{v}), (u + \underline{\varepsilon} \cdot \underline{v})\rangle] \Phi_k(du) (P * T_k)^m (d\underline{v})$$
$$= \hat{\chi}(t, h_k) \int \exp\left[\frac{1}{2} \sum_{i,j=1}^{m} R(t, h_k)(x_i, x_j) \varepsilon_i \varepsilon_j\right] P^m(d\underline{x})$$

*where* $R(t, h_k)(x, y)$ *is defined in* (1.11).

(ii) *for all t fulfilling* $|t| \leqq n^{(s-2)/10}$ *we have*

$$|\hat{\chi}(t, h) - \hat{\chi}(t, h')| \leqq c |t| (1 + |t|^4) n^{-(s-2)/2} |\hat{\chi}(t, h)|.$$

(iii) $|\hat{\chi}_s^{(n)}(t, h)| \leqq c(1 + |t|^{6(s-3)}) |\hat{\chi}(t, h)|$.

(iv) $|\hat{\chi}_s^{(n)}(t, h) - \hat{\chi}_s^{(n)}(t, h')| \leqq c(|t| + |t|^{6(s-3)+3}) n^{-(s-2)/2} |\hat{\chi}(t, h)|$.

(3.12) *Remark.* $\lim_{k \to \infty} \hat{\chi}_s^{(n)}(t, h_k) = \hat{\chi}_s^{(n)}(t, h)$.

*Proof.* (i) Let $I_k$ denote the l.h.s. of (3.11). For $P^m$-a.a. $(x_1, \ldots, x_m)$ there exists a vector $b \in \mathbb{R}^k$ such that $\sum_{i=1}^{m} \varepsilon_i T_k(x_i) = C_k^{1/2} b$. Introducing $v = C_k^{-1/2} u$ as new variable we obtain (w.l.g. let $C_k$ be nonsingular)

(3.13) $I_k = \int \exp[it\langle D_k(v+b), (v+b)\rangle] \varphi_{\mathrm{Id}}(v) d^k v$

where $D_k = C_k^{1/2} B_k C_k^{1/2}$, $d^k v$ denotes the $k$-dimensional Lebesgue measure and $\varphi_{\mathrm{Id}}(v)$ denotes the density of the $k$-dimensional standard normal distribution. Assume that the symmetric matrix $D_k$ has eigenvalues $\mu_i$, $i = 1, \ldots, k$, and denote by $z_i$ resp. $a_i$, $i = 1, \ldots, k$, the coordinates of $v, b \in \mathbb{R}$ with respect to an orthonormal basis of eigenvectors of $D_k$. Hence $\langle D_k(v+b), (v+b)\rangle = \sum_{i=1}^{k} \mu_i (z_i + a_i)^2$ and by

G.V. Martinov (1975, p. 790, (18))

$$(3.14)\quad I_k = \exp\left[it\sum_{j=1}^{k}\mu_j\right]\left[\prod_{j=1}^{k}(1-2it\mu_j)\exp(2it\mu_j)\right]^{-1/2} \times \exp\left[it\sum_{j=1}^{k}\mu_j(1-2it\mu_j)^{-1}a_j^2\right].$$

The first factor of (3.14) is equal to $\exp[it\int h_k(x,x)P(dx)]$. (See (3.5).) Since $\mu_j$, $j=1,\dots,k$, are the eigenvalues of the Hilbert-Schmidt operator $H_k: L^2(X,P)\circlearrowleft$ with kernel $h_k$ we obtain

$$(3.15)\quad I_k = \hat{\chi}(t,h_k)\exp[it\langle D_k(1-2itD_k)^{-1}b,b\rangle].$$

Since by definition of $D_k$ and $b$ (see (1.11))

$$(3.16)\quad it\langle D_k(1-2itD_k)^{-1}b,b\rangle = it\langle B_k(1-2itC_kB_k)^{-1}C_k^{1/2}b, C_k^{1/2}b\rangle = \sum_{i,j=1}^{m} R(t,h_k)(x_i,x_j)\varepsilon_i\varepsilon_j,$$

this together with (3.15) proves part (i).

(ii) Denote by $\mu_j'$, $j=1,\dots,k$, the eigenvalues of the operator on $L^2(X,P)$ induced by the kernel $h_k'$, $k\geqq k_0$ (see 3.8). Using $|\exp(x)-1|\leqq|x|\exp(|x|)$, $x\in\mathbb{C}$, we have

$$(3.17)\quad |\hat{\chi}(t,h_k)-\hat{\chi}(t,h_k')|\leqq|J_k|\exp(|J_k|)\,|\hat{\chi}(h_k,t)|$$

where

$$J_k = \frac{1}{2}\sum_{i=1}^{k}[\log(1-2it\mu_i)-\log(1-2it\mu_i')]$$

$$=(-i)\int_0^t\sum_{i=1}^{k}[\mu_i(1-2ir\mu_i)^{-1}-\mu_i'(1-2ir\mu_i')^{-1}]\,dr$$

$$(3.18)\quad = -\frac{1}{2}t^{-1}\int_0^t\int(R(r,h_k)-R(r,h_k'))(x,x)P(dx)\,dr.$$

Since by (1.11)

$$(3.19)\quad R(t,h)(\cdot,\cdot) = 2it(h(\cdot,\cdot)+\int h(\cdot,z)R(t,h)(\cdot,z)P(dz))$$

we have

$$(3.20)\quad (R(t,h)-R(t,h'))(x,y) = 2it((h-h')(x,y) + \int[R(t,h)(x,z)(h-h')(z,y)+R(t,h')(z,y)(h-h')(z,x) + R(t,h)(x,z)(h-h')(z,w)R(t,h')(w,y)]\,P^2(dz,dw)).$$

Therefore (3.7) implies

$$(3.21)\quad \begin{aligned}|\int (R(t,h_k)-R(t,h'_k))(x,x)\,P(dx)| &\leqq |2t|\,n^{-(s-1)/2}\\ &\quad +|2t|\,|\int (h_k-h'_k)(z,x)\,[(R(t,h_k)+R(t,h'_k))(z,x)\\ &\quad +R(t,h_k)(z,w)\,R(t,h'_k)(w,x)]\,P^3(dz,dw,dx)|\\ &\leqq |2t|\,[n^{-(s-2)/2}+n^{-(s-2)/2}(\gamma_s+\gamma'_s+\gamma_s\gamma'_s)]\end{aligned}$$

where $\gamma_s^s=\int |R(t,h_k)|^s\,dP^2$ and $\gamma_s'^s=\int |R(t,h'_k)|^s\,dP^2$. Here we have used (3.7) with $r=s-2$, (3.8) and Hölder's inequality.

Since $\|(\mathrm{Id}-2itH_k)^{-1}\|\leqq 1$ for any symmetric operator, where $\|\cdot\|$ denotes the supremum operator norm in the Hilbert space $L^2(X,P)$, it follows from (1.11) that

$$(3.22)\quad |R(t,h)(x,y)|\leqq |2t|\,|h(x,y)|+|2t|^2\,\|h(\cdot,x)\|_s\,\|h(\cdot,y)\|_s.$$

Hence

$$\gamma'_s+\gamma_s\leqq c\,|t|\,(1+|t|).$$

The relations (3.18) and (3.21) imply

$$(3.23)\quad |J_k|\leqq c\,|t|\,(1+|t|^4)\,n^{-(s-2)/2}$$

thus proving part (ii) for $h_k$ and $h'_k$, $k\geqq k_0$.

The relation $\lim_{k\to\infty}\hat{\chi}(t,h_k)=\hat{\chi}(t,h)$ can be proved along about the same line as (3.17)-(3.23), but it also follows from the continuity of $h\to\hat{\chi}(t,h)$ in the strong $L^s(X^2,P^2+P*d)$ topology. See N. Dunford and J.T. Schwarz (1963, XI, p. 1036, Theorem 2.6). This proves part (ii).

From (3.17) with $h'_k\equiv 0$ and $h_k$ replaced by $h$ we obtain by (3.18)

$$(3.24)\quad |\hat{\chi}(t,h)|\leqq \exp\left[-2\int_0^{|t|}\sum_{j=1}^{\infty}\lambda_j^2(1+4r^2\lambda_j^2)^{-1}\,r\,dr\right].$$

This proves after some elementary computations the examples (2.2).

(iii) Let $\alpha=(\alpha_1,\dots,\alpha_m)$, $2\leqq\alpha_i\leqq s-1$, $m\leqq 3(s-2)/2$ and

$$(3.25)\quad \sum_{i=1}^{m}(\alpha_i-2)\leqq s-3.$$

Then it is sufficient to estimate $\gamma_\alpha(t,h)$ in (1.10). By the generalized Hölder inequality we obtain after integration in $x_1$

$$(3.26)\quad \begin{aligned}|\gamma_\alpha(t,h)|&\leqq c\sum{}^*\,\delta_s^{r_{11}}\int|\prod_{i,j\neq 1}R(t,h)(x_i,x_j)^{r_{ij}}|\\ &\quad\times\|R(t,h)(x_2,\cdot)\|_s^{r_{12}+r_{21}}\dots\|R(t,h)(x_m,\cdot)\|_s^{r_{1m}+r_{m1}}\,P^{m-1}(d\underline{x})\end{aligned}$$

where $\delta_s^s=\int|R(t,h)(x_1,x_1)|^s\,P(dx_1)$.

Hence after $m-1$ further integrations we obtain by (1.11) and (3.22)

(3.27) $|\gamma_\alpha(t,h)| \leqq c(\delta_s+\gamma_s)^{|\alpha|} \leqq c\,|t|^{|\alpha|}(1+|t|)^{|\alpha|}$

thus proving part (iii).

(iv) By (1.11), part (ii) and (iii) it is sufficient to estimate $\gamma_\alpha(t,h)-\gamma_\alpha(t,h')$. Hence, it is sufficient to consider the integrals

(3.28) $$\int (R(t,h)-R(t,h)-R(t,h'))(x_{i_0},x_{j_0})\,\Pi^* R(t,h)(x_i,x_j)^{\bar r_{ij}} \times \Pi^{**} R(t,h)(x_i,x_j)^{\bar{\bar r}_{ij}} P^m(d\underline{x})$$

where $\bar r_{ij}$, $\bar{\bar r}_{ij} \in \mathbb{N}_0$ and $\Pi^*, \Pi^{**}$ denote products over certain subsets of $i,j=1,\dots,m$.

Using (3.20) and (3.7) with $r=\min\{s+1-\alpha_i\colon i=1,\dots,m\}$ we may proceed as in (3.26)-(3.27) and obtain the following estimate for (3.28)

$$c n^{-(r-1)/2}(|t|^{|\alpha|}+|t|^{2|\alpha|+3}).$$

Since $\gamma_\alpha(t,h)$ occurs in $\hat\chi_s^{(n)}(t,h)$ with coefficients of order $n^{-\alpha_0/2}$ where $\alpha_0=\sum_{i=1}^m(\alpha_i-2)\leqq s-3$, and $r+\alpha_0\geqq s-2$, part (iv) is proved.

Exactly in the same way one shows that $\lim_{k\to\infty}\gamma_\alpha(t,h_k)=\gamma_\alpha(t,h)$, thus proving Remark (3.12).

The following three lemmas are basic for the proof of our results.

Let $H$ be a separable normed linear space. Let $\|\cdot\|$ denote a continuous seminorm defined on $H$.

Assume that there exists a constant $\gamma>0$ such that for every integer $n$ and vectors $x_1,\dots,x_n\in H$

(3.29) $\int \|\eta_1 x_1+\dots+\eta_n x_n\|^2 N^n(d\underline{\eta}) \leqq \gamma(\|x_1\|^2+\dots+\|x_n\|^2)$,

where $N|\mathscr{B}^1$ denotes the standard normal distribution on $\mathbb{R}$. (A Banach space is called convex of type 2 if its norm fulfills condition (3.29))

(3.30) **Lemma.** (i) *Let $\Phi|\mathscr{B}$ denote a centered Gaussian p-measure defined on the Borel $\sigma$-field $\mathscr{B}$ of $H$. Then we have for all $\alpha\geqq 0$*

$$\int \exp(\alpha\|x\|)\,\Phi(dx) \leqq \exp(c_1\alpha^2 m_2+c_2) \text{ with constants } 0<c_1,\ c_2<e^{27},$$

*where $\|\cdot\|$ denotes an arbitrary continuous seminorm and $m_i=\int\|x\|^i\,\Phi(dx)$, $i=1,2$.*

(ii) *Assume that $Q_i|\mathscr{B}$ are mean zero p-measures fulfilling $Q_i\{x\colon\|x\|\geqq b\}=0$ for $i=1,\dots,n$ and some $b>0$. Furthermore, assume that condition (3.29) is fulfilled. Then there exists a constant $0<c<\exp(28)$ such that*

(3.31) $$\int \exp(\alpha\|x_1+\dots+x_n\|)\,Q_1(dx_1)\dots Q_n(dx_n) \leqq \exp(\exp(c\gamma\alpha^2 b^2)\int(\|x\|^2+1)(Q_1+\dots+Q_n)(dx)+c),\quad \alpha\geqq 0.$$

*Proof.* (i) M.X. Fernique (1970) proved

(3.32) $$\Phi\{x\colon\|x\|>r\} \leqq q(u)\exp\left[-\tfrac{1}{24}r^2u^{-2}\ln\frac{q(u)}{1-q(u)}\right]$$

for $r>u>0$ and $q(u)=\Phi\{x: \|x\| \leqq u\}>1/2$. Let $u=\exp(13)m_2^{1/2}$, then $q(u)\geqq 1-\exp(-26)$. By partial integration and (3.32) we have

$$\begin{aligned}\int \exp(\alpha r)\, dq(r) &\leqq 1+\int (1_{[0,u]}(r)+(1-q(u))\,1_{[u,\infty)}(r))\alpha \exp(\alpha r)\, dr\\ &\leqq 1+\alpha u[\exp(\alpha u)+\int \exp(\alpha u r-r^2)\, dr]\\ &\leqq \exp(c_1 m_2 \alpha^2+c_2)\end{aligned}$$

for some constants $0<c_1,\ c_2<\exp(27)$.

(ii) Let $P_n|\mathscr{B}^n=Q_1\times\ldots\times Q_n|\mathscr{B}^n$ and denote the l.h.s. of (3.31) by $d_n(\alpha)$. Then

$$d_n(\alpha)\leqq\int \exp((2\pi)^{1/2}\alpha\,\|\eta_1 x_1+\ldots+\eta_n x_n\|)\, P_n(d\underline{x})\, N^n(d\underline{\eta}). \tag{3.33}$$

In order to prove this inequality, let $F|\mathscr{B}^1$ denote the $p$-measure with $F\{\varepsilon=\pm 1\}=1/2$. Since $N\times F\{(\eta,\varepsilon): |\eta|\,\varepsilon<z\}=N\{\eta: \eta<z\}$ for all $z\in\mathbb{R}$, we can replace the integration in (3.33) with respect to $N^n$ by $N^n\times F^n$. Therefore, Jensen's inequality together with $\int |x|\, N(dx)=2(2\pi)^{-1/2}$ imply that the r.h.s. of (3.33) is larger than

$$\int \exp(2\alpha\,\|\varepsilon_1 x_1+\ldots+\varepsilon_n x_n\|)\, P_n(d\underline{x})\, F^n(d\varepsilon).$$

By $\int \|x+a\|\, Q_i(dx)\geqq \|a\|$ and Jensen's inequality we obtain for every $n\geqq k\geqq 1$

$$\begin{aligned}&\int \exp(2\alpha\,\|x_1+\ldots+x_k-x_{k+1}\ldots-x_n\|)\, P_n(d\underline{x})\\ &\geqq \max\{\int \exp(2\alpha\,\|x_1+\ldots+x_k\|)\, P_n(d\underline{x}),\ \int \exp(2\alpha\,\|x_{k+1}+\ldots+x_n\|)\, P_n(d\underline{x})\}.\end{aligned} \tag{3.34}$$

By Jensen's inequality the r.h.s. of (3.34) is larger than $d_n(\alpha)$, thus proving the inequality (3.33).

Let $x_1,\ldots,x_n$ denote fixed vectors in $H$. We apply part (i) of this Lemma to the Gaussian $p$-measure $\Phi|\mathscr{B}$ induced by $N^n|\mathscr{B}^n$ and the map $\underline{\eta}\to\eta_1 x_1+\ldots+\eta_n x_n$ and obtain by (3.33) and (3.29)

$$d_n(\alpha)\leqq\int \exp(c\gamma\alpha^2(\|x_1\|^2+\ldots+\|x_n\|^2)+c)\, P_n(d\underline{x}),\ \ c=2\pi c_1+c_2.$$

Hence,

$$\begin{aligned}\int \exp(c\gamma\alpha^2\,\|x\|^2)\, Q_i(dx) &\leqq 1+\exp(c\gamma b^2\alpha^2)\, m_{2,i}\\ &\leqq \exp(\exp(c\,\alpha^2 b^2\gamma)\, m_{2,i}),\end{aligned}$$

where $m_{2,i}=\int \|x\|^2\, Q_i(dx)$, concludes the proof of part (ii).

The assumption (3.29) holds for $H=\mathbb{R}^k$ and the seminorm

$$\|x\|_s=(\int |\langle x, B_k u\rangle|^s (P*T_k)(du))^{1/s},\qquad s\geqq 3$$

with $\gamma$ depending only on $s$, since the spaces $L^s(X,P)$, $s\geqq 2$ are of type 2. See e.g. N.C. Jain (1975, p. 120, Corollary 3.3 and p. 118, Theorem 3.1). These remarks also apply to the seminorm $\|\cdot\|_s'$ which is defined by replacing $B_k$ by $B_k'$ in the definition above.

Let $H'|\mathscr{B}^k = H * t_n|\mathscr{B}^k$, where $t_n(x)=x$ if $\|x\|_s \leqq n^{1/2}$ and $t_n(x)=0$ otherwise, denote the truncation of a $p$-measure $H|\mathscr{B}^k$ at $n^{1/2}$.

Define $Q_k|\mathscr{B}^k = P * T_k|\mathscr{B}^k$. Denote by $Q'_{k,n}|\mathscr{B}^k$ resp. $Q_{k,n}|\mathscr{B}^k$ the convolution of $n$ copies of $Q'_k(n^{1/2}\cdot)|\mathscr{B}^k$ resp. $Q_k(n^{1/2}\cdot)|\mathscr{B}^k$. The following remark is a consequence of Lemma (3.30) and of the inequality $\|m'_n\|_s \leqq n^{-1/2}(2\beta_s)^{1/s}$ for $k \geqq k_0$ (see (3.8)), where $m'_n$ denotes the mean of $Q'_k$.

(3.35) *Remark.* For every integers $p$ and $s \geqq 3$ there exist constants $c(p)$ and $c(s)$, independent of $n$ and $k$, such that for $k \geqq k_0$

(i) $(Q'_{k,n}+\Phi_k)\{x \in \mathbb{R}^k \colon \|x\|_s > c(s)\log n\} \leqq n^{-6(s-2)}$.

A similar result holds for $\|x\|'_s$ and $Q_{k,n}|\mathscr{B}^k$.

(ii) $\int \|x\|_s'^p (Q_{k,n}+\Phi_k)(dx) \leqq c(p)$.

Let $\bar{H}|\mathscr{B}^k$, $\bar{H}(A) = \int\limits_{x-y\in A} H^2(dx,dy)$, denote the symmetrization and let $H(v)$ $H(v) = \int \exp(i\langle v, x\rangle) H(dx)$ denote the c.f. of $H|\mathscr{B}^k$. Define

(3.36) $$\hat{\chi}_{i,n}(t;k;\underline{\varepsilon}) = \int \exp[it\langle B'_k(u+\underline{\varepsilon}\cdot\underline{v}), u+\underline{\varepsilon}\cdot\underline{v}\rangle]\, F_{i,n}(du)$$
$$(Q_k+\Phi_k)(dv_1)\, Q_k^{m-1}(dv_2,\dots,dv_m)$$

where $\underline{v}=(v_1,\dots,v_m)\in(\mathbb{R}^k)^m$, $\underline{\varepsilon}\in[0,1]^m$, $\underline{\varepsilon}\cdot\underline{v}$ is defined as in Lemma (3.10)(i) and $F_{i,n}|\mathscr{B}^k$, $i=0,\dots,n$, denotes the convolution of $i$ copies of $Q_k(n^{1/2}\cdot)|\mathscr{B}^k$ with $(n-i)$ copies of $\Phi_k(n^{1/2}\cdot)|\mathscr{B}^k$.

(3.37) **Lemma.** (i) *Let $H_i|\mathscr{B}^k$, $i=1,2$, denote two p-measures. Then*

$$\left|\int \exp it\langle B_k(u+w+v), u+w+v\rangle\, H_1(du)\, H_2(dw)\right| \leqq \left[\int \hat{\bar{H}}_1(2tB_k w)\, \bar{H}_2(dw)\right]^{1/4}$$

*for every $v \in \mathbb{R}^k$, $t \in \mathbb{R}$.*

*Define*

$$\hat{\chi}_{n,r}(t,h_k) = \inf_{1\leqq p \leqq [r/2]} \left[\int \hat{\bar{Q}}_k(2tn^{-1}B_k(v_1+\dots+v_p))^{r-p}\bar{Q}_k^p(dv_1,\dots,dv_p)\right]^{1/2}.$$

(ii) *Then we have for $\alpha=(\alpha_1,\dots,\alpha_m)$, $0 \leqq \alpha_i \leqq s$, $i=1,\dots,m$ and $k \geqq k_0$*

$$\sup\{|D^\alpha \hat{\chi}_{i,n}(t;k;n^{-1/2}\varepsilon_1,0,\dots,0)| \colon \varepsilon_1\in[0,1], i=0,\dots,n\}$$
$$\leqq c(|t|^{|\alpha|}+|t|^{2|\alpha|})\left[\hat{\chi}_{n,[n/2]}(t,h'_k)^{1/2} + \left|\hat{\chi}\left(\frac{t}{2},h'_k\right)\right|^{1/2}\right].$$

(iii) For $k \geqq k_0$ *and* $s \geqq 4$

$$|\hat{\chi}_n(t,h_k) - \hat{\chi}_n(t,h'_k)| \leqq cn^{-(s-2)/2}t^2(1+t^2)[\hat{\chi}_{n,n-4}(t,h'_k)^{1/4} + \hat{\chi}_{n,n-4}(t,h_k)^{1/4}]$$

*Proof.* (i) We shall prove a more general result than (i) by estimating the partial derivatives with respect to $\underline{\varepsilon}$:

$$D^{\alpha}[\int \exp[itB_k\langle u+w+\underline{\varepsilon}\cdot\underline{v}\rangle]\, H_1(du)H_2(dw)]|_{\varepsilon=(\varepsilon_1 n^{-1/2},0,\dots,0)}$$

where for reasons of simplicity $B_k\langle u\rangle=\langle B_k u,u\rangle$. The derivative of the integrand is a sum of terms of the type

$$(3.38)\quad (n^{-1/2}\varepsilon_1)^{\eta}(it)^r d_{(r_{ij})}\prod_{i=1}^{m}\langle B_k'(u+w),v_i\rangle^{(r_{0i}+r_{0i})}\prod_{i,j=1}^{m}\langle B_k' v_i,v_j\rangle^{r_{ij}}$$

where $\eta, d_{(r_{ij})}, r_{ij}\in\mathbb{N}_0$, $\eta\delta_{i1}+\alpha_i=\sum_{j=0}^{m}(r_{ij}+r_{ji})$, $i=1,\dots,m$ and $|\alpha|/2\leqq r\leqq|\alpha|$.

Hence, it is sufficient to estimate

$$(3.39)\quad \left|\int\prod_{i=1}^{m}\langle B_k' u,v_i\rangle^{r_i}\langle B_k' w,v_i\rangle^{s_i}\exp[itB_k'\langle u+w+\underline{\varepsilon}\cdot\underline{v}\rangle]\,H_1\times H_2(du,dw)\right|.$$

By Hölder's inequality (3.39) is smaller than

$$(3.40)\quad [\int|M_n(\underline{v},w)|^2 H_2(dw)]^{1/2}[\int\prod_i\langle B_k' w,v_i\rangle^{2s_i}H_2(dw)]^{1/2}$$

where $M_n(\underline{v},w)=\int\exp[itB_k'\langle u+w+\underline{\varepsilon}\cdot\underline{v}\rangle]\prod_i\langle B_k' u,v_i\rangle^{r_i}H_1(du)$. Interchanging the integrations in $w$ and $u$ in the first factor of (3.40) yields the following estimate of this factor

$$(3.41)\quad [\int\hat{H}_2(2tB_k'(u_1-u_2))\,\theta(u_1,u_2,\underline{v})\prod_i(\langle B_k' u_1,v_i\rangle\langle B_k' u_2,v_i\rangle)^{r_i}H_1^2(du_1,du_2)]^{1/2}$$

where $|\theta(u_1,u_2,\underline{v})|=1$.

Applying Hölder's inequality to (3.41) we obtain the estimate

$$(3.42)\quad [\int\hat{\bar{H}}_2(2tB_k' u)\,\bar{H}_1(du)]^{1/4}[\int\prod_i\langle B_k' u,v_i\rangle^{2r_i}H_1(du)]^{1/2}.$$

For $\alpha=0$, this together with (3.40) and (3.41) proves part (i).

(ii) Choosing $H_j$, $j=1,2$, equal to a convolution of an appropriate number of copies of $Q_k(n^{1/2}\cdot)|\mathscr{B}^k$ resp. $\Phi_k(n^{1/2}\cdot)|\mathscr{B}^k$ such that $H_1 * H_2|\mathscr{B}^k=F_{i,n}|\mathscr{B}^k$ we can estimate the first factor of (3.42) by

$$\hat{\chi}_{n,i}(t,h_k')^{1/2}\qquad\text{for } i\geqq[n/2] \text{ resp.}$$

$$(3.43)\quad \inf_{1\leqq m\leqq n-i}[\int\Phi_k(4tm^{1/2}n^{-1}B_k' u)^{n-i-m}\,\Phi_k(du)]^{1/4}$$

$$\leqq\left|\hat{\chi}\left(\frac{t}{2},h_k'\right)\right|^{1/2}\qquad\text{for } i<[n/2]$$

using $|\hat{Q}_k(v)|\leqq 1$ and with the notations of 3.17

$$(3.44)\quad \int \exp\left(-\tfrac{1}{2}\langle C_k B'_k x, B'_k x\rangle\right) \Phi_k(dx) = \prod_{i=1}^{k} (1+t^2 \mu_i'^2)^{-1/2} = \left|\hat{\chi}\left(\frac{t}{2}, h'_k\right)\right|^2.$$

By (3.36), (3.38), (3.40) and (3.42) we have to estimate

$$(3.45)\quad n^{-\eta/2} \int \prod_{i,j} \langle B'_k v_i, v_j\rangle^{r_{ij}} \left[\int \prod_i \langle B'_k u, v_i\rangle^{2r_i} \langle B'_k w, v_i\rangle^{2s_i} H_1(du) H_2(dw)\right]^{1/2} (Q_k+\Phi_k)(dv_1)\, Q_k^{m-1}(dv_2, \ldots, dv_m)$$

where

$$(3.46)\quad \sum_{j=1}^{k} (r_{ij}+r_{ji}) + r_i + s_i = \alpha_i + \eta\,\delta_{i1}, \qquad i=1,\ldots,m,$$

and $H_i$, $i=1,2$, are chosen as in the previous argument. Consider first the integration in (3.45) with respect to $\Phi_k(dv_1)$:

$$(3.47)\quad \int \prod_i \langle B'_k v_1, v_i\rangle^{2(r_{1i}+r_{i1})} \langle B'_k u, v_1\rangle^{2r_1} \langle B'_k w, v_1\rangle^{2s_1} \Phi_k(dv_1) \leqq c \prod_{i=2}^{m} \|v_i\|_s'^{2(r_{1i}+r_{i1})} \|u\|_s'^{2r_1} \|w\|_s'^{2s_1}$$

using the generalized Hölder inequality together with

$$(3.48)\quad \int \langle B'_k u, v_1\rangle^{2r} \Phi_k(dv_1) = c\langle C_k B'_k u, B'_k u\rangle^r \leqq c\|u\|_s'^{2r}$$

(see (3.35)) and

$$(3.49)\quad \int \langle B'_k v_1, v_1\rangle^{2r} \Phi_k(dv_1) \leqq c(r).$$

With the notations of (3.13) this integral can be written as

$$\int \left(\sum_{i=1}^{k} \mu'_i x_i^2\right)^{2r} \varphi_{\mathrm{Id}}(\underline{x})\, d^k \underline{x} = \sigma(\mu'_1, \ldots, \mu'_k).$$

Since $\sigma(\cdot)$ is a symmetric homogenous polynomial of degree $2r$ in the variables $\mu'_1, \ldots, \mu'_k$ and coefficients *depending on* $2r$ *only*, it is well known that $\sigma(\cdot)$ can be written as homogeneous polynomial with weighted degree $2r$ and coefficients *not depending on* $k$ in the variables $\sum_{i=1}^{k} \mu_i'^p$, $p=1,\ldots,2r$. Hence,

$$\left|\sum_{i=1}^{k} \mu_i'^p\right| \leqq \left[\left(\int (h'_k \circ d)\, dP\right)^2 + \int h_k'^2\, dP^2\right]^{p/2},$$

(3.7), (3.8) and $\beta_s=1$ imply (3.49).

Using Hölder's inequality in (3.45) and (3.47), we integrate with respect to $H_1(du)$ and $H_2(dw)$. These integrals are bounded by constants. See Remark

(3.35)(ii). Hence, the second summand of (3.45) is smaller than

(3.50) $$c n^{-\eta/2} \int \Big| \prod_{i,j \neq 1} \langle B_k' v_i, v_j \rangle^{r_{ij}} \Big| \prod_i \|v_i\|_s^{\prime r_{1i}+r_{i1}} Q_k^{m-1}(dv_2, \ldots, dv_m).$$

Since $n^{-1/2}|\langle B_k' v_i, v_j\rangle|$ resp. $n^{-1/2}\|v_i\|_s'$ are bounded, $Q_k^2$ resp. $Q_k$-a.e. by 4, the relation (3.50) together with $\alpha_i \leqq s$ for $i=1, \ldots, m$ implies as in the prof of Lemma (3.10)(iii) that this part of (3.45) can be estimated by a constant.

Next consider integration in (3.45) with respect to $Q_k(dv_1) \ldots Q_k(dv_m)$. The inner integral in (3.45) with respect to $H_1(du)$ and $H_2(dw)$ can be estimated for $r \geqq 1$ with the help of

(3.51) $$\int \langle B_k' u, v_i \rangle^{2r} H_1(du) \leqq c \|v_i\|_s^{\prime 2}(1+\|v_i\|_s^{\prime 2r-2}) \quad Q_k\text{-a.e.}$$

If $H_1$ is a convolution of copies of $Q_k(n^{1/2} \cdot)|\mathscr{B}^k$ this can be proved along about the line of F. Götze and C. Hipp (1978, p. 76, Lemma 4.6) since $|\langle B_k' u, v_i\rangle| \leqq 4n^{1/2}$ $Q_k^2$-a.e. and

$$\int \langle B_k' u, v_i \rangle^2 Q_k(du) \leqq \|v_i\|_s^{\prime 2}.$$

If $H_1$ is a convolution of copies of $\Phi_k(n^{1/2} \cdot)$ see (3.48). The mixed case can be obtained from the inequality

$$|a+b|^{2r} \leqq 2^{2r-1}(|a|^{2r}+|b|^{2r}).$$

Finally, the integration with respect to $Q_k^m(d\underline{v})$ can be treated by the arguments used before.

Counting the minimal and maximal number of factors $|t|$ the estimation of (3.45) by a constant together with (3.42), (3.43) and (3.44) proves part (ii) of Lemma (3.37).

(iii) Let $i, j, m, l \in \{1, \ldots, n\}$ and define $(i,j) \leqq (m,l)$ iff $i<m$ or $i=m$, and $j \leqq l$. Then

(3.52) $$\hat{\chi}_n(h_k, t) - \hat{\chi}_n(h_k', t) = \sum_{j,l=1}^n \hat{\chi}_{jl}(x_j, x_l)[1-\exp(itn^{-1}(h'-h)(x_j, x_l))]\, P(dx_j, dx_l)$$

where

$$\hat{\chi}_{jl}(x_j, x_l) = \int \exp\Big[itn^{-1} \sum_{m,p=1}^n h_{mp}^{jl}(x_m, x_p)\Big] \prod_{m \neq j,l} P(dx_m)$$

and

$$h_{mp}^{jl} = \begin{cases} h_k & \text{if } (m,p) \geqq (j,l) \\ h_k' & \text{otherwise.} \end{cases}$$

Consider first the case $j=l$. Then

(3.53) $$\sum_{j=1}^n \Big| \int \hat{\chi}_{jj}(x_j, x_j)[1-\exp(itn^{-1}(h'-h)(x_j, x_j))]\, P(dx_j) \Big| \leqq \Big( \sum_{j=1}^n \sup_{x_j} |\hat{\chi}_{jj}(x_j, x_j)| \Big) c|t| n^{-1} n^{-(s-1)/2}$$

using (3.7) with $r=s$ and $|1-\exp(x)|\leqq|x|$ for $x\in\mathbb{C}$, $\operatorname{Re}x=0$. The summands on the r.h.s. of (3.52) with $j\neq l$ can be estimated by

$$(3.54)\quad |\int \hat{\chi}_{jl}(x,y)(1+itn^{-1}(h-h')(x,y)-\exp[itn^{-1}(h'-h)(x,y)])\,P^2(dx,dy) \\ +|t|\,|\int \hat{\chi}_{jl}(x,y)(h'-h)(x,y)\,P^2(dx,dy)|.$$

Since $|1+ia-\exp(ia)|\leqq a^2/2$ for $a\in\mathbb{R}$, the first summand in (3.54) is smaller than

$$(3.55)\quad c\sup_{x,y}|\hat{\chi}_{jl}(x,y)|\,|t|^2 n^{-2} n^{-(s-2)/2}$$

using (3.7) with $r=s-2$.

Define the functions $\hat{\chi}_{j,l}(\varepsilon,\eta,x_j,x_l)$ by replacing $h^{jl}_{mp}(\cdot,\cdot)$ in the definition of $\hat{\chi}_{j,l}(x_j,x_k)$ by $\varepsilon_m\varepsilon_p h^{jl}_{mp}(\cdot,\cdot)$ with $\varepsilon_m=\varepsilon$ for $m=j$, $\varepsilon_m=\eta$ for $m=l$ and $\varepsilon_m=1$ otherwise. Note that $\hat{\chi}_{jl}(1,1,\cdot,\cdot)=\hat{\chi}_{jl}(\cdot,\cdot)$. Let $f(\varepsilon,\eta)$ denote the second summand of (3.54) with $\hat{\chi}_{jl}(\cdot,\cdot)$ replaced by $\hat{\chi}_{jl}(\varepsilon,\eta,\cdot,\cdot)$. Since $(D^{(i,0)}f)(\varepsilon,0)=(D^{(0,i)}f)(0,\eta)=0$ for every $\varepsilon,\eta\in\mathbb{R}$ and $i=0,1,2$ we obtain by twofold Taylor expansion

$$(3.56)\quad f(1,1)=(D^{(1,1)}f)(\delta,\delta)+\tfrac{1}{2}\delta[(D^{(2,1)}f)(\delta,\delta^*)+(D^{(1,2)}f)(\delta^*,\delta)]$$

for some $0\leqq\delta^*\leqq\delta\leqq 1$. Using similar arguments as in the proof of part (i) in order to estimate $(D^{\alpha}\hat{\chi}_{jl})(\varepsilon,\eta,\cdot,\cdot)$ for $\alpha=(1,1)$, $(2,1)$ and $(1,2)$ (for $4\leqq s<6$ use the Hölder inequality in (3.42) with exponent $\frac{1}{4}$) we obtain after some computations

$$(3.57)\quad |f(1,1)|\leqq cn^{-2}t^2(1+t^2)\,n^{-(s-2)/2}\,[\hat{\chi}_{n,n-4}(t,h_k)^{1/4}+\hat{\chi}_{n,n-4}(t,h'_k)^{1/4}]$$

using (3.50), (3.7), (3.42) and the estimate $[\hat{\chi}_{n,n-4}(t,h_k)^2+\hat{\chi}_{n,n-4}(t,h'_k)^2]^{1/8}$ for

$$\inf_{2\leqq m\leqq n}\left[\int\prod_{q=1}^{m}\left|\hat{Q}_k\left(2tn^{-1}\sum_{p=1}^{n-m}B_{qp}v_p\right)\right|^4\bar{Q}_k^{n-m}(dv_1,\ldots,dv_{n-m})\right]^{1/8}$$

where $B_{qp}=B_k$ resp. $=B'_k$ iff $h^{jl}_{qp}=h_k$ resp. $=h'_k$. These arguments applied to (3.53), (3.55) prove together with (3.57) and (3.54) part (iii) of Lemma (3.37).

(3.58) **Lemma.** *For all* $n\in\mathbb{N}$ *we have*

$$\hat{\chi}_{n,n_0}(t,h)^{1/2}\leqq c\begin{cases}|\hat{\chi}(t/8,h)|^{1/2}+n^{-3(s-2)/2} & \text{for } |t|\leqq n^{1/7}\\ n^{-3(s-2)/2} & \text{for } n^{1/7}\leqq|t|\leqq n/\eta_n\end{cases}$$

*where* $[n/2]\leqq n_0\leqq n$ *and* $\eta_n=c(s\log n)^6\gamma^5_{n,s}$.

*Proof.* Let $Q'_k|\mathscr{B}^k$ denote the $p$-measure obtained by truncation of $Q_k|\mathscr{B}^k$ at $x\in\mathbb{R}^k$ with $\|x\|_s>m^{1/2}$, $m\in\mathbb{N}$. (See (3.35).) Since for $s\geqq 3$ and $k\geqq k_0$ (see 3.8) $\hat{Q}'_k(v)\leqq\hat{Q}_k(v)+6m^{-s/2}$ we have

$$(3.59)\quad \hat{Q}'_k(v)^m\leqq c\hat{Q}_k(v)^m+cm^{-m}\qquad\text{for every } v\in\mathbb{R}^k$$

and similarly

$$(3.60)\quad \hat{Q}_k(v)^m\leqq c\hat{Q}'_k(v)+cm^{-m}.$$

Using (3.59) in order to estimate the integrand in $\hat{\chi}_{n,n_0}(t,h)$ we obtain by interchanging the integrations

$$(3.61)\quad \begin{aligned}&\hat{\chi}_{n,n_0}(t,h_k)\\ &\leqq c\left[\int \hat{\hat{Q}}_k(2tn^{-1}B_k(v_1+\ldots+v_m))^{n_0-m}\,\bar{Q}'_k(dv_1,\ldots,dv_m)\right]^{1/2}+cm^{-m}.\end{aligned}$$

Let $H|\mathscr{B}^k$ denote a $p$-measure. Then for $k\geqq k_0$ and $1\leqq p\leqq n/2$ and all $r$ fulfilling $|r|\leqq\frac{3}{2}(cs\log n)^{-3}\gamma_n^{-2}p^{1/2}$, $\gamma_n\in\mathbb{R}_+$, $\gamma_n\geqq 2$, we have

$$(3.62)\quad \int \hat{\hat{Q}}_k(rp^{-1/2}B_kx)^p\,H(dx)\leqq\varphi_H(r)+e\varphi_H(\gamma_n)+H\{x:\ \|x\|_s>cs\log n\}$$

where $\varphi_H(r)=\int\hat{H}(rB_kx)\,\Phi_k(dx)$.

In order to prove (3.62), we expand the c.f. of $Q_k$ around 0 and obtain the following inequality

$$|\hat{Q}_k(B_kx)|^2\leqq 1-\langle C_kB_kx,B_kx\rangle+\left[\tfrac{8}{6}\gamma_n^2\,\|x\|_s^3\right]\gamma_n^{-2}$$

since $\|x\|_s^3\geqq\int|\langle B_kx,a\rangle|^3\,Q_k(dx)$ by Hölder's inequality. Furthermore, $\langle C_kB_kx,B_kx\rangle\leqq\|x\|_s^2$ and $1-a\leqq\exp(-a)$ for $a\leqq\frac{3}{2}$ imply

$$(3.63)\quad |\hat{Q}_k(rp^{-1/2}B_kx)|^{2p}\leqq\exp\left(-\tfrac{1}{2}r^2\langle C_kB_kx,B_kx\rangle\right)$$

for all $r$ and $x$ fulfilling $|r|\,\|x\|_s^3\leqq\frac{3}{2}\gamma_n^{-2}p^{1/2}$ and $x\notin A_n\cup E_n$ where $A_n=\{x\in\mathbb{R}^k:\ \langle C_kB_kx,B_kx\rangle\leqq\gamma_n^{-2}\}$ and $E_n=\{x\in\mathbb{R}^k:\ \|x\|_s>cs\log n\}$. Splitting the domain of integration in (3.62) we obtain by (3.63) the following estimate for all $r$ fulfilling the condition of (3.62)

$$\begin{aligned}&\int(1_{A_n}(x)+1_{E_n}(x)+\hat{\Phi}_k(rB_kx)\,1_{\mathbb{R}^k\setminus A_n\setminus E_n})\,H(dx)\\ &\quad\leqq\int\exp\left[1-\gamma_n^2\langle C_kB_kx,B_kx\rangle\right]H(dx)+H(E_n)+\varphi_H(r)\end{aligned}$$

by interchanging integrations. This immediately proves (3.62). We apply (3.62) with $H|\mathscr{B}^k=\bar{Q}_k'^{m}(m^{1/2}\cdot)|\mathscr{B}^k$, $p=n_0-m$ and $r=tn^{-1}(n_0-m)^{1/2}$. By Remark (3.35)(i) with $h$ replaced by $h_k$ we obtain for $k\geqq k_0$ (see (3.8))

$$(3.64)\quad \hat{\chi}_{n,n_0}(t,h_k)^2\leqq n^{-6(s-2)}+cm^{-m}+c\varphi_H(tn^{-1}(n_0-m)^{1/2}m^{1/2})+e^1c\varphi_H(\gamma_n)$$

for all $t$ fulfilling

$$(3.65)\quad |t|\leqq\tfrac{3}{2}(cs\log n)^{-3}\gamma_n^{-2}m^{-1/2}n.$$

In order to estimate $\varphi_H(r)$ in (3.64) apply (3.62) with $H|\mathscr{B}^k=\Phi_k|\mathscr{B}^k$ and $r=tn^{-1}(n_0-m)^{1/2}n^{1/2}$ resp. $r=\gamma_n$. Hence by (3.44) for $k\geqq k_0$

$$(3.66)\quad \begin{aligned}\hat{\chi}_{n,n_0}(t,h_k)^2\leqq{}& n^{-6(s-2)}+c\,|\hat{\chi}(tn^{-1}(n_0-m)^{1/2}m^{1/2}/2,h_k)|^2\\ &+e(e+2)\,c\,|\hat{\chi}(\gamma_n/2,h_k)|^2+(e+1)\,\Phi_k(E_n)+cm^{-m}.\end{aligned}$$

Note that by Remark (3.35)(i) $\Phi_k(E_n)=n^{-6(s-2)}$ for $k\geqq k_0$. The estimate (3.66) holds for all $t, \gamma_n$ and $m$ fulfilling (3.65) and

$$(3.67)\quad \begin{aligned} &|t| \leqq \tfrac{2}{3}(cs\log n)^{-3}\gamma_n^{-2}n(n_0-m)^{-1/2},\\ &|\gamma_n| \leqq \tfrac{2}{3}(cs\log n)^{-3}\gamma_n^{-2}m^{1/2}.\end{aligned}$$

We now choose the 'free' parameters $m$ and $\gamma_n$ in dependence of $n$ and $t$.

By (1.4) $|t| \to |\hat{\chi}(t,h)|$ is monotone decreasing and $|\lambda_{a(s)}|>0$ implies $|\hat{\chi}(t,h)| \leqq (1+|\lambda_{a(s)}|\,|t|)^{-24(s-2)}$. Therefore the sequence $\gamma_{n,s}$ defined by

$$(3.68)\quad |\hat{\chi}(\gamma_{n,s}|\lambda_{a(s)}|^{-1},h)| = n^{-3(s-2)}$$

is increasing with $\gamma_{n,s} \leqq cn^{1/8}$. Let $t_n = \tfrac{3}{2}(cs\log n)^{-3}\gamma_{n,s}^{-2}n^{1/2}$. Choose

$$(3.69)\quad \gamma_n=\gamma_{n,s} \quad\text{and}\quad m=m_{t,n}=\begin{cases}[n/4] & \text{if } |t|\leqq n^{1/7}\\ [nt^{-2}\gamma_{n,s}^2] & \text{if } n^{1/7}\leqq |t|\leqq t_n.\\ [nt_n^{-2}\gamma_{n,s}^2] & \text{if } |t|>t_n\end{cases}$$

Then $m_{t,n} \geqq m_n = \tfrac{2}{9}(cs\log n)^6\gamma_{n,s}^6$ for sufficiently large $n$ implies $m_{t,n}^{-m_{t,n}} \leqq cn^{-6(s-2)}$. Furthermore conditions (3.65) and (3.67) are fulfilled *for* $|t| \leqq t_n$ *and sufficiently large n*. (For condition (3.65), use $\gamma_{n,s}=O(n^{1/8})$.)

Hence, (3.66) together with Remark (3.12) imply in the limit $k\to\infty$

$$(3.70)\quad |\hat{\chi}_{n,n_0}(t,h)|^{1/2} \leqq \begin{cases} c|\hat{\chi}(t/8,h)|^{1/2}+cn^{-3(s-2)/2} & \text{if } |t|\leqq n^{1/7}\\ cn^{-3(s-2)/2} & \text{if } n^{1/7}\leqq|t|\leqq t_n\end{cases}$$

The condition

$$(3.71)\quad t_n \leqq |t| \leqq \tfrac{3}{2}n^1(cs\log n)^{-3}\gamma_{n,s}^{-2}m_n^{-1/2} = n^1/\eta_n$$

implies (3.65). In (3.64) the function $r\to\varphi_H(r)=\int\exp(-\tfrac{1}{2}r^2\langle C_kB_kx,B_kx\rangle)\,H(dx)$ *is nonincreasing for every k*. Therefore $\varphi_H(tn^{-1}n^{1/2}m_n^{1/2}) \leqq cn^{-6(s-2)}$ for *all* $|t|\geqq t_n$ *and* $k\to\infty$. Hence $|\hat{\chi}_{n,n_0}(t,h)| \leqq cn^{-3(s-2)}$ for $t$ fulfilling (3.71) which proves the lemma.

(3.72) **Corollary.**

$$|\hat{\chi}_n(t,h)-\hat{\chi}_s^{(n)}(t,h)| \leqq c\begin{cases} |t|\,\{(1+|t|^{6(s-2)})|\hat{\chi}(ct,h)|^{1/2} \\ \quad +(1+|t|^2)|\hat{\chi}(ct,h)|^{1/4}+n^{-3(s-2)/2}\} & \textit{for } |t|\leqq n^{1/7}\\ |t|(1+t^{6(s-2)})|\hat{\chi}(t,h)|^{1/2} \\ \quad +\begin{cases} n^{-3(s-2)/2} & \textit{for } n^{1/7}\leqq|t|\leqq n/\eta_n\\ |\hat{\chi}_{n,n}(t,h)|^{1/2} & \textit{otherwise.}\end{cases}\end{cases}$$

*Proof.* From Theorem 3.1 in Götze (1979) applied with $H=\mathbb{R}^k$, $Q=P*T_k$, $\Phi=\Phi_k$ and $f(x)=\exp[it\langle B_k'x,x\rangle]$ and Lemma (3.10)(i) we obtain

$$(3.73)\quad \left|\hat{\chi}_n(t,h_k')-\sum_{j=0}^{s-3}n^{-j/2}\left(P_j(D)\exp\left[\tfrac{1}{2}\sum_{m,l=1}^{p}R(t,h_k')(x_m,x_l)\varepsilon_m\varepsilon_l\right]\right)\right|_{\varepsilon=0}$$
$$\leqq cn^{-(s-2)/2}\sup\{|D^\alpha\hat{\chi}_{i,n}(t,k,n^{-1/2}\varepsilon_1,0,\dots,0)|,\ \varepsilon_1\in[0,1],\hat{\chi}(t,h_k')|$$
$$|\alpha|\leqq 3(s-2),\ \alpha_i\leqq s,\ i=1,\dots,p\leqq 3(s-2)/2\}.$$

Computing the derivatives with respect to $\varepsilon_1, \dots, \varepsilon_p$ in (3.73) we obtain the expansion $\hat{\chi}_s^{(n)}(t, h'_k)$ defined in (3.10). Lemma (3.58) together with Lemma (3.37)(ii) imply (using $\lim_{k\to\infty} \hat{\chi}_{n,n_0}(t, h'_k) = \hat{\chi}_{n,n_0}(t, h')$) that the remainder term of (3.73) is smaller than

$$(3.74) \quad n^{-(s-2)/2} c(t)(1+|t|^{6(s-2)}) \left( \left| \hat{\chi}\left(\frac{t}{8}, h'\right) \right|^{1/2} + n^{-3(s-2)/2} \right).$$

(Note that Lemma (3.58) holds for $|t| \leqq n^{1/7}$ and $h'$, too.)

Lemma (3.10)(ii) and (iv), Lemma (3.37)(iii) and Lemma (3.58) together imply Corollary (3.72) for $|t| \leqq n^{1/7}$. For $|t| > n^{1/7}$ we have

$$|\hat{\chi}_n(t, h) - \hat{\chi}_s^{(n)}(t, h)| \leqq |\hat{\chi}_n(t, h)| + |\hat{\chi}_s^{(n)}(t, h)|.$$

Hence, Lemma (3.10)(iii) together with Lemma (3.37)(i) and Lemma (3.58) prove the Corollary for $s \geqq 4$.

For $s = 3$ we don't need truncation. Using an obvious modification of the proof of Lemma (3.37)(i), Lemma (3.37)(ii) holds for $h_k$, too. By the same arguments as above the case $s = 3$ can be proved with the help of (3.73) where $h'_k$ is replaced by $h_k$ and Lemma (3.58).

## 4. Proof of Theorems

By a well known Theorem of Esseen on inversion of c.f. (see e.g. V.V. Petrov (1975, p. 109, Theorem 2)) we have

$$\begin{aligned}(4.1) \quad I &= \sup_r |P^n\{w_n^2 < r\} - \hat{\chi}_s^{(n)}(r, h)| \\ &\leqq c \int 1_{[-T_n, T_n]}(t)\, |t|^{-1} |\hat{\chi}_n(t, h) - \hat{\chi}_s^{(n)}(r, h)|\, dt + c\, T_n^{-1} \sup_r \left| \frac{\partial}{\partial r} \chi_s^{(n)}(r, h) \right|\end{aligned}$$

for $T_n = n^{(s-2)/2}$. Note that Lemma (3.10)(iii) and (1.4) imply

$$(4.2) \quad \left| \frac{\partial}{\partial r} \chi_s^{(n)}(r, h) \right| \leqq c \int |\hat{\chi}_s^{(n)}(t, h)|\, dt \leqq c |\lambda_{a(s)}|^{-(6(s-3)+2)}.$$

It is now a routine matter to estimate the first summand on the r.h.s. of (4.1) by means of Corollary (3.72), splitting the domain of integration $[-T_n, T_n]$ into $A_n = [-n^{-1/7}, n^{1/7}]$, $[-n/\eta_n, n/\eta_n] \setminus A_n$ and $B_n = [-T_n, T_n] \setminus [-n/\eta_n, n/\eta_n]$, using Condition (2.8) which implies $\hat{\chi}_{n,n}(t, h) \leqq n^{-(s-2)/2} (\log n)^{-1}$ for $t \in B_n$ together with

$$\int |t|^{6(s-2)} |\hat{\chi}(t, h)|^{1/2}\, dt \leqq c |\lambda_{a(s)}|^{-6(s-2)-1}$$

and

$$\int [1_{A_n}(t) |t|^{6(s-2)} n^{-(s-2)} + 1_{B_n}(t) |t|^{-1}]\, dt \leqq c\, s \log n.$$

This together with (4.1) and (4.2) proves Theorem (2.9) for $s \geqq 4$ and $\beta_s = \int |h|^s dP^2 + \int |h \circ d|^s dP = 1$.

Similarly Theorem (2.3) is proved with $T_n = n/\eta_n$ resp. $T_n = n^{1/2}$. Note that $|\lambda_{31}| > 0$ implies $\eta_n \leqq n^{1/2}$.

If $\beta_s \neq 1$ replace $h$ by $h\beta_s^{-1/s}$. Hence $\lambda_k$ is replaced by $\lambda_k \beta_s^{-1/s}$ which proves together with $|\lambda_k|^{-1} \beta_s^{1/s} \geqq 1$ the remainder terms in Theorems (2.3) and (2.9). The Remark (2.6) can be proved with the help of Lemma 4 in Petrov (1975, p. 140) which shows $\hat{\chi}_{n,n}(t,h) = o(n^{-2})$ for $t \in B_n$, $s=4$. (See (1.6) for $|t| \leqq c$.)

*Proof of Remark* (2.7). Assume (3.2) for $h_k(x,y) = \langle T_k B T_k x, y \rangle$, $T_k$ linear. Then Lemma (3.10)(i) with $\underline{\varepsilon} = 0$ and Remark (3.12) with $s=3$ apply to $\Phi | \mathscr{A}$. Hence we obtain $\int \exp[it\langle Bx, x\rangle] \Phi(dx)$ as limit of the l.h.s. of (3.11) for $k \to \infty$. Furthermore, we have $\int \langle Bx, x\rangle (\Phi - P)(dx) = 0$ (use finite dimensional approximations and the Theorem of Monotone Convergence). Since the operators defined on $L^2(X,P)$ resp. $L^2(X,\Phi)$ pertaining to the kernel $\langle Bx, y\rangle$ have the same set of eigenvalues as $BC\colon X \to X$, the limit of the r.h.s. of (3.11) for $k \to \infty$ is equal to $\hat{\chi}(t,h)$ (defined in (1.4)), thus proving Remark (2.7).

*Proof of Remark* (2.12). Since

$$(4.3) \quad \Big|\int_{I_x} \exp(ith(\cdot,x))\, dP\Big| \leqq 4|t|^{-1}\delta^{-1} \quad \text{and} \quad P(I_x) \geqq \eta$$

after change of variables and partial integration, it is sufficient to consider the case where $|t|$ is small. Hence the relation (1.6) follows from Petrov (1975, p. 140, Lemma 4) together with the fact that (4.3) yields a positive lower bound for the variance of $h(\cdot, x)$ uniformly for $P$-a.e. $x \in [0,1]$.

## References


Anderson, T.W., Darling, D.A.: Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes. Ann. Math. Statist. **23**, 193–212 (1952)

Bellman, R.: A Brief Introduction to Theta Functions. New York: Holt, Rinehart and Winston 1961

Bikjalis, A.: Remainder terms in asymptotic expansions for characteristic functions and their derivatives. Litovsk. Mat. Sb. **7**, 571–582. Selected Translations in Statist. and Probab. **11**, 149–162 (1973)

Callaert, H., Janssen, P.: The Berry-Esseen theorem for U-statistics. Ann. Statist. **6**, 417–421 (1978)

Callaert, H., Janssen, P., Veraverbeke, N.: An Edgeworth expansion for U-statistics. [To appear Ann. Statist. 1978]

Carleman, T.: Zur Theorie der linearen Integralgleichungen. Math. Z. **9**, 196–217 (1921)

Csörgö, S.: On an asymptotic expansion for the von Mises $\omega^2$ statistic. Acta Sci. Math. **38**, 45–67 (1976)

Dunford, N., Schwartz, J.T.: Linear Operators. Part II. New York: Wiley 1963

Esseen, C.G.: Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. Acta Math. **77**, 1–125 (1945)

Fernique, M.X.: Intégrabilité des vecteurs Gaussiens. C. R. Acad. Sci. Paris Sér. A **270**, 1698–1699 (1970)

Götze, F.: On Edgeworth expansions in Banach spaces. Preprints in Statistics 51, University of Cologne (1979)

Götze, F., Hipp, C.: Asymptotic expansions in the central limit theorem under moment conditions. Z. Wahrscheinlichkeitstheorie verw. Gebiete **42**, 67–87 (1978)

Hansen, E.R.: A Table of Series and Products. Engleward Cliffs, N.J.: Prentice-Hall 1975

Jain, N.C.: Central limit theorem in a Banach space. In *Probability in Banach spaces.* Proceedings of the First International Conference on Probability in Banach spaces, (A. Beck, Ed.), pp. 113–130. Lecture Notes in Mathematics **526**. Berlin-Heidelberg-New York: Springer 1975

Kandelaki, N.P.: On a limit theorem in Hilbert space. News of the Computing Centre of the Academy of Sci. Georgian SSSR **1**, 46–55 (Russian) (1965)

Kiefer, J.: Skorohod embedding of multivariate R.V.'s and the sample D.F. Z. Wahrscheinlichkeitstheorie verw. Gebiete **24**, 1–35 (1972)

Kuelbs, J., Kurtz, T.: Berry-Esseen estimates in Hilbert space and an application to the law of the iterated logarithm. Ann. Probability **2**, 387–407 (1974)

Martinov, G.V.: Computation of distribution functions of quadratic forms of normally distributed random variables. Theor. Probability Appl. **20**, 782–793 (1975)

Nikitin, Ya. Yu.: Estimates of the rate of convergence for some limit theorems and statistical criteria. Soviet Math. Dokl. **13**, 189–192 (1972)

Oberbettinger, F., Badii, L.: Tables of Laplace Transforms. Berlin-Heidelberg-New York: Springer 1973

Orlov, A.I.: The rate of convergence of the distribution of the Smirnov – von Mises statistic. Theor. Probability Appl. **19**, 737–757 (1974)

Paulauskas, V.I.: On the rate of convergence in the central limit theorem in certain Banach spaces. Theor. Probability Appl. **21**, 754–769 (1976)

Petrov, V.V.: Sums of Independent Random Variables. Berlin-Heidelberg-New York: Springer 1975

Rosenkrantz, W.A.: A rate of convergence for the von Mises statistic. Trans. Amer. Math. Soc. **139**, 329–337 (1969)

Sazonov, V.V.: On $\omega^2$-criterion, Sankhyā, ser. A. **30**, 205–210 (1968)

Sazonov, V.V.: An improvement of a convergence-rate estimate. Theor. Probability Appl. **14**, 640–651 (1969)

Stephens, M.A.: The distribution of the goodness-of-fit statistic $U_N^2$. I. Biometrika **50**, 303–313 (1963)

Stephens, M.A.: The distribution of the goodness-of-fit statistic $U_N^2$. II. Biometrika **51**, 393–397 (1964)

von Mises, R.: On the asymptotic distribution of differentiable statistical functions. Ann. Math. Statist. **18**, 309–348 (1947)

Watson, G.S.: Goodness-of-fit tests on a circle. Biometrika **48**, 109–114 (1961)

Received July 12, 1979