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Toeplitz Minimal Flows Which Are Not Uniquely Ergodic

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Summary. We study minimal symbolic dynamical systems which are orbit closures of Toeplitz sequences. We construct $0-1$ subshifts of this type for which the set of ergodic invariant measures has any given finite cardinality, is countably infinite or has cardinality of the continuum.

The first example of a minimal flow which is not uniquely ergodic was found by Markov (cf. Nemytskii and Stepanov, 1960, p. 512). A paper of Oxtoby (1952) includes a particularly elegant example of such a flow, obtained as the orbit closure of a point in $\{0, 1\}^{\mathbb{Z}}$. Later Jacobs and Keane (1969) defined a class of almost periodic 0–1 sequences, called *Toeplitz sequences*, which includes Oxtoby's sequence. Although the orbit closure of a *regular* Toeplitz sequence (see Sect. 2) is always uniquely ergodic, Markley and Paul (1979) have shown that in a certain sense most non-regular Toeplitz sequences yield minimal flows which are not uniquely ergodic.

We consider the problem of describing the invariant measures on the orbit-closure of a Toeplitz sequence. We generalize the definition of Toeplitz sequences to sequences in a compact symbol space Σ. In Sect. 2 we identify the maximal equicontinuous factor of the flow; this was done by Eberlein (1970) for regular Toeplitz 0–1 sequences. In Sect. 3, we determine the ergodic measures for Oxtoby's flow; there are exactly two. We construct analogous flows in $\Sigma^{\mathbb{Z}}$ for which the set of ergodic measures has the same cardinality as Σ. In Sect. 4 we construct Toeplitz 0–1 sequences for which the orbit closure has the measure structure of a skew product, with the maximal equicontinuous factor as base and a freely chosen subshift of $\{0, 1\}^{\mathbb{Z}}$ as fiber. This construction yields minimal flows for which the set of ergodic measures has any given finite cardinality, is countably infinite or has cardinality of the continuum.

The last section contains computations of entropy for our examples. We find Toeplitz flows with entropy arbitrarily close to log 2. Markley and Paul (1979) show that a non-regular Toeplitz flow "usually" has positive entropy.

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1. Preliminaries

We summarize some basic definitions and results; we refer the reader to Oxtoby (1952) and Ellis (1969) for more details.

By *flow* we will mean a pair (X, T) where X is a compact metrizable space and T is a homeomorphism of X to itself. (X, T) is *minimal* if X has no proper closed T-invariant subset. $\mathscr{B}(X)$ will denote the σ-algebra of Borel sets of X. An *invariant measure* for (X, T) is a probability measure μ on $\mathscr{B}(X)$ with $\mu(T^{-1}B)=\mu(B)$ for all $B \in \mathscr{B}(X)$; the measure is *ergodic* if every T-invariant Borel set has measure 0 or 1. The invariant measures for (X, T) form a non-empty closed set, and the ergodic measures are exactly the extreme points of this set. The flows is *uniquely ergodic* if it admits only one (ergodic) invariant measure.

A compact topological group G is *monothetic* if some $g \in G$ generates a dense subgroup of G; g is called a (topological) *generator* of G. G is necessarily abelian. We also denote by g the translation $h \mapsto h+g$ on G. Then (G, g) is a minimal flow, and the Haar measure on G is the unique invariant measure.

The flow (Y, S) is a *factor* of (X, T) if there is a continuous map π of X onto Y with $\pi \circ T = S \circ \pi$; if π is a homeomorphism then (X, T) and (Y, S) are *isomorphic* as flows. Every minimal flow (X, T) has a *maximal equicontinuous factor* (Ellis and Gottschalk, 1960). This can be characterized (up to flow isomorphism) as a factor π: $(X, T) \to (G, g)$, G a compact metrizable monothetic group with generator g, such that for any other such factor π': $(X, T) \to (G', g')$ we have a factor map φ: $(G, g) \to (G', g')$ with $\varphi \circ \pi = \pi'$.

The importance of the maximal equicontinuous factor to the problem of determining the invariant measures on (X, T) can be seen as follows: if μ is an invariant measure on (X, T) then $\mu \circ \pi^{-1}$ is an invariant measure on (G, g), so it must be equal to the Haar measure m. If $B \in \pi^{-1}(\mathscr{B}(G))$, then $\mu(B) = \mu \circ \pi^{-1}(\pi(B)) = m(\pi(B))$. Thus the invariant measures on (X, T) all coincide on the σ-algebra $\pi^{-1}(\mathscr{B}(G)) \subset \mathscr{B}(X)$.

We will use the following fact from Paul (1976).

Proposition 1.1. *Let (X, T) be a minimal flow and π: $(X, T) \to (G, g)$ a factor map, with G a compact metrizable monothetic group with generator g. If for some $x \in X$ we have $\pi^{-1}(\pi(X)) = \{x\}$, then (G, g) is the maximal equicontinuous factor of (X, T).*

2. Toeplitz Sequences

We will generalize the usual definition of Toeplitz $0-1$ sequences to sequences in $X = \Sigma^{\mathbb{Z}}$, where Σ is a compact metric space. We write elements of X as $x = (x(n))$. The metric

$$|x, y| = \sum_{n=-\infty}^{\infty} 2^{-|n|} |x(n), y(n)|$$

gives the product topology on X. S will denote the left shift homeomorphism, $Sx(n)=x(n+1)$. The *orbit* of x is $\mathcal{O}(x)=\{S^n x\colon n\in\mathbb{Z}\}$.

For $x\in X$, $p\in\mathbb{N}$ and $\sigma\in\Sigma$ we set

$$\mathrm{Per}_p(x, \sigma)=\{n\in\mathbb{Z}\colon x(n')=\sigma \text{ for all } n'\equiv n \bmod p\}$$

$$\mathrm{Per}_p(x)=\bigcup_{\sigma\in\Sigma} \mathrm{Per}_p(x, \sigma)$$

$$\mathrm{Aper}(x)=\mathbb{Z}\backslash(\bigcup_{p\in\mathbb{N}} \mathrm{Per}_p(x)).$$

By the *p-skeleton* of x we will mean the part of x which is periodic with period p. To make this precise, we define the p-skeleton to be the sequence obtained from x by replacing $x(n)$ by a new symbol "__" for all $n\notin \mathrm{Per}_p(x)$.

Definition. The sequence $\eta\in X$ is a *Toeplitz sequence* if $\mathrm{Aper}(\eta)=\emptyset$.

It is not hard to extend the basic results of Jacobs and Keane (1969) and Eberlein (1970) to this setting. We include proofs in this section for completeness and to introduce ideas we will use later.

Periodic sequences are Toeplitz sequences. Every Toeplitz sequence is *almost periodic*: that is, every block occuring in η appears with bounded gap between sucessive occurences. (In fact, every block in η is in the p-skeleton of η for some p.) Hence $(\bar{\mathcal{O}}(\eta), S)$ is a minimal flow (Oxtoby, 1952). From now on we will consider only non-periodic Toeplitz sequences.

If $p|q$ then $\mathrm{Per}_p(x)\subset \mathrm{Per}_q(x)$. We call p an *essential period* of x, $p\in\mathscr{P}(x)$, if $p|q$ for every q satisfying $\mathrm{Per}_p(x, \sigma)=\mathrm{Per}_p(x, \sigma)-q$ for all $\sigma\in\Sigma$. Thus $p\in\mathscr{P}(x)$ if and only if the p-skeleton of x is not periodic with any smaller period. The following is easily verified:

Proposition 2.1. *If p and q are essential periods of x, so is their least common multiple.*

Definition. A *period structure* for a non-periodic Toeplitz sequence η is an increasing sequence $(p_i)_{i\in\mathbb{N}}$ of natural numbers satisfying

(i) p_i is an essential period of η for all i,

(ii) $p_i|p_{i+1}$,

(iii) $\bigcup_{i=1}^{\infty} \mathrm{Per}_{p_i}(\eta)=\mathbb{Z}$.

Every non-periodic Toeplitz sequence has a period structure. For example, order the elements of $\mathscr{P}(\eta)$ and let q_i be the least common multiple of the first i elements of $\mathscr{P}(\eta)$. A period structure is obtained by deleting repeated terms from the sequence (q_i).

We can now describe the maximal equicontinuous factor of the flow $(\bar{\mathcal{O}}(\eta), S)$. Fix a period structure (p_i) for η. Since $p_i|p_{i+1}$, we have a sequence of group homomorphisms

$$\mathbb{Z}/p_1\mathbb{Z} \xleftarrow{\varphi_1} \mathbb{Z}/p_2\mathbb{Z} \xleftarrow{\varphi_2} \dots \leftarrow \mathbb{Z}/p_i\mathbb{Z} \xleftarrow{\varphi_i} \mathbb{Z}/p_{i+1}\mathbb{Z} \leftarrow \dots$$

where $\varphi_i(\eta)$ is the residue of n modulo p_i. We let G be the inverse limit group, $G = \varprojlim \mathbb{Z}/p_i\mathbb{Z}$. That is,

$$G = \{(n_i): n_i \in \mathbb{Z}/p_i\mathbb{Z} \text{ and } n_j \equiv n_i \bmod p_i \text{ for } i<j\}$$

and $(n_i)+(m_i)=(n_i+m_i)$, where n_i+m_i is taken modulo p_i. We denote by $\hat{1}$ the element (1) in G, and $\hat{n} = n \cdot \hat{1}$ for $n \in \mathbb{Z}$. The metric

$$|(n_i), (m_i)| = \max\left\{\frac{1}{i+1} : n_i \neq m_i\right\}$$

gives the usual inverse limit topology on G. G is a compact monothetic group with generator $\hat{1}$.

Theorem 2.2. *$(G, \hat{1})$ is the maximal equicontinuous factor of $(\bar{\mathcal{O}}(\eta), S)$.*

Remark. It can be seen algebraically that G is independent of the choice of period structure (p_i). In fact, G can be obtained without resorting to period structures as $\varprojlim \mathbb{Z}/p\mathbb{Z}$, where the inverse limit is taken over all $p \in \mathscr{P}(\eta)$, respecting the homomorphisms $\mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ for $p|q$ (cf. Jacobson, 1980, p. 72–74).

For each $i \in N$, $n \in \mathbb{Z}/p_i\mathbb{Z}$ we set $A_n^i = \{S^m\eta: m \equiv n \bmod p_i\}$.

Lemma 2.3. (i) *$\bar{A}_n^i$ is exactly the set of all $\omega \in \bar{\mathcal{O}}(\eta)$ with the same p_j-skeleton as $S^n\eta$.*

(ii) *$\{\bar{A}_n^i: n \in \mathbb{Z}/p_i\mathbb{Z}\}$ is a partition of $\bar{\mathcal{O}}(\eta)$ into relatively open sets.*

(iii) *$\bar{A}_n^i \supset \bar{A}_m^j$ for $i<j$ and $m \equiv n \bmod p_i$.*

(iv) *$S\bar{A}_n^i = \bar{A}_{n+1}^i$.*

Proof. Let $\omega \in \bar{A}_n^i$; we will show ω has the same p_i-skeleton as $S^n\eta$. Clearly $\mathrm{Per}_{p_i}(\omega, \sigma) \supset \mathrm{Per}_{p_i}(S^n\eta, \sigma)$ for all $\sigma \in \Sigma$. Suppose $k \in \mathrm{Per}_{p_i}(\omega, \sigma) \setminus \mathrm{Per}_{p_i}(S^n\eta, \sigma)$; then we can find $k' \equiv k \bmod p_i$, $j \geqq i$ and $\tau \neq \sigma$ with $k' \in \mathrm{Per}_{p_j}(S^n\eta, \tau)$. Then for any $\omega' \in A_n^i$, $\omega'(k'') = \tau$ for some $k'' \equiv k \bmod p_i$, $0 \leqq k'' < p_j$. Since $\omega(k'') = \sigma$ for all $k'' \equiv k \bmod p_i$, we obtain

$$|\omega, \omega'| \geqq 2^{-p_j}|\sigma, \tau| \quad \text{for all } \omega' \in A_n^i.$$

This contradicts $\omega \in \bar{A}_n^i$.

A similar argument shows that if $n \neq m \bmod p_i$ then A_n^i and A_m^i are a positive distance apart. For $S^n\eta$ and $S^m\eta$ must have different p_i-skeletons since p_i is an essential period of η. We can find $k \in \mathrm{Per}_{p_i}(S^m\eta, \sigma) \cap \mathrm{Per}_{p_j}(S^n\eta, \tau)$, with $i<j$ and $\sigma \neq \tau$. We deduce as above that $|\omega, \omega'| \geqq 2^{-p_j}|\sigma, \tau|$ for all $\omega \in A_m^i$ and $\omega' \in A_n^i$. Since $\{A_n^i: n \in \mathbb{Z}/p_i\mathbb{Z}\}$ is a finite partition of $\mathcal{O}(\eta)$, (ii) is true and the rest of (i) follows. Parts (iii) and (iv) are immediate.

Proof of Theorem 2.2. For $g = (n_i) \in G$ we set

$$A_g = \bigcap_{i=0}^{\infty} \bar{A}_{n_i}^i.$$

By Lemma 2.3, A_g is the intersection of a nested sequence of closed sets, so it is closed and non-empty. It also follows from the lemma that $\{A_g\colon g\in G\}$ is a partition of $\bar{\mathcal{O}}(\eta)$, and $SA_g=A_{g+\hat{1}}$. We define the factor map $\pi\colon(\bar{\mathcal{O}}(\eta),S)\to(G,\hat{1})$ by $\pi^{-1}(g)=A_g$. The map π is continuous because sets $B^j_m=\{(n_i)\in G\colon n_j=m\}$ are a basis for G, and $\pi^{-1}(B^j_m)=\bar{A}^j_m$.

To prove that $(G,\hat{1})$ is the maximal equicontinuous factor, we show $\pi^{-1}(\pi(\eta))=\{\eta\}$ and apply Proposition 1.1. If $\omega\in A_{\hat{0}}=\pi^{-1}(\pi(\eta))$, then $\omega\in A^i_0$ for all i, and so ω has the same p_i-skeleton as η for all i. Since η is a Toeplitz sequence, $\omega=\eta$.

Corollary 2.4. *$\pi(\omega)=\pi(\omega')$ if and only if ω and ω' have the same p_i-skeleton for all $i\in\mathbb{N}$. In particular, π is one-to-one on the set of Toeplitz sequences in $\bar{\mathcal{O}}(\eta)$.*

We now turn to the problem of determining the invariant measure on $(\bar{\mathcal{O}}(\eta),S)$. We let m denote the Haar measure on G. As we observed in Sect. 1, any invariant Borel measure on $\bar{\mathcal{O}}(\eta)$ coincides with $m\circ\pi$ on $\pi^{-1}(\mathcal{B}(G))$. For $i\in\mathbb{N}$, $n\in\mathbb{Z}/p_i$ we have $\bar{A}^i_n\in\pi^{-1}(\mathcal{B}(G))$, and $m\circ\pi(\bar{A}^i_n)=\frac{1}{p_i}$ since $\{\bar{A}^i_n\colon n\in\mathbb{Z}/p_i\mathbb{Z}\}$ is a partition of $\bar{\mathcal{O}}(\eta)$ and $S\bar{A}^i_n=\bar{A}^i_{n+1}$.

Since $\mathrm{Per}_{p_i}(\eta)$ is periodic, it has a density in $\mathbb{Z}$ given by

$$d_i=\frac{1}{p_i}\cdot\#\{n\in\mathbb{Z}/p_i\mathbb{Z}\colon n\in\mathrm{Per}_{p_i}(\eta)\}.$$

The d_i are increasing; we set $d=\lim\limits_{i\to\infty}d_i$.

Definition. The Toeplitz sequence η is *regular* if $d=1$.

We let $\mathcal{T}=\mathcal{T}(\eta)$ denote the set of Toeplitz sequences in $\bar{\mathcal{O}}(\eta)$, and $C=C(\eta)=\{\omega\in\bar{\mathcal{O}}(\eta)\colon 0\notin\mathrm{Aper}(\omega)\}$.

Proposition 2.5. *C, $\mathcal{T}\in\pi^{-1}(\mathcal{B}(G))$. We have $m\circ\pi(C)=d$, and*

$$m\circ\pi(\mathcal{T})=\begin{cases}1 & \text{if } d=1\\ 0 & \text{if } d<1.\end{cases}$$

Proof. For $i\in N$ we set $C_i=\{\omega\in\bar{\mathcal{O}}(\eta)\colon 0\in\mathrm{Per}_{p_i}(\omega)\}$. Then C_i is the union of those $\bar{A}^i_n$ for which $0\in\mathrm{Per}_{p_i}(S^n\eta)$, that is, $n\in\mathrm{Per}_{p_i}(\eta)$. Hence $C_i\in\pi^{-1}(\mathcal{B}(G))$ and $m\circ\pi(C_i)=d_i$. Then $C=\bigcup\limits_{i=1}^{\infty}C_i\in\pi^{-1}(\mathcal{B}(G))$; since the C_i are nested, $m\circ\pi(C)=\lim\limits_{i\to\infty}d_i=d$. We have $\mathcal{T}=\bigcap\limits_{n\in\mathbb{Z}}S^nC\in\pi^{-1}(\mathcal{B}(G))$. If $d=1$, $m\circ\pi(\mathcal{T})=1$; if $d<1$ then $m\circ\pi(C)<1$, so $m\circ\pi(\mathcal{T})=0$ since $\mathcal{T}$ is shift invariant.

Theorem 2.6. (Jacobs and Keane). *If η is a regular Toeplitz sequence then $(\bar{\mathcal{O}}(\eta),S)$ is uniquely ergodic.*

Proof. Let μ be an invariant measure on $(\bar{\mathcal{O}}(\eta),S)$. For $B\in\mathcal{B}(\bar{\mathcal{O}}(\eta))$ we write $B=(B\cap\mathcal{T})\cup(B\setminus\mathcal{T})$. Since π is $1-1$ on $\mathcal{T}$, $B\cap\mathcal{T}=\pi^{-1}(\pi(B\cap\mathcal{T}))$ and $\mu(B\cap\mathcal{T})=m\circ\pi(B\cap\mathcal{T})$. Since $\mu(\mathcal{T})=m\circ\pi(\mathcal{T})=1$, $\mu(B\setminus\mathcal{T})=0$. Thus $\mu(B)=m\circ\pi(B\setminus\mathcal{T})$. This determines μ uniquely.

3. Oxtoby Sequences

We construct Toeplitz sequences in $\Sigma^{\mathbb{Z}}$ analogous to Oxtoby's example. For $\Sigma = \{0, 1\}$ our construction differs slightly from Oxtoby's; the change was made for notational convenience and does not materially affect any results or proofs.

Construction. Let $(p_i)_{i\in\mathbb{N}}$ be a fixed sequence of natural numbers with $p_i | p_{i+1}$ and $p_i \geqq 3$, $\frac{p_{i+1}}{p_i} \geqq 3$ for all $i \in \mathbb{N}$. Fix a dense sequence $(\sigma_i)_{i\in\mathbb{N}}$ in Σ with the property that every element of the sequence appears infinitely often in the sequence. (For $\Sigma = \{0, 1, \ldots, r-1\}$ we may take $\sigma_i \equiv i \bmod r$.)

We define the sequence $\eta \in \Sigma^{\mathbb{Z}}$ by inductive steps. The first step is to set $\eta(n) = \sigma_1$ for all $n \equiv -1$ or $0 \bmod p_1$. For each $k \in \mathbb{Z}$ we set $J(1, k) = [kp_1 + 1, (k+1)p_1 - 1)$. Step 2 is to set $\eta(n) = \sigma_2$ for all $n \in J(1, k)$ with $k \equiv -1$ or $0 \bmod \frac{p_2}{p_1}$. In general, for $i \in \mathbb{N}$ we let $J(i, k)$ denote the set of $n \in [kp_i, (k+1)p_i)$ for which $\eta(n)$ has not yet been defined at the end of the i^{th} step. The $(i+1)^{\text{th}}$ step is to set $\eta(n) = \sigma_{i+1}$ for $\eta \in J(i, k)$ with $k \equiv -1$ or $0 \bmod \frac{p_{i+1}}{p_i}$.

Sequences η defined as above will be called *Oxtoby sequences.* Note that after the i^{th} step η is defined on all of $[-p_{i-1}, p_i]$. The construction is periodic at each step, so η is a Toeplitz sequence. For $i < j$ and $m \in \mathbb{Z}$, $J(j, m)$ is a union of sets of the from $J(i, k)$. $\mathrm{Per}_{p_i}(\eta)$ is exactly the set on which η is defined at the end of the i^{th} step, since the sets $J(i, k)$ are translates of one another by multiplies of p_i and are filled with different symbols at different stages of the construction. It is clear that the p_i-skeleton is not periodic with any smaller period, so (p_i) is a period structure for η.

We continue to use the notation of Sect. 2. In particular, $\pi\colon (\bar{\mathcal{O}}(\eta), S) \to (G, \hat{1})$ is the maximal equicontinuous factor.

Proposition 3.1. *The Oxtoby sequence η is regular if and only if*

$$\sum_{i=1}^{\infty} \frac{p_i}{p_{i+1}} \quad \textit{diverges.}$$

Proof. Recall that $d_i = \frac{1}{p_i} \# \{\eta \in \mathbb{Z}/p_i\mathbb{Z} \colon \eta \in \mathrm{Per}_{p_i}\eta\}$. Then $d_1 = \frac{2}{p_1}$. For $i \geqq 1$,

$$d_{i+1} = d_i + (1 - d_i)\frac{2p_i}{p_{i+1}}$$

since $\frac{2p_i}{p_{i+1}}$ is the proportion of $[0, p_{i+1}) \setminus \mathrm{Per}_{p_i}(\eta)$ which is in $\mathrm{Per}_{p_{i+1}}(\eta)$. Hence

$$\begin{aligned} 1 - d_{i+1} &= (1 - d_i)\left(1 - \frac{2p_i}{p_{i+1}}\right) \\ &= \left(1 - \frac{2}{p_1}\right) \prod_{j=1}^{i} \left(1 - \frac{2p_j}{p_{j+1}}\right) \end{aligned}$$

by induction. Thus $\lim_{i\to\infty} d_i=1$ if and only if this product tends to 0, which happens exactly when $\sum_{i=1}^{\infty} \frac{p_i}{p_{i+1}}$ diverges.

Theorem 3.2. *If the Oxtoby sequence η is not regular, then the set of ergodic invariant measures on $(\bar{\mathcal{O}}(\eta), S)$ is in one-to-one correspondence with Σ.*

Before proving this theorem, we must establish some notation and two lemmas.

Notation. By Corollary 2.4, for each $g\in G$ the sets $\operatorname{Per}_{p_i}(\omega)$ and $\operatorname{Aper}(\omega)$ are independent of the choice of $\omega\in\pi^{-1}(g)$. We will sometimes denote these sets $\operatorname{Per}_{p_i}(g)$ and $\operatorname{Aper}(g)$.

Lemma 3.3. (i) For all $\omega\in\bar{\mathcal{O}}(\eta)$, $\omega(n)$ is constant on $\operatorname{Aper}(\omega)$.

(ii) For each $g\in G$ and $\sigma\in\Sigma$ there is an $\omega\in\pi^{-1}(g)$ with $\omega(n)=\sigma$ for all $n\in\operatorname{Aper}(g)$.

Proof. (i) Let $\pi(\omega)=g=(n_i)$. For each i, ω has the same p_i-keleton as $S^{n_i}\eta$ and so $[-n_i, p_i-n_i)\cap\operatorname{Aper}(\omega)\subset[-n_i, p_i-n_i)\setminus\operatorname{Per}_{p_i}(S^{n_i}\eta)=J(i,0)-n_i$. If $S^m\eta\in A^i_{n_i}$ then $m=n_i+kp_i$ for some $k\in\mathbb{Z}$; then $S^m\eta(n)$ is constant on $J(i,0)-n_i$ since $\eta(n)$ is constant on $J(i,k)$. Since $\omega\in\bar{A}^i_{n_i}$, $\omega(n)$ must also be constant on $J(i,0)-n_i$. Hence $\omega(n)$ is constant on $[-n_i, p_i-n_i)\cap\operatorname{Aper}(\omega)$ for all i. If $-n_i\to-\infty$ and $p_i-n_i\to\infty$ we can conclude that $\omega(n)$ is constant on $\operatorname{Aper}(\omega)$. If either of these conditions fail, it is easy to see that $g=\hat{m}$ for some $m\in\mathbb{Z}$; then $\omega=S^m\eta$ and $\operatorname{Aper}(\omega)=\emptyset$.

(ii) Let $g=(n_i)\in G$ and $\sigma\in\Sigma$. The sequences $S^{n_i}\eta$ all have the same p_j-skeleton for $i>j$, so for each $n\in\operatorname{Aper}(g)$, $S^{n_i}\eta(n)$ is eventually constant. $S^{n_i}\eta(n)=\sigma_{i+1}$ for $n\in J(i,0)-n_i$, which contains $[-n_i, p_i-n_i)\cap\operatorname{Aper}(g)$. We choose $i_1<i_2<\ldots<i_j<\ldots$ with $\sigma_{i_j}\to\sigma$. Then $S^{n_{i_j-1}}\eta$ converges to the desired ω as $j\to\infty$.

Set $Z=G\times\Sigma$, with the product topology. We define a flow T on Z by $T(g,\sigma)=(g+\hat{1},\sigma)$. The ergodic measures on (Z,T) are exactly those of the form $m_\sigma=m\times\delta_\sigma$, where m is the Haar measure on G and δ_σ is the point measure on Σ, $\delta_\sigma(\{\sigma\})=1$. We define a map $\varphi\colon Z\to\bar{\mathcal{O}}(\eta)$ by mapping (g,σ) to the unique $\omega\in\pi^{-1}(g)$ with $\omega(n)=\sigma$ for all $n\in\operatorname{Aper}(g)$. The map φ is $1-1$ except on $\varphi^{-1}(\mathcal{T})$, and $\varphi\circ T=S\circ\varphi$. It can be seen that φ is *not* continuous, but we do have:

Lemma 3.4. *φ is bimeasurable.*

Proof. First we show φ is measurable. Sets of the form

$$U=U(\sigma,\varepsilon)=\{\omega\in\bar{\mathcal{O}}(\eta)\colon |\omega(0),\sigma|<\varepsilon\}$$

and their translates by powers of S form a sub-basis for $\mathcal{B}(\bar{\mathcal{O}}(\eta))$, so it suffices to show $\varphi^{-1}(U)\in\mathcal{B}(Z)$. We write $U=(U\cap C)\cup(U\setminus C)$. (Recall $C=\{\omega\in\bar{\mathcal{O}}(\eta)\colon 0\notin\operatorname{Aper}(\omega)\}\in\pi^{-1}(\mathcal{B}(G))$.) For $j=0,1,\ldots$ we set

$$\begin{aligned} V_j&=\{\omega\in\bar{\mathcal{O}}(\eta)\colon 0\in\operatorname{Per}_{p_i}(\omega,\sigma_j) \text{ for some } i\}\\ &=\bigcup\{\bar{A}^i_n\colon 0\in\operatorname{Per}_{p_i}(S^n\eta,\sigma_j)\}\in\pi^{-1}(\mathcal{B}(G)).\end{aligned}$$

Then

$$U\cap C=\bigcup\{V_j\colon |\sigma,\sigma_j|<\varepsilon\}\in\pi^{-1}(\mathscr{B}(G))$$
$$\varphi^{-1}(U\cap C)=\pi(U\cap C)\times\Sigma\in\mathscr{B}(Z)$$
$$U\setminus C=\{\omega\colon |\omega(0),\sigma|<\varepsilon\}\setminus C$$
$$\varphi^{-1}(U\setminus C)=(G\setminus\pi(C))\times\{\tau\in\Sigma\colon |\tau,\sigma|<\varepsilon\}\in\mathscr{B}(Z)$$

and so $\varphi^{-1}(U)\in\mathscr{B}(Z)$.

To show φ^{-1} is measurable, we note that sets of the form

$$W=W(i,n,\sigma,\varepsilon)=\pi(\bar{A}_n^i)\times\{\tau\colon |\sigma,\tau|<\varepsilon\}$$

generate $\mathscr{B}(Z)$. If we set $B=U\setminus C$, where U and C are as above, then

$$S^l B=\{\omega\in\bar{\mathcal{O}}(\eta)\colon l\in\operatorname{Aper}(\omega) \text{ and } |\sigma,\omega(l)|<\varepsilon\}$$

$$\varphi(W)=\bar{A}_n^i\cap[\mathscr{T}\cup(\bigcup_{l\in\mathbb{Z}}S^l B)]\in\mathscr{B}(\bar{\mathcal{O}}(\eta)).$$

Proof of Theorem 3.2. For each $\sigma\in\Sigma$ we define μ_σ on $\mathscr{B}(\bar{\mathcal{O}}(\eta))$ by $\mu_\sigma(B)=m_\sigma(\varphi^{-1}(B))$. It is easy to see that μ_σ is an ergodic measure on $(\bar{\mathcal{O}}(\eta),S)$ with $\mu_\sigma(\varphi(G\times\{\sigma\}))=1$. If $\tau\neq\sigma$ then

$$\varphi(G\times\{\sigma\})\cap\varphi(G\times\{\tau\})=\mathscr{T};$$

since $\mu_\sigma(\mathscr{T})=\mu_\tau(\mathscr{T})=0$, μ_σ and μ_τ are distinct measures.

Finally, suppose that μ is an ergodic measure on $(\bar{\mathcal{O}}(\eta),S)$. The formula $\nu(A)=\mu(\varphi(A))$ defines an ergodic measure on (Z,T). Hence $\nu=m_\sigma$ for some $\sigma\in\Sigma$. If $B\in\mathscr{B}(\bar{\mathcal{O}}(\eta))$ then $B\setminus\mathscr{T}=\varphi(\varphi^{-1}(B\setminus\mathscr{T}))$ and

$$\begin{aligned}\mu(B)&=\mu(B\setminus\mathscr{T})=\nu(\varphi^{-1}(B\setminus\mathscr{T}))=m_\sigma(\varphi^{-1}(B\setminus\mathscr{T}))\\&=\mu_\sigma(B\setminus\mathscr{T})=\mu_\sigma(B).\end{aligned}$$

Thus $\mu=\mu_\sigma$.

4. Toeplitz Sequences Constructed from Subshifts

In Sect. 3 we obtained minimal flows with arbitrarily many ergodic measures at the expense of working with an arbitrary compact symbol space Σ. In this section we show how to do this while remaining in $\{0,1\}^{\mathbb{Z}}=X$.

Let (Y,S) be a subshift of (X,S) containing at least two points. For $r\in\mathbb{N}$ we let $\mathrm{Bl}_r(Y)$ denote the set of r-blocks occurring in Y, and β_r its cardinality. In the construction of η which follows, if $b=b_1b_2\dots b_r\in\mathrm{Bl}_r(Y)$ and $J=\{n_1,n_2,\dots,n_r\}\subset\mathbb{Z}$ with $n_1<n_2<\dots<n_r$, then by *filling the set J with the block b* we shall mean setting $\eta(n_1)=b_1,\dots,\eta(n_r)=b_r$.

Construction. As with Oxtoby sequences, we construct η in steps.

Step 1. Choose $p_1>2$ and set $\eta(n)=0$ for $n\equiv -1 \bmod p_1$, $\eta(n)=1$ for $n\equiv 0 \bmod p_1$.

Step 2. We let $J(1,k)=[kp_1+1,(k+1)p_1-1)$ for $k\in\mathbb{Z}$. $J(1,k)$ has cardinality $r_1=p_1-2$. For $k=-1,0,\ldots,\beta_{r_1}-2$ we fill each $J(1,k)$ with a different element of $\mathrm{Bl}_{r_1}(Y)$. We choose $p_2>p_1\beta_{r_1}$, $p_1|p_2$. For $k'\equiv k \bmod \frac{p_2}{p_1}$, $k\in[-1,\beta_{r_1}-2]$ we fill $J(1,k')$ with the block that was used to fill $J(1,k)$, so that η has period p_2 where it is defined. (Since $p_2>p_1\beta_{r_1}$, η remains undefined on some sets $J(1,k)$.)

Step $i+1$. Let $J(i,k)$ be the set of $n\in[kp_i,(k+1)p_i)$ for which η has not been defined after the i^{th} step, with $\#J(i,k)=r_i$. For $k=-1,0,\ldots,\beta_{r_i}-2$ we fill each $J(i,k)$ with a different block in $\mathrm{Bl}_{r_i}(Y)$. We then choose $p_{i+1}>p_i\beta_{r_i}, p_i|p_{i+1}$ and fill $J(i,k')$ in the same way as $J(i,k)$ for $k'\equiv k \bmod \frac{p_{i+1}}{p_i}$, $k\in[-1,\beta_{r_i}-2]$.

Since $\beta_{r_i}\geqq 2$ for all i, after the i^{th} step η is defined on $[-p_i,p_i]$. If $1\leqq j\leqq r_i$ there are at least two r_i-blocks of Y which differ in the j^{th} coordinate, so the j^{th} place in $J(i,k)$ is not filled with the same element of $\{0,1\}$ for all k. It follows that $\mathrm{Per}_{p_i}(\eta)$ is exactly the set of integers on which η is defined by the end of the i^{th} step. Thus (p_i) is a period structure for the Toeplitz sequence η.

Proposition 4.1. *The sequence η is regular if and only if*

$$\sum_{i=1}^{\infty}\frac{p_i\beta_{r_i}}{p_{i+1}}$$

diverges.

We omit the proof, which is similar to that of Proposition 3.1. Note that for any Y, we can choose (p_i) to make this sum converge. For the rest of the section η will be a non-regular Toeplitz sequence constructed as above. $G,\pi,\mathcal{T},C$ are as defined in Sect. 2. We set

$$\begin{aligned}D&=\{\omega\in\bar{\mathcal{O}}(\eta)\colon \mathrm{Aper}(\omega)\text{ is a 2-sided infinite sequence}\}\\&=\bar{\mathcal{O}}(\eta)\setminus(\liminf_{k\to\infty} S^kC\cup\liminf_{k\to\infty} S^{-k}C)\in\pi^{-1}(\mathcal{B}(G)).\end{aligned}$$

Lemma 4.2. *For η non-regular, $m\circ\pi(D)=1$.*

Proof. Since $\liminf S^kC$ is an S-invariant set in $\pi^{-1}(\mathcal{B}(G))$, it must have $m\circ\pi$-measure 0 or 1. For η non-regular, $m(\pi(C))=d<1$, so $m\circ\pi(\liminf S^kC)=0$. The same holds for $\liminf S^{-k}C$.

Let $Z=G\times Y$ with the product topology, and let $\theta\colon G\to\{0,1\}$ be the indicator function of $G\setminus\pi(C)$; that is $\theta(g)=1$ if $0\in\mathrm{Aper}(g)$ and $\theta(g)=0$ otherwise. We define a map $T\colon Z\to Z$ by

$$T(g,y)=(g+\hat{1},S^{\theta(g)}y).$$

Thus T is a "piecewise power" skew product. (See Belinskaya, 1974.) T is a bimeasurable bijection. (It is not a homeomorphism.)

We define $\varphi\colon Z\to\{0,1\}^{\mathbb{Z}}$ by sending (g,y) to the sequence x defined as follows. For $n\in\mathbb{Z}\setminus\mathrm{Aper}(g)$ we let $x(n)=\omega(n)$ where $\omega\in\pi^{-1}(g)$. (This is independent of the choice of ω.) For $n\in\mathrm{Aper}(g)$, we set $i(n)=m\in\mathbb{N}\cup\{0\}$ if n is the

$(m+1)^{\text{th}}$ smallest element of $\mathrm{Aper}(g)\cap(\mathbb{N}\cup\{0\})$, and $i(n)=-m\in\mathbb{Z}^-$ if n is the m^{th} greatest element of $\mathrm{Aper}(g)\cap\mathbb{Z}^-$. We then set $x(n)=y i(n))$. Thus we have "filled" $\mathrm{Aper}(g)$ with as much of y as will fit. If $g\in\pi(D)$ all of y is used, so φ is $1-1$ on $\pi(D)\times Y$. We have positioned y in $\mathrm{Aper}(g)$ so that $y(0)$ fills the first non-negative place in $\mathrm{Aper}(g)$. It is not hard to see that $Sx=\varphi(g+\hat{1},y)$ if $0\notin\mathrm{Aper}(g)$ and $Sx=\varphi(g+\hat{1},Sy)$ if $0\in\mathrm{Aper}(g)$; that is, $S\circ\varphi=\varphi\circ T$.

Lemma 4.3. $\varphi(Z)=\bar{\mathcal{O}}(\eta)$.

Proof. The proof is similar to that of Lemma 3.3. We first show $\varphi(Z)\supset\bar{\mathcal{O}}(\eta)$. If $\omega\in\mathcal{T}$, $\pi(\omega)=g$, then $\varphi(g,y)=\omega$ for all $y\in Y$. If $\omega\notin\mathcal{T}$, $\pi(\omega)=g=(n_i)$, then the intervals $[-n_i,p_i-n_i)$ increase to fill $\mathbb{Z}$. For each i, $[-n_i,p_i-n_i)\cap\mathrm{Aper}(\omega)$ is either empty or equal to $[-n_i,p_i-n_i)\setminus\mathrm{Per}_{p_i}(\omega)$. The latter set is filled by an r_i-block of Y in each $S^m\eta\in A^i_{n_i}$, since it is of the form $J(i,k)-m$ for some $k\in\mathbb{Z}$. Hence it is filled by an r_i-block of Y in $\omega\in\bar{A}^i_{n_i}$. It follows that $\mathrm{Aper}(\omega)$ is filled by a sequence, or part of a sequence, in Y.

Now let $x=\varphi(g,y)$ with $g=(n_i)\in G$, $y\in Y$. For each $i\in N$ we pick $k_i\in\mathbb{Z}$ so that the r_i-block of Y which fills $J(i,k_i)$ in η matches the r_i-block of y which fills $[-n_i,p_i-n_i)\setminus\mathrm{Per}_{p_i}(g)$ in x. Set $m_i=k_i p_i+n_i$; then $S^{m_i}\eta\to x$ and $x\in\bar{\mathcal{O}}(\eta)$.

Lemma 4.4. *φ is bimeasurable.*

We omit the proof, which is similar to that of Lemma 3.4.

Theorem 4.5. *Let η be a non-regular Toeplitz sequence constructed as above from (Y,S). There is a one-to-one correspondence between the ergodic T-invariant Borel measures ν on Z and the ergodic measures on $(\bar{\mathcal{O}}(\eta),S)$ given by $\nu\leftrightarrow\nu\circ\varphi^{-1}$.*

Proof. If ν is a T-invariant Borel measure on Z then $\nu\circ\varphi^{-1}$ is an invariant measure on $(\bar{\mathcal{O}}(\eta),S)$. Since φ is one-to-one on $\varphi^{-1}(D)$ and $\nu(\varphi^{-1}(D))=\nu(\pi(D)\times\Sigma)$ $=m\circ\pi(D)=1$, $\nu\circ\varphi^{-1}$ is ergodic if and only if ν is. If $\nu\circ\varphi^{-1}=\nu'\circ\varphi^{-1}$, then for $B\in\mathcal{B}(Z)$, $\nu(B)=\nu(B\cap\varphi^{-1}(D))=\nu\circ\varphi^{-1}(\varphi(B)\cap D)=\nu\circ\varphi^{-1}(\varphi(B)\cap D)=\nu'(B)$, so $\nu=\nu'$. Finally, if μ is an invariant measure on $(\bar{\mathcal{O}}(\eta),S)$ then $\nu=\mu\circ\varphi$ is a T-invariant Borel measure on Z and $\mu=\nu\circ\varphi^{-1}$.

Remark. For each invariant Borel measure ν on Z, φ is a conjugacy between the measure-theoretic dynamical systems $(Z,\mathcal{B}(Z),\nu,T)$ and $(\bar{\mathcal{O}}(\eta),\ \mathcal{B}(\bar{\mathcal{O}}(\eta)),$ $\nu\circ\varphi^{-1}S)$.

If λ is an invariant measure on (Y,S) then $m\times\lambda$ is a T-invariant Borel measure on Z. If λ is ergodic, $m\times\lambda$ need not be. (See Example 4.7.) Distinct ergodic measures λ,λ' on (Y,S) are mutually singular, so $m\times\lambda$ and $m\times\lambda'$ are mutually singular and can be decomposed into mutually singular ergodic measures. Hence Z always admits at least as many ergodic measures as Y.

Example 4.6. Take $Y=\{0,1\}^{\mathbb{Z}}$. The set of ergodic measures on (Y,S) has cardinality of the continuum, so the same is true for $(\bar{\mathcal{O}}(\eta),S)$.

Remark. For $n\in\mathbb{N}$ we set

$$\theta_n(g)=\sum_{k=0}^{n-1}\theta(g+\hat{k})$$

for $g \in G$. Then

$$T^n(g, y) = (g + \hat{n}, S^{\theta_n(g)} y)$$

for all $(g, y) \in Z$. $\theta_n(g)$ is simply the cardinality of $\mathrm{Aper}(g) \cap [0, n)$. If $g \in A_0^i$ then $\mathrm{Aper}(g) \cap [0, p_i)$ is either $J(i, 0)$ or $\emptyset$, and $\theta_{p_i}(g) = r_i$ or 0.

Example 4.7. Let $1 < r \in N$. Define $y_r \in \{0, 1\}^{\mathbb{Z}}$ by $y_r(n) = 1$ if $r | n$, $y_r(n) = 0$ otherwise. We set $Y = \mathcal{O}(y_r) = \bar{\mathcal{O}}(y_r)$; (Y, S) is a cyclic permutation of r points and is uniquely ergodic. We construct a Toeplitz sequence η from (Y, S), choosing $p_1 = r + 2$ so that $r_1 = r$. We claim that $(\bar{\mathcal{O}}(\eta), S)$ admits exactly r ergodic measures.

For $k = 0, 1, \ldots, r-1$ we set

$$F_k = \pi(\bar{A}_0^1) \times \{S^k y_r\}$$
$$F_k' = F_k \cup T F_k \cup \ldots \cup T^{p_1 - 1} F_k.$$

If $z = (g, y) \in F_k$, then by the remark above $T^{p_1} z = (g + \hat{p}_1, S^r y)$ or $(g + \hat{p}_1, y)$. Hence $T^{p_1} F_k = F_k$, and each F_k' is a closed T-invariant subset of Z. The sets F_k' partition Z. It is not hard to see that F_k' supports a unique ergodic measure v_k given by

$$v_k(B) = m(\pi \circ \varphi(B \cap F_k')).$$

Thus $(\bar{\mathcal{O}}(\eta), S)$ has exactly r ergodic measures.

Example 4.8. We let $s_1 \in \mathbb{N}$ and $s_{i+1} = s_i \cdot 2^{s_i + i}$ for $i \in N$. Let $y_{s_i} \in \{0, 1\}^{\mathbb{Z}}$ be defined as in Example 4.7, and set

$$Y = \left(\bigcup_{i=1}^{\infty} \mathcal{O}(y_{s_i}) \right) \cup \{y_0\},$$

where $y_0(n) = 0$ for all $n \in \mathbb{Z}$. Y is a closed shift-invariant set. We construct η from the subshift (Y, S); we take $p_1 = s_1 + 2$ and at the $(i+1)^{\text{th}}$ step we shoose $p_{i+1} = p_i(2^{s_i + i} - \beta_{r_i})$. A little calculation shows that for this choice of (p_i), $s_i = \# J(i, k) = r_i$ for all i. We have

$$\sum_{i=1}^{\infty} \frac{p_i \beta_{r_i}}{p_{i+1}} = \sum_{i=1}^{\infty} \frac{\beta_{r_i}}{2^{r_i + i} - \beta_{r_i}} < \sum_{i=1}^{\infty} \frac{1}{2^i - 1} < \infty$$

since $\beta_{r_i} < 2^{r_i}$, so η is not regular.

We can partition Z into T-invariant closed sets E_i where $E_0 = G \times \{y_0\}$, $E_i = G \times \mathcal{O}(y_{s_i})$ for $i \in \mathbb{N}$. Following Example 4.7, for each $i \in \mathbb{N}$ we set

$$F_k^i = \pi(A_0^i) \times \{S^k y_{s_i}\}$$
$$F_k^{i\prime} = F_k^i \cup T F_k^i \cup \ldots \cup T^{p_i - 1} F_k^i$$

for $k = 0, 1, \ldots, s_i - 1$. The $F_k^{i\prime}$ partition E_i, and each supports a unique ergodic measure. Thus Z admits countably many ergodic T-invariant Borel measures, and the set of ergodic measures on $(\bar{\mathcal{O}}(\eta), S)$ is countably infinite.

5. Calculation of Entropy

From our analysis it is easy to compute the topological entropy of each of the flows we have constructed. First, we use the well-known variational principle of Dinaburg (1970) and Goodman (1971) that the topological entropy $h(X, T)$ of a flow (X, T) is the supremum over all invariant measures ν of the metric entropy $h_\nu(X, T)$. For each of our flows η, we have a map $\varphi\colon Z \to \bar{\mathcal{O}}(\eta)$ (where $Z = G \times \Sigma$ for the examples of Sect. 3 and $Z = G \times Y$ for Sect. 4) which for each T-invariant measure ν is an isomorphism of the measure-theoretic dynamical systems $(Z, \mathscr{B}(Z), \nu, T)$ and $(\bar{\mathcal{O}}(\eta), \mathscr{B}(\bar{\mathcal{O}}(\eta)), \nu \circ \varphi^{-1}, S)$, and so preserves metric entropy. Since $\nu \leftrightarrow \nu \circ \varphi^{-1}$ is a one-to-one correspondence of invariant measures, we must have $h(Z, T) = h(\bar{\mathcal{O}}(\eta), S)$.

Each of the flows (Z, T) described in Sect. 3 is a product of the flow $(G, \hat{1})$ and the trivial flow on $\Sigma^{\mathbb{Z}}$. The entropy of a product is the sum of the entropies of the factors, and translation on a compact group has entropy 0. Hence the flows in Sect. 3 all have 0 entropy.

Formulas for the entropy of a piecewise power skew product appear in Belinskaya (1974) and Newton (1969). The version which is most convenient for us is in a more recent paper of Marcus and Newhouse (1979). We state a special case of the theorem to avoid making new definitions.

Theorem 5.1. *Let (X, R) and (Y, S) be flows with finite topological entropy, and let $T\colon X \times Y \to X \times Y$ be given by $T(x, y) = (Rx, S^{\theta(x)}y)$, where θ is a Borel-measurable integer-valued function on X. Let $\pi_1\colon X \times Y \to X$ denote the natural projection. If ν is invariant on (X, R),*

$$\sup h_\mu(X \times Y, T) = h_\nu(X, R) + h(Y, S)\left|\int \theta(x)\, d\nu\right|$$

where the sup is taken over all T-invariant measures μ with $\pi_1(\mu) = \nu$.

We apply this to the flows of Sect. 4, taking $(X, R) = (G, \hat{1})$ and Y, S, θ as in Sect. 4. Then for every T-invariant μ, $\pi_1(\mu) = m$, the Haar measure on G. Thus Theorem 5.1 reduces to

$$\begin{aligned} h(Z, T) &= h_m(G, \hat{1}) + h(Y, S)\left|\int \theta(x)\, dm\right| \\ &= 0 + h(Y, S)\, m(G \setminus \pi(C)) \\ &= (1 - d)\, h(Y, S). \end{aligned}$$

Hence the flows of Examples 4.7 and 4.8 have entropy 0. The flow of Example 4.6 has entropy $(1-d)\log 2$. It is possible to make d arbitrarily small by choosing a rapidly increasing period structure (p_i) in the construction. This yields:

Corollary 5.2. *There exist Toeplitz $0-1$ sequences with entropy arbitrarily close to* $\log 2$.

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