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The Student t-Distribution of Any Degree of Freedom is Infinitely Divisible

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1. Introduction

Let $\phi_\nu(x)=K_{\nu-1}(x^{1/2})/x^{1/2}K_\nu(x^{1/2})$, where $K_\nu(u)$ is the modified Bessel function, as defined, e.g., in [6] and [1]. It has been shown by Kelker [5] and by Ismail and Kelker [4] that the complete monotonicityof $\phi_\nu(x)$ is the necessary and sufficient condition for the infinite divisibility of the Student t-distribution with $k=2\nu$ degrees of freedom, and also for that of $Y_k=(\chi_k^2)^{-1}$, where χ_k^2 is a chi-square variable with k degrees of freedom. Ismail and Kelker conjectured that $\phi_\nu(x)$ is, in fact, completely monotonic for all real $\nu\geq 0$. From this conjecture would follow the infinite divisibility of the mentioned distributions for odd $k=2n+1$, by taking $\nu=n+1/2$, and for even $k=2n$, by taking $\nu=n$.

It is the purpose of this paper to prove a theorem which is somewhat stronger than, and implies that conjecture. For the particular case of $k=2n+1$, the conjecture has already been proved in [3]. That case of odd k is also covered by the present work, but the connection between the results proved here and those of [3] is not entirely trivial. The link is provided by Corollary 3 (Section 6).

The notations for Bessel functions are the standard ones, as used, e.g., in [1], or [6]. Whenever a needed formula occurs in [1], it will be quoted by its number there. So, e.g., [1; 29.3.37] will recall that

$$\mathscr{L}^{-1}((s^{1/2}+a)^{-1})=(\pi t)^{-1/2}-a e^{a^2 t}\operatorname{erfc}(a t^{1/2}), \qquad \text{etc.}$$

Here and in the rest of the paper $\mathscr{L}^{-1}$ stands for the inverse Laplace transform, while $\mathscr{L}$ stands for the direct one. The reference for complete monotonicity is [7], that for general properties of Bessel functions is [6].

While working on this problem I have benefited greatly from conversations and correspondence with interested colleagues, in particular with S. Kotz and B. Epstein. A referee also made a useful suggestion. I take this opportunity to gratefully acknowledge my indebtedness to them all.

2. Main Results

Theorem. *For real* $\nu \geqq 0$, *the function*

$$\phi_\nu(x) = K_{\nu-1}(x^{1/2})/x^{1/2} K_\nu(x^{1/2}) \tag{1}$$

is the Stieltjes transform of the function

$$g_\nu(x) = 2\{\pi^2 x (J_\nu^2(x^{1/2}) + Y_\nu^2(x^{1/2}))\}^{-1}.$$

Corollary 1. $\phi_\nu(x)$ *is the Laplace transform of a completely monotonic function, positive for* $x > 0$.

Corollary 2. *The Student t-distribution and the distribution* Y_k *(defined in Section 1) are infinitely divisible for all degrees of freedom k.*

Corollary 3. *For* $x > 0$,

$$J_{n+1/2}^2(x^{1/2}) + J_{-(n+1/2)}^2(x^{1/2}) = 2\pi^{-1} x^{-(n+1/2)} \prod_{j=1}^n (x + \alpha_j^2)$$

the product being taken over all the zeros α_j *of* $K_{n+1/2}(u)$.

Corollary 4. *For* $\nu \geqq 0$, $t > 0$

$$\frac{4}{\pi^2} \int_0^\infty \frac{t\,dx}{x(J_\nu^2(x) + Y_\nu^2(x))(x^2 + t^2)} = \frac{2}{\pi^2} \int_0^\infty \frac{t\,dx}{x(J_\nu^2(x^{1/2}) + Y_\nu^2(x^{1/2}))(x + t^2)} = \frac{K_{\nu-1}(t)}{K_\nu(t)}.$$

The proof of the theorem is given in Section 3–5. Corollary 1 follows from the fact that the Stieltjes transform is the iterated Laplace transform, so that the theorem is equivalent to the statement

$$\mathscr{L}^{-1}(\phi_\nu) = \mathscr{L}(g_\nu) = G_\nu(t).$$

By Bernstein's Theorem (see [7], pp. 160–161), $G_\nu(t)$ is completely monotonic, because it is the Laplace transform of the positive function $g_\nu(x)$; in particular, $G_\nu(t) > 0$ for $t > 0$.

Corollary 2 follows from Corollary 1, on account of the quoted results of [5, 4] and Bernstein's Theorem.

Corollary 4 follows from the Theorem, if we write $\phi_\nu(x)$ as a Stieltjes transform of $g_\nu(x)$ and then replace $x^{1/2}$ by t.

Corollary 3 follows almost trivially, by comparing (1) with the results of [3]. In fact, Corollary 3 may well be known, but as it could not be located in the literature, a simple, direct proof of it is provided in Section 6. By using it, the main result of [3] follows also from the present theorem.

3. Two Lemmas

We recall the following known facts that can be found in Chapter *XV* of [6]. The functions $K_\nu(u)$, while in general not uniform in the complex plane, become uniform in the plane cut along the negative real axis ("cut plane", for short) (see [1], p. 358). For $\nu - 1/2$ not an integer they have infinitely many zeros

$\alpha_j (j=1,2,\ldots)$. None of these has $|arg\,\alpha_j| \leqq \frac{\pi}{2}$; exactly $2m$ simple zeros have $\frac{\pi}{2} < |\arg\alpha_j| < \pi$, while all other zeros have $|arg\,\alpha_j| > \pi$. Here $2m$ is the even integer closest to $\nu - 1/2$. If $\nu - 1/2 = n$, an integer, then $K_\nu(u)$ has exactly n zeros. In particular, for n odd (i.e., for $\nu = 2k - 1/2$, $k \in Z^+$), $K_\nu(u)$ has an odd number of zeros, of which exactly one is real and negative. In all other cases with real $\nu \geqq 0$, the zeros α_j of $K_\nu(u)$ with $\pi/2 < |arg\,\alpha_j| < \pi$ occur in pairs of complex conjugate ones.

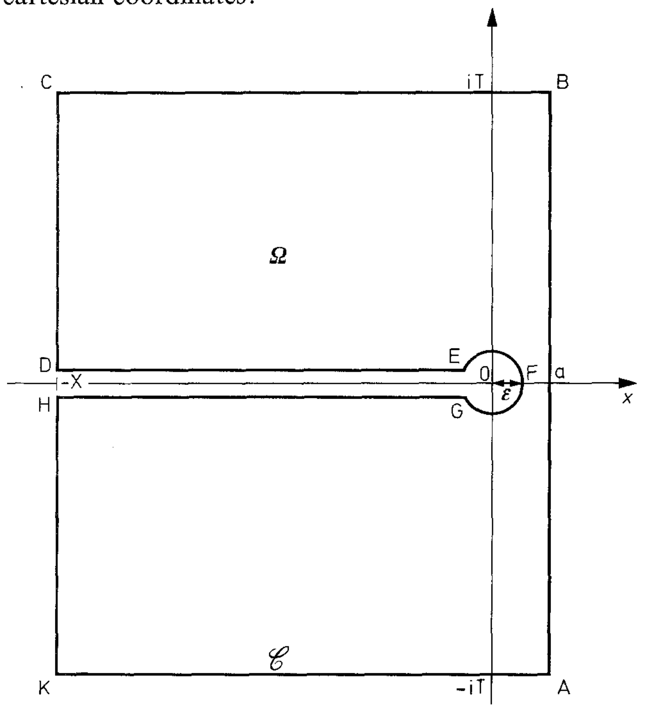
Lemma 1. *The function* $K_{\nu-1}(u)/u K_\nu(u) - \sum_{j=1}^{2m} \{\alpha_j(\alpha_j - u)\}^{-1}$ *is single-valued and holomorphic in the cut plane.*

Proof. Define the function $\psi(u) = \psi_\nu(u)$ by $\psi(u) = K_{\nu-1}(u)/u K_\nu(u) - \sum_{j=1}^{2m} A_j(u-\alpha_j)^{-1}$ where $A_j/(u-\alpha_j)$ is the principal part of $K_{\nu-1}(u)/u K_\nu(u)$ at the simple pole $u = \alpha_j$. Here and in what follows, the subscript ν will be suppressed, whenever possible. Then $\psi(u)$ is single-valued in the cut plane because the functions that define it are; it also is holomorphic there because at each α_j the poles of $\phi(u)$ are cancelled by the pole of $A_j/(u-\alpha_j)$. In order to find A_j, we observe that

$$\lim_{u\to\alpha_j} \{(u-\alpha_j) K_{\nu-1}(u)/u K_\nu(u)\} = K_{\nu-1}(\alpha_j)/\alpha_j K'_\nu(\alpha_j) = -K_{\nu-1}(\alpha_j)/\alpha_j K_{\nu-1}(\alpha_j)$$
$$= -\alpha_j^{-1},$$

by use of [1; 9.6.26]. Hence, $A_j = -\alpha_j^{-1}$ and Lemma 1 follows.

Let $\varepsilon > 0$ be arbitrarily small and select $a > \varepsilon$; also, let T, X and q be (arbitrarily large) real and positive. Consider the following set of points, defined by their cartesian coordinates:



$A(a, -T)$, $B(a, T)$, $C(-X, T)$, $D(-X, \varepsilon^q)$, $E(-\varepsilon, \varepsilon^q)$, $F(\varepsilon, 0)$, $G(-\varepsilon, -\varepsilon^q)$, $H(-X, -\varepsilon^q)$, $K(-X, -T)$. The contour $\mathscr{C}$, passing through these points, consists of the straight line segments AB, BC, CD and DE; the arc of circle EFG with center at the origin and radius ε; and the straight line segments GH, HK and KA. We shall integrate various functions around $\mathscr{C}$ and then take the limits as $q \to \infty$, $X \to \infty$, $T \to \infty$, $\varepsilon \to 0$, in that order, in as far as the order is relevant. In the limit, the segments DE and HG become the negative real axis and DE should be considered as running along the upper rim, and HG along the lower rim of the cut plane. The functions $K_\nu(u)$, $K_{\nu-1}(u)$ and $\psi(u)$ are single-valued in the closed, simply connected portion of the complex plane enclosed by $\mathscr{C}$, that we shall denote by Ω.

Lemma 2. *For* $z = r e^{i\theta}$, *with* $0 \leq |\theta| < \pi$, *define* $z^{1/2}$ *by* $z^{1/2} = r^{1/2} e^{i\theta/2}$, *single-valued in* Ω; *then*

$$I = \int_{\mathscr{C}} \phi(z) e^{zt} dz = 0$$

Proof. As $\phi(z)$ is single-valued in the cut plane, hence in Ω, it follows from Lemma 1 that

$$I = \int_{\mathscr{C}} \psi(z^{1/2}) e^{zt} dz + \sum_{j=1}^{2m} \alpha_j^{-1} \int_{\mathscr{C}} (\alpha_j - z^{1/2})^{-1} e^{tz} dz.$$

The first integral vanishes, because the integrand is single-valued and holomorphic in Ω. The same is true also of the other integrals, because $\alpha_j - z^{1/2} \neq 0$ in Ω. Indeed, there $|\arg z^{1/2}| < \frac{\pi}{2}$, while $|\arg \alpha_j| > \frac{\pi}{2}$ for all $j = 1, 2, \ldots, 2m$. The proof of Lemma 2 is complete. One may remark that the proof is valid even in the case of $\nu - 1/2 = 2n - 1$, an odd integer; indeed, the simple, real, negative zero α_0 of $K_\nu(u) = K_{2n-1/2}(u)$ falls outside $\mathscr{C}$.

4. Parenthetical Remark

It is worthwhile to verify computationally that $\int_{\mathscr{C}} (\alpha - z^{1/2})^{-1} e^{tz} dz = 0$ for $\alpha = \alpha_j$ ($j = 1, 2, \ldots, n$), because the same pattern of proof will be used in the somewhat more difficult, but analogous, computation of $\int_{\mathscr{C}} \phi(z) e^{zt} dz$.

Let $q \to \infty$, $X \to \infty$, $T \to \infty$ and observe that first the integrals along CD and HK, then those along BC and KA vanish in the limit, while the integral along AB becomes $2\pi i \mathscr{L}^{-1}((\alpha - z^{1/2})^{-1})$; the latter is easily computed by use of [1; 29.3.37] and equals $-2\pi i \{(\pi t)^{-1/2} + \alpha e^{\alpha^2 t} \operatorname{erfc}(-\alpha t^{1/2})\}$. The equality to be verified now reduces to

$$\{\int_{DE} + \int_{EFG} + \int_{GH}\} (\alpha - z^{1/2})^{-1} e^{tz} dz = 2\pi i \{(\pi t)^{-1/2} + \alpha e^{\alpha^2 t} \operatorname{erfc}(-\alpha t^{1/2})\}. \tag{2}$$

For $\varepsilon \to 0$, $\int_{EFG} (\alpha - z^{1/2})^{-1} e^{tz} dz \to \alpha^{-1} \int_{|z|=\varepsilon} dz = 0$ and the first member of (2) reduces to

$$\begin{aligned}\int_0^\infty \{(\alpha - i r^{1/2})^{-1} - (\alpha + i r^{1/2})^{-1}\} e^{-rt} dr &= 2i \int_0^\infty r^{1/2} (r + \alpha^2)^{-1} e^{-rt} dr \\ &= 2i \mathscr{L}(r^{1/2} (r + \alpha^2)^{-1}).\end{aligned}$$

By using [1; 29.3.114 and 29.2.9], with $(\alpha^2)^{1/2} = -\alpha$ (needed for $\mathrm{Re}(\alpha^2)^{1/2} > 0$), we find that

$$\mathscr{L}(r^{1/2}(r+\alpha^2)^{-1}) = -\frac{d}{dt}\left\{e^{\alpha^2 t}\cdot\frac{2}{\sqrt{\pi}}\int_{-\alpha t^{1/2}}^{\infty} e^{-u^2}\,du\right\}\cdot(-\pi/\alpha)$$
$$= \pi\alpha e^{\alpha^2 t}\operatorname{erfc}(-\alpha t^{1/2}) + (\pi/t)^{1/2}$$

and the left hand side of (2) becomes $2i\pi^{1/2}t^{-1/2} + 2\pi i\alpha e^{\alpha^2 t}\operatorname{erfc}(-\alpha t^{1/2})$, i.e., precisely the right hand side; equality (2) is proved.

5. Proof of the Theorem

By Lemma 2, $I = \int_{\mathscr{C}} \phi(z) e^{zt}\,dz = 0$. For $|u| \to 0$, $|\arg u| < 3\pi/2$, it follows from [1; 9.7.2] that $K_\nu(u) \sim (\pi/2u)^{1/2} e^{-u}$; hence, the integrand of I becomes $(1+o(1))z^{-1/2}e^{zt}$. It follows, in particular, that

$$\left|\int_K^H \phi(z) e^{zt}\,dz\right| = \left|\int_C^D \phi(z) e^{tz}\,dz\right| = (1+o(1))(\pi/2)^{1/2}\int_0^T X^{-1/2} e^{-Xt}\,dy$$
$$= (1+o(1))(\pi/2)^{1/2}\,T X^{-1/2} e^{-Xt} \to 0$$

for $t \geqq 0$, any $T > 0$ and $X \to \infty$. The integral $\int_B^C \phi(z) e^{tz}\,dz$ now becomes

$$\int_a^{-\infty} \phi(x+iT)\, e^{t(x+iT)}\,dx$$

and

$$\left|\int_B^C \phi(z) e^{zt}\,dz\right| = \left|\int_A^K \phi(z) e^{zt}\,dz\right| < (1+o(1))(\pi/2)^{1/2}\, T^{-1/2}\int_{-a}^{\infty} e^{-xt}\,dx$$
$$\leqq C(T^{1/2}t)^{-1} \to 0$$

for any $t > 0$ and $T \to \infty$. By observing also that $(2\pi i)^{-1}\int_A^B \phi(z) e^{tz}\,dz \to \mathscr{L}^{-1}(\phi)$, it follows that

$$\mathscr{L}^{-1}(\phi) + \frac{1}{2\pi i}\,\psi(t) = 0 \tag{3}$$

where

$$\psi(t) = \left(\int_{DE} + \int_{EFG} + \int_{GH}\right)\phi(z) e^{tz}\,dz. \tag{4}$$

We shall evaluate the integrals in (4) and prove

Lemma 3. $\psi(t) = -4i\pi^{-1}\mathscr{L}(\{x(J_\nu^2(x^{1/2}) + Y_\nu^2(x^{1/2}))\}^{-1})$.

If we substitute this in (3), we obtain that

$$\mathscr{L}^{-1}(\phi(z)) = 2\pi^{-2}\mathscr{L}(\{x(J_\nu^2(x^{1/2}) + Y_\nu^2(x^{1/2})\}^{-1}).$$

This finishes the proof of (1).

In order to estimate the integrals in (4), we first observe that, by [1; 9.6.9] as $|z| \to 0$, $K_\nu(z) \sim \frac{1}{2}\Gamma(z)(2/z)^\nu$, so that $\phi(z) = (1+o(1))(\Gamma(\nu-1)/\Gamma(\nu))(z/2)z^{-1} = (1+o(1))(2(\nu-1))^{-1}$ and

$$\int_{EFG} \phi(z) e^{tz}\,dz = \frac{1+o(1)}{2(\nu-1)}\int_{-\pi}^{\pi} e^{t\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta \to 0 \quad \text{as } \varepsilon \to 0.$$

Next, with $z=re^{i\pi}$,

$$\int_{DE}\phi(z)e^{tz}\,dz=\int_{\infty}^{0}\frac{K_{\nu-1}(r^{1/2}e^{i\pi/2})e^{-rt}}{r^{1/2}e^{i\pi/2}K_{\nu}(r^{1/2}e^{i\pi/2})}e^{i\pi}\,dr=\int_{0}^{\infty}\frac{K_{\nu-1}(e^{i\pi/2}r^{1/2})e^{-rt}}{(ir^{1/2})K_{\nu}(e^{i\pi/2}r^{1/2})}\,dr.$$

Similarly,

$$\int_{G}^{H}\phi(z)e^{tz}\,dz=-\int_{0}^{\infty}\frac{K_{\nu-1}(e^{-i\pi/2}r^{1/2})}{(-ir^{1/2})K_{\nu}(e^{-i\pi/2}r^{1/2})}e^{-rt}\,dr$$

and their sum is

$$-i\int_{0}^{\infty}\left(\frac{K_{\nu-1}(-ir^{1/2})}{K_{\nu}(-ir^{1/2})}+\frac{K_{\nu-1}(ir^{1/2})}{K_{\nu}(ir^{1/2})}\right)\frac{e^{-tr}}{r^{1/2}}\,dr.$$

We now replace the $K_{\nu}(\pm ir^{1/2})$ by $H_{\nu}^{(s)}(r^{1/2})$ $(s=1,2)$, by using [1; 9.6.4] and obtain

$$-\int_{0}^{\infty}\left(\frac{H_{\nu-1}^{(1)}(r^{1/2})}{H_{\nu}^{(1)}(r^{1/2})}-\frac{H_{\nu-1}^{(2)}(r^{1/2})}{H_{\nu}^{(2)}(r^{1/2})}\right)\frac{e^{-tr}}{r^{1/2}}\,dr$$

or, replacing the Hankel functions by $J_{\nu}(r^{1/2})\pm iY_{\nu}(r^{1/2})$ (see [1; 9.1.3 and 9.1.4])

$$-\int_{0}^{\infty}\left\{\frac{J_{\nu-1}(r^{1/2})+iY_{\nu-1}(r^{1/2})}{J_{\nu}(r^{1/2})+iY_{\nu}(r^{1/2})}-\frac{J_{\nu-1}(r^{1/2})-iY_{\nu-1}(r^{1/2})}{J_{\nu}(r^{1/2})-iY_{\nu}(r^{1/2})}\right\}\frac{e^{-tr}}{r^{1/2}}\,dr$$
$$=-2i\int_{0}^{\infty}\frac{J_{\nu}(r^{1/2})Y_{\nu-1}(r^{1/2})-J_{\nu-1}(r^{1/2})Y_{\nu}(r^{1/2})}{J_{\nu}^{2}(r^{1/2})+Y_{\nu}^{2}(r^{1/2})}\,\frac{e^{-tr}}{r^{1/2}}.$$

By using [1; 9.1.16] the last expression is seen to equal

$$-\frac{4i}{\pi}\int_{0}^{\infty}\frac{e^{-tr}}{r\{J_{\nu}^{2}(r^{1/2})+Y_{\nu}^{2}(r^{1/2})\}}\,dr.$$

With this, Lemma 3 is proved.

It is clear that $g(x)=2\{\pi^{2}x(J_{\nu}^{2}(x^{1/2})+Y_{\nu}^{2}(x^{1/2}))\}^{-1}>0$ for $x>0$. Hence, $G(t)=\mathscr{L}(g)=\int_{0}^{\infty}e^{-xt}g(x)\,dx$ is positive for $t>0$ and completely monotone. Consequently, also $\phi(x)=\mathscr{L}(G)=\int_{0}^{\infty}e^{-xt}G(t)\,dt$ is positive for $x>0$ and completely monotone. In both cases one may (but hardly needs to) invoke the easier half of Bernstein's theorem ([7], pp. 160–161). The proof of the Theorem is complete.

6. Discussion of Corollary 3

For any non-negative integer n, set

$$P_{n}(u)=(2u/\pi)^{1/2}e^{u}u^{n}K_{n+1/2}(u); \tag{5}$$

then (see [6] or [1], Chapter 9) $P_{n}(u)$ is a polynomial of exact degree n. In fact, it is the Bessel Polynomial of degree n in Burchnall's normalization (see [2]).

It follows that if $\nu=n+1/2$,

$$\phi_{n+1/2}(x)=K_{n-1/2}(x^{1/2})/x^{1/2}K_{n+1/2}(x^{1/2})=P_{n-1}(x^{1/2})/P_{n}(x^{1/2})$$

and it is in this representation of $\phi_\nu(x)$, for ν half an odd integer, that the problem was formulated in [4] and solved, in that particular case of odd $k=2\nu=2n+1$, in [3].

The result obtained there is equivalent to

$$\mathscr{L}^{-1}(\phi_{n+1/2})=\mathscr{L}(\pi^{-1}x^{n-1/2}\prod_{j=1}^{n}(x+\alpha_j^2)^{-1}). \tag{6}$$

Corollary 3 now follows by comparing (1) and (6), while also taking into account the uniqueness of the Laplace transform.

On the other hand, if one can prove Corollary 3 directly, then the result of [3] is an immediate consequence of (2). It is quite likely that Corollary 3 is, in fact, known. However, as no proof of it could be located in the literature and in view of its simplicity, a direct poof is given here for completeness.

The zeros of $P_n(u)$ and of $K_{n+1/2}(u)$ are the same and the leading coefficient of $P_n(u)$ is one; hence, $P_n(u)=\prod_{j=1}^{n}(u-\alpha_j)$. It follows that $P_n(ix^{1/2})P_n(-ix^{1/2})=\prod_{j=1}^{n}(x+\alpha_j^2)$, or, by using (5),

$$2\pi^{-1}x^{n+1/2}K_{n+1/2}(ix^{1/2})K_{n+1/2}(-ix^{1/2})=\prod_{j=1}^{n}(x+\alpha_j^2). \tag{7}$$

According to [1; 9.6.4], we may replace $K_{n+1/2}(ix^{1/2})$ by

$$-\tfrac{1}{2}\pi i e^{-(1/2)\pi i(n+1/2)}H_{n+1/2}^{(2)}(x^{1/2})$$

and $K_{n+1/2}(-ix^{1/2})$ by $\frac{1}{2}\pi i e^{(1/2)\pi i(n+1/2)}H_{n+1/2}^{(1)}(x^{1/2})$, and then $H_{n+1/2}^{(v)}(x^{1/2})$ $(v=1,2)$ by $J_{n+1/2}(x^{1/2})+(-1)^{v-1}iY_{n+1/2}(x^{1/2})$. After these substitutions the left hand side of (7) becomes $\frac{1}{2}\pi x^{n+1/2}\{J_{n+1/2}^2(x^{1/2})+Y_{n+1/2}^2(x^{1/2})\}$, or, by [1; 9.1.2], $\frac{1}{2}\pi x^{n+1/2}\{J_{n+1/2}^2(x^{1/2})+J_{-(n+1/2)}^2(x^{1/2})\}$ and this finishes the proof of the Corollary.

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