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Recurrent and Transient Sets for 3-dimensional Random Walks

By

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Abstract. Before Itô and McKean characterized the recurrent and transient sets for the simple random walk in 3 dimensions, it was thought that a condition of the form

$$\sum_{b \in B} f(|b|) = +\infty \tag{A}$$

might be necessary and sufficient for B to be recurrent. Their characterization has been extended to hold for an arbitrary 3-dimensional aperiodic random walk with zero mean and finite second moments; in this paper it is used to show that for such a random walk no condition of type (A) can be necessary and sufficient for B to be recurrent, and to find the best possible conditions of type (A) which are necessary or sufficient for B to be recurrent.

§ 1

In what follows $a, b, c,$ will be members of the 3-dimensional lattice L_3 of points with integer coordinates and S_n will denote the position at time n of a particle performing an aperiodic random walk on L_3. It is known (Blackwell [1]) that for any subset B of L_3 $P\{S_n \in B$ for an infinite number of values of $n \mid S_0 = a\} = P^*(B)$ is independent of a and can take only the values 0 or 1, and B is said to be recurrent or transient according as $P^*(B)$ is 1 or 0. For the simple random walk Itô and McKean proved

$$\sum_{n=1}^{\infty} 2^{-n} \hat{C}(B_n) = +\infty \Leftrightarrow P^*(B) = 1\,, \tag{1.1}$$

where

$$\hat{C}(B_n) = \sum_{b \in B_n} P\{S_r \notin B_n, \text{all } r \geqq 1 \mid S_0 = b\}$$

is the discrete capacity of B_n, the intersection of B and the spherical shell $\{2^n \leqq |a| < 2^{n+1}\}$. (1.1) actually decides the issue for every aperiodic 3-dimensional random walk with zero mean and finite second moments, for Spitzer ([4] p. 321) has established that a subset B of L_3 is either recurrent for each such random walk or transient for each such walk.

Since Itô and McKean showed that

$$k_1 C(\hat{A}) \leqq \hat{C}(A) \leqq k_2 C(\hat{A})\,, \tag{1.2}^\star$$

where $C(\hat{A})$ is the Newtonian capacity of the set $\hat{A}$ derived from the subset A of L_3 by centreing at each point of A a unit cube with edges parallel to the coordinate axes, we may replace $\hat{C}(B_n)$ by $C(\hat{B}_n)$ in (1.1), and it is this version of (1.1) that we shall use.

★ $k_1, k_2, \ldots$ denote positive constants.

The following facts about the capacities of certain solids are easily checked:

Lemma 1.3. *The capacity of any solid consisting of n non-intersecting unit cubes is not less than* $k_3 n^{1/3}$.

Lemma 1.4. *If* C_α^β *is the capacity of a rectangular block of dimensions* $\beta \times \beta \times \beta\alpha$,

$$k_4 \frac{\alpha\beta}{\log\alpha} \leqq C_\alpha^\beta \leqq k_5 \frac{\alpha\beta}{\log\alpha} \quad \text{for} \quad \alpha \geqq 1\,.$$

Lemma 1.5. *The capacity of a solid consisting of n unit cubes whose centres are collinear and equally spaced at a distance* $2\beta + 2$ *apart and whose faces are parallel is not less than* $k_6 n$ *provided* $\beta > \log n$.

We also need some information about series of positive terms:

Lemma 1.6. *Given an arbitrary monotone sequence* (λ_n) *of positive terms with* $\lim_{n\to+\infty} \lambda_n = +\infty$ *there exists for each* $\alpha > 1$ (β_n) *with*

$$0 \leqq \beta_n \leqq 1\,, \tag{1.7}$$

$$\sum_{n=1}^{\infty} \beta_n < +\infty\,, \tag{1.8}$$

$$\sum_{n=1}^{\infty} \beta_n^\alpha \lambda_n = +\infty\,. \tag{1.9}$$

Proof. Let $m_0 = 0$ and m_j for $j \geqq 1$ be the first n for which $\lambda_n > j^\alpha$ and $m_j > m_{j-1}$. Let $\beta_n = \frac{1}{2 j^{1+\delta}}$ when $n = m_j$ and $\beta_n = 0$ when $n \neq m_j$ for any j. Then (1.7) is clearly satisfied and since

$$\sum_{n=1}^{\infty} \beta_n = \sum_{j=1}^{\infty} \frac{1}{2\, j^{1+\delta}}\,,$$

$$\sum_{n=1}^{\infty} \beta_n^\alpha \lambda_n = \sum_{j=1}^{\infty} \frac{1}{2^\alpha j^{\alpha\delta}}\,,$$

(1.8) and (1.9) are also satisfied for each choice of δ in $0 < \delta < \alpha^{-1}$.

Lemma 1.10. *Given an arbitrary monotone sequence* (λ_n) *of positive terms with* $\lim_{n\to\infty} \lambda_n = 0$ *there exists an increasing sequence of positive integers* (n_j) *with*

$$\sum_{j=1}^{\infty} \frac{1}{n_j} = +\infty\,, \tag{1.11}$$

$$\sum_{j=1}^{\infty} \frac{\lambda\, n_j}{n_j} < +\infty\,. \tag{1.12}$$

Proof. Let l_r be the first value of n for which $\lambda_n < 1/r$, let $m_0 = h_0 = 0$ and define h_r and m_r inductively for $r \geqq 1$ by

$$m_r = \max\{2r, l_r, m_{r-1} + h_{r-1} + 1\}\,, \tag{1.13}$$

$$\sum_{s=0}^{h_r} \frac{1}{(m_r + s)} < \frac{1}{r} \leqq \sum_{s=0}^{h_r+1} \frac{1}{(m_r + s)}\,. \tag{1.14}$$

Now let (n_j) consist of all integers of the form $m_r + s$, $0 \leqq s \leqq h_r$, arranged in increasing order. Then (1.12) holds, for by (1.13) and (1.14),

$$\sum_{j=1}^{\infty} \frac{\lambda_{n_j}}{n_j} = \sum_{r=1}^{\infty} \sum_{s=m_r}^{m_r+h_r} \frac{\lambda_s}{s} \leqq \sum_{r=1}^{\infty} \lambda_{m_r} \sum_{s=0}^{h_r} \frac{1}{m_r+s} \leqq \sum_{r=1}^{\infty} \frac{1}{r^2}.$$

Also we have $m_r \geqq 2r$, from (1.13), and therefore

$$\sum_{s=0}^{h_r} \frac{1}{(m_r+s)} = \sum_{s=0}^{h_r+1} \frac{1}{(m_r+s)} - \frac{1}{(m_r+h_r+1)} > \frac{1}{2r},$$

whence,

$$\sum_{j=1}^{\infty} \frac{1}{n_j} = \sum_{r=1}^{\infty} \sum_{s=0}^{h_r} \frac{1}{(m_r+s)} > \frac{1}{2} \sum_{r=1}^{\infty} \frac{1}{r},$$

so that (1.11) holds.

§ 2

In this section we prove that there is no positive valued function f with the property

$$\sum_{b \in B} f(b) = +\infty \Leftrightarrow P^*(B) = 1. \tag{2.1}$$

(According to BREIMAN [2], this has been proved by P. ERDÖS and B. H. MURDOCH [unpublished]).

We use a 3-dimensional version of the argument by which BREIMAN [2] proved the same proposition for a 1-dimensional random walk.

Let A_n be $\{a : 2^n \leqq |a| < 2^{n+1}\}$, and $I(n, m, \alpha)$ be the rectangular block of lattice points

$$\{a : 2^n + m < a_1 \leqq 2^n(1+\alpha) + m, 0 < a_2 \leqq 2^n\alpha, -2^{n-1} < a_3 \leqq 2^{n-1}\}$$

where a_1, a_2, and a_3 are the coordinates of a.

We have

Lemma 2.2. *If* $I = \bigcup_{r=1}^{\infty} I_r$, *where* $I_r = I(n_r, m_r, \alpha_r)$ *and* (n_r) *is an increasing sequence of positive integers,* (m_r) *a sequence of non-negative integers and* (α_r) *a sequence of real numbers satisfying, for each* $r \geqq 1$,

$$I_r \subseteqq A_{n_r} \quad \text{and} \quad 2^{n_r}\alpha_r > 1, \tag{2.3}$$

then

$$\sum_{r=1}^{\infty} \left\{\log \frac{1}{\alpha_r}\right\}^{-1} = +\infty \Leftrightarrow P^*(I) = 1. \tag{2.4}$$

Proof. Note that α_r is necessarily less than $\frac{1}{2}\sqrt{3}$ (otherwise $I_r \nsubseteq A_{n_r}$) and $\hat{I}_r$ is a solid rectangular block $[2^{n_r}\alpha_r] \times [2^{n_r}\alpha_r] \times 2^{n_r}$ (where $[x]$ denotes the largest integer $\leqq x$). Lemma 1.4 tells us that $\sum_{r=1}^{s} 2^{-n_r} C(\hat{I}_r)$ converges together with $\sum_{r=1}^{s} \left\{\log \frac{2^{n_r}}{[2^{n_r}\alpha_r]}\right\}^{-1}$, and this plainly converges together with $\sum_{r=1}^{s} \left\{\log \frac{1}{\alpha_r}\right\}^{-1}$; thus (2.4) is a consequence of (1.1) and (1.2).

We now argue by contradiction, and assume the existence of a positive f satisfying (2.1). Writing

$$F(n, m, \alpha) = \sum_{a \in I(n,m,\alpha)} f(a),$$

we define for $0 \leqq \alpha \leqq \alpha_0$

$$g(\alpha) = \liminf_{n \to +\infty} \left\{ \inf_{m: I(n,m,\alpha) \subseteqq A_n} F(n, m, \alpha) \right\} \tag{2.5}$$

where $\frac{1}{2}\sqrt{3} \geqq \alpha_0 = \sup\{\alpha : \text{for every } n \geqq 1\ \exists\, m \quad \text{with} \quad I(n, m, \alpha) \subseteqq A_n\} > 0$.

Lemma 2.6. *Within its range of definition $g(\alpha)$ is non-decreasing and $g(\alpha + \beta) \geqq g(\alpha) + g(\beta)$. Also $g(\alpha) < \infty$ for $0 \leqq \alpha \leqq \gamma$ for some $\gamma > 0$, and hence $g(\alpha) \leqq k_7 \alpha$ in this range.*

Proof. The first assertion is obvious. As for the second, take $\alpha > 0$, $\beta > 0$, such that $\alpha + \beta \leqq \alpha_0$ and write $m' = m + [2^n \alpha]$. Then, since $I(n, m, \alpha)$ and $I(n, m', \beta)$ are disjoint subsets of $I(n, m, \alpha + \beta)$,

$$F(n, m, \alpha + \beta) \geqq F(n, m, \alpha) + F(n, m', \beta) \quad \text{for all } n \text{ and } m. \tag{2.7}$$

Now $\{m : I(n, m, \alpha + \beta) \subseteqq A_n\} \subseteqq \{m : I(n, m, \alpha) \subseteqq A_n\}$, $\{m' : I(n, m, \alpha + \beta) \subseteqq A_n\} \subseteqq \{m' : I(n, m', \beta) \subseteqq A_n\}$, so for each n,

$$\inf_{m: I(n,m,\alpha+\beta) \subseteqq A_n} F(n, m, \alpha) \geqq \inf_{m: I(n,m,\alpha) \subseteqq A_n} F(n, m, \alpha), \tag{2.8}$$

$$\inf_{m: I(n,m,\alpha+\beta) \subseteqq A_n} F(n, m', \beta) \geqq \inf_{m: I(n,m,\beta) \subseteqq A_n} F(n, m, \beta). \tag{2.9}$$

(2.7), (2.8) and (2.9) in (2.5) yield

$$\begin{aligned} g(\alpha + \beta) &\geqq \liminf_{n \to +\infty} \left\{ \inf_{m: I(n,m,\alpha) \subseteqq A_n} F(n, m, \alpha) + \inf_{m: I(n,m,\beta) \subseteqq A_n} F(n, m, \beta) \right\} \\ &\geqq g(\alpha) + g(\beta). \end{aligned}$$

Suppose now that $g(\alpha) = +\infty$ for all $\alpha \in (0, \alpha_0)$. Then given any $\alpha_r \in \langle 0, \alpha_0)$ with $\sum_{r=1}^{\infty} \left\{\log \frac{1}{\alpha_r}\right\}^{-1} < +\infty$ we see that $F(n, 0, \alpha_r) \to +\infty$ as $n \to +\infty$ for each r. We can thus find an increasing sequence of positive integers (n_r) such that $2^n \alpha_r > 1$ and $F(n, 0, \alpha_r) \geqq k_8 > 0$ for all $n \geqq n_r$. Then by Lemma 2.2 $I = \bigcup_{r=1}^{\infty} I(n_r, 0, \alpha_r)$ is transient yet $\sum_{a \in I} f(a) = \sum_{r=1}^{\infty} F(n_r, 0, \alpha_r) = +\infty$: this contradiction of (2.1) implies that $g(\gamma) < +\infty$ for some $\gamma > 0$. Taking any $\alpha \in (0, \gamma)$ write $\gamma = n\alpha + \beta$, where $0 \leqq \beta < \alpha$. Then the first assertion of the Lemma gives $g(\gamma) \geqq g(n\alpha) \geqq n g(\alpha)$, and therefore

$$\frac{g(\alpha)}{\alpha} \leqq \frac{g(\gamma)}{n\alpha} \leqq \frac{2 g(\gamma)}{\gamma} = k_7 < \infty.$$

Plainly we can find $\alpha_r \in \langle 0, \gamma)$ such that $\sum_{r=1}^{\infty} \left\{\log \frac{1}{\alpha_r}\right\}^{-1} = +\infty$ and $\sum_{r=1}^{\infty} \alpha_r < +\infty$.

Lemma 2.6 then implies the existence of (n_r), (m_r) such that $I = \bigcup_{r=1}^{\infty} I(n_r, m_r, \alpha_r)$

satisfies the conditions of Lemma 2.2 and $F(n_r, m_r, \alpha_r) \leqq 2k_7\alpha_r$ for all r. The conclusion of Lemma 2.2 is that $P^*(I) = 1$ and yet $\sum_{a\in I} f(a) = \sum_{r=1}^{\infty} F(n_r, m_r, \alpha_r) < +\infty$, which is the required contradiction.

§ 3

Our sufficient condition is $\{\sum_{b\in B}{}' \text{ denotes } \sum_{b\in B:|b|>0}\}$

$$\sum_{b\in B}{}' \frac{1}{|b|^3} = +\infty \Rightarrow P^*(B) = 1\,. \tag{3.1}$$

To prove (3.1), let $B_n = B_\wedge A_n$ have N_n points, and $\sum_{b\in B}{}' \frac{1}{|b|^3} = +\infty$. Then since each $b\in B_n$ has $|b| \geqq 2^n$, $\sum_{n=1}^{\infty} N_n 2^{-3n} = +\infty$, and therefore $\sum_{n=1}^{\infty} N_n^{1/3} 2^{-n} = +\infty$. By Lemma 1.3 $C(\hat{B}_n) \geqq k_3 N_n^{1/3}$, so that $\sum_{n=1}^{\infty} 2^{-n} C(\hat{B}_n) = +\infty$, and $P^*(B) = 1$.

(3.1) is the best possible sufficient condition of type (A) in the following sense: given an arbitrary positive-valued function f with $\lim_{x\to+\infty} f(x) = +\infty$, there is a set B with $\sum_{b\in B}{}' f(|b|)/|b|^3 = +\infty$ and $P^*(B) = 0$.

To establish this, notice that we can take the given function f to be monotone: for if not, $g(x) = \inf_{y\geqq x} f(y)$ is monotone, and any transient set with $\sum_{b\in B}{}' g(|b|)/|b|^3 = +\infty$ necessarily has $\sum_{b\in B}{}' f(|b|)/|b|^3 = +\infty$. Putting $\lambda_n = f(2^n)$ and $\alpha = 3$ in Lemma 1.6 yields a sequence (β_n) with

$$0 \leqq \beta_n \leqq 1\,, \tag{3.2}$$

$$\sum_{n=1}^{\infty} \beta_n < +\infty\,, \tag{3.3}$$

$$\sum_{n=1}^{\infty} \beta_n^3 f(2^n) = +\infty\,. \tag{3.4}$$

Letting B_n be the set of all lattice points lying within a sphere of radius $\beta_n 2^{n-1}$ centred at the point $(3.2^{n-1}, 0, 0)$, it is easy to see that $B = \bigcup_{n=1}^{\infty} B_n$ is transient. For $\hat{B}_n$ is certainly contained within a sphere of radius $\beta_n 2^{n-1} + 1$, so that $C(\hat{B}_n) \leqq \beta_n 2^{n-1} + 1$ and, by (3.3)

$$\sum_{n=1}^{\infty} 2^{-n} C(\hat{B}_n) \leqq \sum_{n=1}^{\infty} \tfrac{1}{2}\beta_n + 2^{-n} < +\infty\,.$$

Moreover $N_n \geqq k_9 (2^{n-1}\beta_n)^3$ so that, by (3.4)

$$\sum_{b\in B}{}' f(|b|)/|b|^3 \geqq \sum_{n=1}^{\infty} N_n f(2^n) 2^{-3n-3} \geqq k_9/64 \sum_{n=1}^{\infty} \beta_n^3 f(2^n) = +\infty\,.$$

Our necessary condition is

$$P^*(B) = 1 \Rightarrow \sum_{b\in B}' \frac{1}{|b|} = +\infty. \tag{3.5}$$

This, of course, is equivalent to

$$\sum_{b\in B}' \frac{1}{|b|} < +\infty \Rightarrow P^*(B) = 0. \tag{3.6}$$

For a random walk of the type we are considering, it is known that $\sum_{n=1}^{\infty} P(S_n = b)$ converges for each b, and SPITZER ([4], p. 308) has shown that $|b| \sum_{n=1}^{\infty} P(S_n = b)$ is bounded for all large enough $|b|$. Thus when $\sum_{b\in B}' \frac{1}{|b|} < +\infty$, $\sum_{n=1}^{\infty} P(S_n \in B) < \infty$, so that $P^*(B) = 0$.

(3.5) is also best possible of its type. For given arbitrary nonnegative f with $\lim_{x\to+\infty} f(x) = 0$, there is a set B with $\sum_{b\in B}' \frac{f(|b|)}{|b|} < +\infty$ and $P^*(B) = 1$.

Again we may take $f(x)$ to be monotone (otherwise consider $g(x) = \sup_{y\geqq x} f(y)$), but this time we put $\lambda_n = f(2^n)$ in Lemma 1.10 and get a sequence of increasing positive integers (n_j) with

$$\sum_{j=1}^{\infty} \frac{1}{n_j} = +\infty, \tag{3.7}$$

$$\sum_{j=1}^{\infty} \frac{f(2^{n_j})}{n_j} < +\infty. \tag{3.8}$$

Let B_n be empty if $n \notin (n_j)$: let B_n for $n = n_j$ consist of all points of the form $(2^n + 2rn, 0, 0)$ with $0 \leqq r \leqq \left[\frac{2^n}{2n}\right] - 1$. Plainly Lemma 1.5 will apply to $\hat{B}_{n_j}$ provided that $\log \left[\frac{2^{n_j}}{2n_j}\right] < n_j - 1$. This is the case if $n_j \geqq 2$, so by Lemma 1.5 and (3.7)

$$\sum_{n=1}^{\infty} 2^{-n} C(\hat{B}_n) \geqq k_6 \sum_{j=2}^{\infty} 2^{-n_j} \left[\frac{2^{n_j}}{2n_j}\right] \geqq \frac{k_6}{4} \sum_{j=2}^{\infty} \frac{1}{n_j} = +\infty.$$

However,

$$\sum_{b\in B}' \frac{f(|b|)}{|b|} \leqq \sum_{j=1}^{\infty} \left[\frac{2^{n_j}}{2n_j}\right] \frac{f(2^{n_j})}{2^{n_j}} \leqq \frac{1}{2} \sum_{j=1}^{\infty} \frac{f(2^{n_j})}{2n_j},$$

so by (3.8) $B = \bigcup_{n=1}^{\infty} B_n$ is the required recurrent set with $\sum_{b\in B}' \frac{f(|b|)}{|b|} < +\infty$.

Naturally we can improve (3.1) if we impose conditions on the set B: for example (see [5])

$$\sum_{b\in B}' \frac{1}{|b|^2} = +\infty \textit{ and } B \textit{ a set of coplanar points} \Rightarrow P^*(B) = 1, \tag{3.9}$$

$$\sum_{b\in B}' \frac{1}{|b| \log |b|} = +\infty \textit{ and } B \textit{ a set of collinear points} \Rightarrow P^*(B) = 1. \tag{3.10}$$

(3.9) and (3.10) are again best possible of their type.

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