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A Classification of a Random Walk Defined on a Finite Markov Chain

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1. Introduction

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables (i.i.d. random variables). If Y_k is real valued then $S_n = \sum_1^n Y_k$ is a random walk on the real line. It is well known, see Feller [4, p. 395] that such a random walk belongs to exactly one of the following four categories,

(i) $\lim_n S_n = \infty$ a.s.

(ii) $\lim_n S_n = -\infty$ a.s.

(iii) $\overline{\lim_n} S_n = +\infty$, $\underline{\lim_n} S_n = -\infty$ a.s.

(iv) For all n, $S_n = 0$ a.s.

To each random walk there is associated a random variable $N = \inf\{n > 0; S_n > 0\}$, the hitting time to the positive half-line, and also a random variable N^-, the hitting time to the negative half-line. It is also well known that those random walks which belong to category (i) above are precisely those for which N is a proper random variable and N^- is improper. Similar statements can be made concerning categories (ii), (iii) and (iv). Spitzer [12, p. 189] developed a necessary and sufficient condition for N to be a proper random variable, namely, the divergence of the series $\sum_1^\infty n^{-1} P(S_n > 0)$.

The object of this paper is to generalise the above results to the situation where the distribution of the increment Y_k depends on the states X_{k-1} and X_k of an underlying ergodic finite state Markov chain $\{X_n\}$.

This type of random walk has been previously studied by Miller [8, 9] and Keilson and Wishart [5, 6] and [7]. A basic difference in this paper from these references is that we do not assume the existence of means for the increments Y_k. We note that if Z_k is the sum of the increments between the k-th and $(k+1)$-st occurrence of the state j in the underlying chain, then $\{Z_k : k = 1, \ldots\}$ are i.i.d. random variables. The methods used in this paper rely heavily on this remark.

2. Preliminaries

Before beginning a description of the process under consideration we shall state the following facts for completeness and future reference. $Y_1, Y_2, \ldots$ are i.i.d. random variables and $S_n = \sum_1^n Y_k$.

Theorem A. *The random walk S_n falls into exactly one of the following four categories.*

(i) $\lim_n S_n = \infty$ *a.s.*

(ii) $\lim_n S_n = -\infty$ *a.s.*

(iii) $\overline{\lim_n} S_n = \infty$, $\underline{\lim_n} S_n = -\infty$ *a.s.*

(iv) *For all n*, $S_n = 0$ *a.s.*

In the following $N = \inf\{n > 0 : S_n > 0\}$ is the hitting time to the positive half-line.

Lemma B. *N is a proper random variable if and only if* $\overline{\lim_n} S_n = +\infty$ *a.s.*

Theorem C. *N is a proper random variable if and only if* $\sum_1^\infty n^{-1} P(S_n > 0) = \infty$.

The process considered in this paper consists of a discrete Markov chain with finite state space $\{1, 2, \ldots, m\}$ called the X-process, taking values $X_0, X_1, \ldots, X_n, \ldots$. Throughout this paper the Markov chain will be ergodic in the sense of Feller [3]. Alongside the X-process is a real valued process called the Y-process which proceeds $Y_0, Y_1, \ldots, Y_n, \ldots$ where $Y_0 = 0$, and for $n \geqq 1$, the distribution of Y_n depends on X_{n-1} and X_n. It follows that given the random variables $\{X_0, \ldots, X_n\}$, the random variables $\{Y_1, \ldots, Y_n\}$ are conditionally independent. Our main concern is the study of the S-process which proceeds $S_0, S_1, \ldots, S_n, \ldots$ where $S_n = \sum_0^n Y_k$. The S-process is known as *a random walk defined on the Markov chain.*

To make this description precise we follow Pyke [10]. Let $Q = (Q_{ij})$ be a matrix valued function on $(-\infty, \infty)$ such that for $i = 1, \ldots, m$, $\sum_{j=1}^m Q_{ij}(\infty) = 1$. The (X, Y)-process is defined to be any two-dimensional stochastic process $\{(X_n, Y_n); n \geqq 0\}$ defined on a complete probability space $(\Omega, \mathfrak{I}, P)$ which satisfies

(i) $Y_0 = 0$ a.s.

(ii) $P\{X_n = k, Y_n \leqq x | \mathfrak{I}_{n-1}\} = Q_{X_{n-1}k}(x)$ a.s. for $n \geqq 1$ where $\mathfrak{I}_n$ denotes the σ-algebra generated by $\{X_0, \ldots, X_n, Y_0, \ldots, Y_n\}$. It should be noted that the X-process is a Markov chain with m states and has transition matrix $Q(\infty)$. As mentioned previously we shall also assume that the Markov chain is ergodic. If we let $S_n = \sum_0^n Y_k$ then the (X, S)-process is a bivariate Markov chain $\{(X_n, S_n); n \geqq 0\}$ such that

$$P\{X_n = k, S_n \leqq x | \mathfrak{I}_{n-1}\} = Q_{X_{n-1}k}(x - S_{n-1}).$$

The (X, S)-process with semi-Markov transition function Q is called *the Markov additive process with kernel Q*. This terminology follows that used by Çinlar [2] in the continuous time case.

Before stating the first lemma we make a few remarks. In some respects the value of X_0 affects the behaviour of the S-process. This can be illustrated by noting that whether the hitting time to the positive half-line is a proper random variable or not may depend on the value of X_0. We will return to this later in the paper. The distribution of X_0 remains unspecified and we will deal with probabilities conditional on X_0. With this in mind, relative to $(\Omega, \mathfrak{I}, P)$, we denote $P\{\cdot|X_0=i\}$ by $P_i\{\cdot\}$. Throughout the paper N and τ with and without subscripts and superscripts will denote stopping times.

In this paper the following lemma plays a crucial role. It is a generalisation of a result of Chung [1, p. 78] which states that if $\{X_n\}$ is a Markov chain and f a function from the state space of the chain to the real line then $\sum_{n=\tau_k+1}^{\tau_{k+1}} f(X_n)$, $k>0$, are i.i.d. random variables, where τ_k, $k>0$, are the times of successive occurrences of the state j in the Markov chain.

To simplify the statement of our lemma we introduce the following notation. Let $\{j_1, \ldots, j_r\}$ be a sequence of states of the state space $\{1, \ldots, m\}$. We define inductively the sequence of stopping times $\{\tau_k: k=1, \ldots, r\}$ by

$$\tau_1 = \inf\{n \geqq 0: X_n = j_1\},$$
$$\tau_k = \inf\{n > \tau_{k-1}: X_n = j_k\}, \qquad k=2, \ldots, r.$$

Let $Z_k = S_{\tau_{k+1}} - S_{\tau_k}$, $k=1, \ldots, r-1$. With the above notation we have:

Lemma 1. *$\{Z_k: k=1, \ldots, r-1\}$ is a set of independent random variables.*

The proof of this is an immediate consequence of the strong Markov property for the bivariate Markov chain $\{(X_n, S_n); n \geqq 0\}$. We remark also that if $j_1 = j_2 = \cdots = j_r$ then the random variables are identically distributed. In future we shall often refer to Z_1 as *the increment from a (j_1, j_2)-block* or simply a *j_1-block* if $j_1 = j_2$.

3. Classification of the Random Walk

In our classification of the random walk defined on the chain, category (iv) of Theorem A is replaced by a modification of the idea of degeneracy introduced by Miller [8].

If $Q_{ij}(\infty) > 0$ then we let

$$F_{ij}(x) = \frac{Q_{ij}(x)}{Q_{ij}(\infty)} = P(Y_n \leqq x | X_{n-1} = i, X_n = j).$$

Definition. The process (X, S) is *degenerate* if there exists constants $\beta_1 \ldots \beta_m$ such that whenever $Q_{ij}(\infty) > 0$ it follows that F_{ij} is the distribution function of a degenerate random variable which takes the value $\beta_j - \beta_i$ a.s. This modification of Miller's definition rules out the possibility of an overall drift of the S-process. It follows that if $X_0 = i$ and $X_n = j$ then $S_n = \beta_j - \beta_i$. We are now in a position to state the analogue of Theorem A for the random walk defined on a Markov chain.

Theorem 1. *For a random walk defined on a finite state ergodic Markov chain there are four mutually exclusive possibilities:*

(i) $\lim_n S_n = +\infty$ *a.s.*

(ii) $\lim_n S_n = -\infty$ *a.s.*

(iii) $\overline{\lim}_n S_n = +\infty$, $\underline{\lim}_n S_n = -\infty$ *a.s.*

(iv) *The (X, S)-process is degenerate.*

Before giving a proof of the theorem we will state and prove the following lemmas.

Lemma 2. *If for some j the increment from a j-block equals zero a.s. then the (X, S)-process is degenerate.*

Proof. Let j be as above and fix i. We consider adjacent (i, j) and (j, i)-blocks having increments W_{ij} and W_{ji} respectively. Now W_{ij} and W_{ji} are independent random variables and $P(W_{ji}+W_{ij}=0)=1$. It follows that there exists a constant C_{ij} such that

$$P(W_{ij}=C_{ij})=P(W_{ji}=-C_{ij})=1.$$

Similarly there exists constants C_{kj} for $k=1, \ldots, m$. If we now consider adjacent (j, k), (k, l) and (l, j) blocks with increments W_{jk}, W_{kl} and W_{lj} respectively where $k, l \neq j$ then we see that

$$P(W_{kl}=-C_{jk}-C_{lj})=1 \quad \text{where } C_{jk}=-C_{kj}.$$

Let us denote by C_{kl} the value $(-C_{jk}-C_{lj})$. By similarly considering a (l, k)-block we can say that for all l and k the increment from a (l, k)-block is a degenerate random variable taking the value C_{lk} and moreover $C_{lk}=-C_{kl}$.

Let us now suppose that a certain $l \to k$ transition is possible and denote by Y_{lk} the increment associated with such a transition. Now $l \to k$ is a particular realisation of an (l, k)-block and therefore $Y_{lk}=C_{lk}$ a.s.

Define $\beta_1, \ldots, \beta_m$ as follows:

$$\beta_1=C \quad \text{an arbitrary constant,}$$

$$\beta_r=C_{r1}+\beta_1=C_{r1}+C \quad \text{for } r=2, \ldots, m.$$

It is immediate that

$$C_{1l}+C_{lk}+C_{k1}=C_{11}$$

from which it follows that

$$C_{lk}=C_{11}+C_{l1}-C_{k1}=0+(\beta_l-\beta_1)-(\beta_k-\beta_1)=\beta_l-\beta_k.$$

Hence

$$Y_{lk}=\beta_l-\beta_k \quad \text{a.s.}$$

This completes the proof.

The sequence $\{S_n: n \geqq 1\}$ is said to be *dominated* if there exists a finite valued function M such that for all n: $|S_n| \leqq M$ a.s.

Lemma 3. *The process (X, S) is degenerate if and only if the sequence $\{S_n: n \geqq 1\}$ is dominated.*

Proof. If the process is degenerate then since $\sup_n S_n = \max_{i,j}(\beta_i-\beta_j)$ it follows that $\{S_n: n \geqq 1\}$ is dominated. Conversely we suppose $\{S_n: n \geqq 1\}$ is dominated. Now for a fixed j we define the following sequence of stopping times $\{\tau_k: k \geqq 0\}$ by

$$\tau_0=0, \quad \tau_k=\inf\{n>\tau_{k-1}: X_n=j\}, \quad k>0.$$

If we let $Z_k=S_{\tau_{k+1}}-S_{\tau_k}$, $k=0, 1, \ldots$ then $S_{\tau_n}=Z_0+\sum_{k=1}^{n-1} Z_k$. From Lemma 1 we see that Z_k, $k=1, 2, \ldots$ are i.i.d. random variables and since $\{S_{\tau_n}: n \geqq 1\}$ is dominated and Z_0 is finite a.s. it follows from Theorem A that $P(Z_k=0)=1$ for $k=1,2,\ldots$. Now Z_k is the increment from a j-block therefore by Lemma 2 we see that the (X,S)-process is degenerate, and the Lemma is proved.

In view of the next Lemma, now is an opportune time to elaborate further the importance of the value of X_0, the initial position of the chain. To do this we define the two stopping times

$$N=\inf\{n>0;\, S_n>0\}, \qquad N^-=\inf\{n>0;\, S_n<0\}.$$

If for all i we have $P_i(N<\infty)=1$ then we say that N is *totally proper*. It is important to note that if $P_i(N<\infty)=1$ for some i then it does not necessarily follow that N is totally proper. This can be seen in the following example.

Let

$$H_a(x)=\begin{cases}0 & \text{if } x<a\\ 1 & \text{if } x\geqq a\end{cases}$$

and we will consider the (X,S)-process in which the X-process is a 2 state Markov chain and the semi-Markov transition function $Q(x)$ is of the form

$$Q(x)=\begin{pmatrix} p\,H_0(x) & q\,H_d(x)\\ r\,H_b(x) & s\,H_c(x)\end{pmatrix}$$

where $p,q,r,s,d>0$ and $b,c<-d$. It can be seen that $P_1(N<\infty)=1$ but $P_2(N<\infty)=0$.

In some respects however the value of X_0 does not affect the process. For example if $\overline{\lim}\, S_n=\infty$ a.s. P_i then $\overline{\lim}\, S_n=\infty$ a.s. P_j for all j. We are now able to state the following lemma.

Lemma 4. *N is totally proper if and only if $\overline{\lim\limits_n}\, S_n=\infty$ a.s.*

Proof. If $\overline{\lim\limits_n}\, S_n=\infty$ a.s. then it is obvious that N is totally proper. Conversely if N is totally proper then we define the sequence of stopping times $\{\tau_k\colon k\geqq 0\}$ inductively by

$$\tau_0=0, \qquad \tau_k=\inf\{n>\tau_{k-1}\colon S_n>S_{\tau_{k-1}}\}, \qquad k>0.$$

By assumption each τ_k is a proper random variable and moreover $\lim\limits_k S_{\tau_k}=\infty$ a.s. It follows from this that $\overline{\lim}\, S_n=\infty$ a.s.

Theorem 1 is now an immediate corollary from the following theorem.

Theorem 2. (i) *N and N^- are both totally proper if and only if $\overline{\lim\limits_n}\, S_n=+\infty$ a.s.* and $\underline{\lim\limits_n}\, S_n=-\infty$ *a.s.*

(ii) *N is totally proper and N^- is not totally proper if and only if $\lim\limits_n S_n=\infty$ a.s.*

(iii) *N is not totally proper and N^- is totally proper if and only if $\lim\limits_n S_n=-\infty$ a.s.*

(iv) *Neither N nor N^- is totally proper if and only if the (X,S)-process is degenerate.*

Proof. (i) This is immediate from Lemma 4.

(ii) Assume N is totally proper and N^- is not totally proper. For each j we define the following sequence of stopping times $\{\tau_k^j: k>0\}$ by

$$\tau_1^j = \inf\{n: X_n = j\},$$
$$\tau_k^j = \inf\{n > \tau_{k-1}^j: X_n = j\}, \qquad k>1.$$

If we let $Z_k^j = S_{\tau_{k+1}^j} - S_{\tau_k^j}$ then τ_k^j is the time of the k-th occurrence of the state j in the Markov chain and Z_k^j is the increment from the k-th j-block. We further define for each j the stopping time N_j to be

$$N_j = \inf\{k>1: S_{\tau_k^j} > S_{\tau_1^j}\}.$$

Similarly we define $N_j^- = \inf\{k>1: S_{\tau_k^j} < S_{\tau_1^j}\}$.

It follows from Lemma 4 that $\overline{\lim_n}\, S_n = +\infty$ a.s. If we suppose that N_j^- is proper for some j then by Lemma B $\underline{\lim}_k\, S_{\tau_k^j} = -\infty$ a.s. therefore $\underline{\lim}_n\, S_n = -\infty$ a.s. and this contradicts, by use of Lemma 4, the fact that N^- is not totally proper. Hence N_j^- is improper for all j. If we now assume that N_j is improper for some j then by Theorem A $Z_k^j = 0$ a.s. and by application of Lemma 2 we see that the process is degenerate. This is also a contradiction therefore N_j is proper for all j. By further application of Lemma B we see that for all j $\lim_k S_{\tau_k^j} = \infty$ a.s. whereby it follows that $\lim_n S_n = \infty$ a.s.

Conversely we assume $\lim_n S_n = \infty$ a.s. The conclusion is a direct consequence of Lemma 4.

(iii) This is proved in a similar manner as (ii).

(iv) If N and N^- are both not totally proper then we show as in part (ii) that N_j and N_j^- are improper for all j.

Whence it follows that $Z_k^j = 0$ a.s. for all j. The fact that (X, S) is degenerate follows by appealing to Lemma 2. If conversely the process (X, S) is degenerate then the fact that N and N^- are both not totally proper is immediate from parts (i), (ii), (iii) and Lemma 3.

4. A Criterion

The aim in this section is to generalize Theorem C to the Markov additive process (X, S). We will prove,

Theorem 3. *Let (X, S) be a Markov additive process in which the Markov chain $\{X_n\}$ is ergodic with a finite state space. A necessary and sufficient condition for N to be totally proper is that*

$$\sum_1^\infty n^{-1} P_i(S_n > 0) = \infty \qquad \textit{for all } i.$$

Before giving a proof we need the following two lemmas.

Lemma 5. *Let $Y, Z_1, Z_2, \ldots$ be a sequence of independent random variables such that $Z_1, Z_2, \ldots$ are also identically distributed and let $S_n = \sum_1^n Z_i$. If $\sum_1^\infty n^{-1} P(S_n > 0) = \infty$ then $\sum_1^\infty n^{-1} P(S_n + Y > 0) = \infty$.*

Proof. We denote by $F(x)$ the distribution function of Y. Now

$$\begin{aligned}\sum_1^\infty n^{-1} P(Y+S_n>0) &= \sum_1^\infty n^{-1} \int_{-\infty}^{\infty} P(S_n > -x)\, dF(x)\\ &= \int_{-\infty}^{\infty} \left(\sum_1^\infty n^{-1} P(S_n > -x)\right) dF(x) = \int_{-\infty}^{\infty} c(x)\, dF(x)\end{aligned}$$

where $c(x) = \sum_1^\infty n^{-1} P(S_n > -x)$.

By application of Theorem 1 of a paper by Rosén [11, p. 324] which states that there is a constant A such that $P\{S_n \in (0,x)\} \leqq A n^{-\frac{1}{2}}$, we see that for $x > 0$

$$\sum_1^\infty n^{-1} P(S_n \in (0,x)) \leqq \sum_1^\infty n^{-\frac{3}{2}} A < \infty \qquad \text{where } A \text{ is a constant}.$$

Since $c(0) = \infty$ it follows that $c(x) = \infty$ for all finite x. Thus the result follows.

We now let τ_k^j be as in Section 3, namely the time of the k-th occurrence of state j. With this notation we have:

Lemma 6. *If for some j, $\sum_{k=1}^\infty k^{-1} P_i(S_{\tau_k^j} > 0) = \infty$, then $\sum_1^\infty n^{-1} P_i(S_n > 0) = \infty$.*

Remark. As a consequence of this lemma we see that to prove the divergence of the series $\sum_1^\infty n^{-1} P_i(S_n > 0)$ it is sufficient to show that for some j, the imbedded series, namely the series obtained by considering the process only when the chain is in state j, diverges.

Proof. The $\tau_1^j, \tau_2^j - \tau_1^j, \tau_3^j - \tau_2^j, \ldots$ are independent random variables and with the exception of τ_1^j they are also identically distributed. Here we are assuming the initial state i is not necessarily equal to j. Since the Markov chain is ergodic the above random variables have finite means and variances.

Let

$$c = \max\{E_i(\tau_1^j), E_i(\tau_2^j - \tau_1^j)\} + 1$$

and

$$\sigma^2 = \max\{\mathrm{Var}_i(\tau_1^j), \mathrm{Var}_i(\tau_2^j - \tau_1^j)\}.$$

Our initial aim is to show

$$\sum_{k=1}^\infty (c\,k)^{-1} \sum_{n > ck} P_i(S_{\tau_k^j} > 0, \tau_k^j = n) < \infty. \tag{1}$$

Now

$$\sum_{k=1}^{\infty}(c\,k)^{-1}\sum_{n>ck}P_i(S_{\tau_k^j}>0,\tau_k^j=n)\leqq\sum_{k=1}^{\infty}(c\,k)^{-1}\sum_{n>ck}P_i(\tau_k^j=n)$$
$$=\sum_{k=1}^{\infty}(c\,k)^{-1}P_i(\tau_k^j>c\,k)$$
$$=\sum_{k=1}^{\infty}(c\,k)^{-1}P_i\{\tau_k^j-E_i(\tau_k^j)>c\,k-E_i(\tau_k^j)\}$$
$$\leqq\sum_{k=1}^{\infty}(c\,k)^{-1}P_i\big(\tau_k^j-E_i(\tau_k^j)>k\big)$$

this follows by definition of c

$$\leqq\sum_{k=1}^{\infty}(c\,k)^{-1}\sigma^2k^{-1}<\infty.$$

The last step follows by application of Čebyšev's inequality. This proves (1). Now

$$\sum_{1}^{\infty}n^{-1}P_i(S_n>0)=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{m}n^{-1}P_i(S_n>0,\tau_k^l=n)$$
$$\geqq\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}n^{-1}P_i(S_{\tau_k^j}>0,\tau_k^j=n)$$
$$\geqq\sum_{k=1}^{\infty}\sum_{ck\geqq n\geqq k}n^{-1}P_i(S_{\tau_k^j}>0,\tau_k^j=n)$$
$$\geqq\sum_{k=1}^{\infty}(c\,k)^{-1}\sum_{ck\geqq n\geqq k}P_i(S_{\tau_k^j}>0,\tau_k^j=n)$$
$$=\sum_{k=1}^{\infty}(c\,k)^{-1}\sum_{n=k}^{\infty}P_i(S_{\tau_k^j}>0,\tau_k^j=n)-\sum_{k=1}^{\infty}(c\,k)^{-1}\sum_{n>ck}P_i(S_{\tau_k^j}>0,\tau_k^j=n)$$
$$=\frac{1}{c}\sum_{k=1}^{\infty}k^{-1}P_i(S_{\tau_k^j}>0)-\sum_{k=1}^{\infty}(c\,k)^{-1}\sum_{n>ck}P_i(S_{\tau_k^j}>0,\tau_k^j=n).$$

It follows from (1) and the assumption of the lemma that $\sum_{1}^{\infty}n^{-1}P_i(S_n>0)=\infty$. We are now in a position to prove the theorem.

Proof of Theorem 3. Assume N is totally proper. Let $X_0=i$ be fixed and N_j be as in Theorem 2 namely

$$N_j=\inf\{k>1\colon S_{\tau_k^j}>S_{\tau_1^j}\}.$$

By the argument used in Theorem 2 it follows that N_j is proper for some j. An application of Theorem C yields the fact that $\sum_{k=1}^{\infty}k^{-1}P_i(S_{\tau_k^j}-S_{\tau_1^j}>0)=\infty$ for that particular j. From Lemma 5 we obtain

$$\sum_{k=1}^{\infty}k^{-1}P_i(S_{\tau_k^j}>0)=\infty.$$

The conclusion that $\sum_{1}^{\infty}n^{-1}P_i(S_n>0)=\infty$ is now an immediate consequence of Lemma 6.

Conversely assume that $\sum_1^\infty n^{-1} P_i(S_n>0)=\infty$ for all i. Let i be arbitrary but fixed. Then

$$\sum_1^\infty n^{-1} P_i(S_n>0)=\sum_{n=1}^\infty \sum_{k=1}^n \sum_{j=1}^m n^{-1} P_i(S_n>0, n=\tau_k^j)$$
$$=\sum_{j=1}^m \sum_{k=1}^\infty \sum_{n=k}^\infty n^{-1} P_i(S_{\tau_k^j}>0, \tau_k^j=n)$$
$$\leqq \sum_{j=1}^m \sum_{k=1}^\infty k^{-1} P_i(S_{\tau_k^j}>0).$$

It follows that at least one of the series $\sum_{k=1}^\infty k^{-1} P_i(S_{\tau_k^j}>0)$ $j=1,\ldots,m$ diverges. We will denote by $\theta(i)$ the least j for which this is so. Similarly we define $\theta(l)$ for $l=1,\ldots,m$.

Define the sequence $\{\theta^k(i)\colon k\geqq 1\}$ inductively by $\theta(i)$ as above, $\theta^k(i)=\theta\big(\theta^{k-1}(i)\big)$ for $k>1$.

The sequence of stopping times $\{\lambda_k\colon k\geqq 0\}$ are defined inductively as follows:

$$\lambda_0=0, \qquad \lambda_k=\inf\{n>\lambda_{k-1}\colon X_n=\theta^k(i)\}, \qquad k>0.$$

If we let $U_k=S_{\lambda_{k+1}}-S_{\lambda_k}$ for $k\geqq 0$ then the elements of the sequence $\{U_k\colon k\geqq 0\}$ are the increments from adjacent $\big(\theta^k,(i),\theta^{k+1}(i)\big)$ blocks. Let τ be the stopping time defined by $\tau=\inf\{\lambda_k\colon U_k\leqq 0\}$.

We consider separately the two possibilities that $P_i(\tau<\infty)=1$ or otherwise.

Firstly we suppose $P_i(\tau<\infty)=1$. We define the sequence of stopping times $\{\tau_k\colon k>0\}$ inductively by

$$\tau_1=\inf\{n>\tau\colon X_n=\theta(X_\tau)\},$$
$$\tau_k=\inf\{n>\tau_{k-1}\colon X_n=\theta(X_\tau)\}, \qquad k>1.$$

From these definitions and that of θ we see that $\sum_1^\infty k^{-1} P_i(S_{\tau_k}-S_\tau>0)=\infty$. Using the definition of τ which implies that $S_{\tau_1}-S_\tau\leqq 0$ we deduce that

$$\sum_1^\infty k^{-1} P_i(S_{\tau_k}-S_{\tau_1}>0)=\infty.$$

It is easy to see that $\sum_1^\infty k^{-1} P_i(S_{\tau_{k+1}}-S_{\tau_1}>0)=\infty$.

Noting that $(S_{\tau_{k+1}}-S_{\tau_1})$ is the sum of k i.i.d. random variables we can use Theorem C and Lemma B to deduce that $\overline{\lim_k}\, S_{\tau_k}=\infty$ a.s. Hence $\overline{\lim_n}\, S_n=\infty$ a.s. and by Lemma 4 it follows that N is totally proper.

Secondly suppose that $P_i(\tau<\infty)<1$ from which it follows that $P_i\{U_k>0;\ k=0,1,\ldots\}>0$. Since the Markov chain has only m states we know that least from $k=m$ onwards the sequence $\{\theta^k(i)\colon k=1,\ldots\}$ will be of a cyclic nature. Hence the distribution functions of the U_k will also be so and since the U_k are independent random variables it follows that at least for $k > m$, $P(U_k > 0) = 1$. Hence $\overline{\lim_k}\, S_{\lambda_k} = \infty$ a.s. P_i therefore $\overline{\lim}\, S_n = \infty$ a.s. P_i and therefore by Lemma 4 we can say that N is totally proper.

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