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On a Mixing Property of Operators in L_p Spaces*

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Let $(X, \mathscr{A})$ be a measurable space. For a measure μ defined on $\mathscr{A}$ and a fixed number p with $1 \leqq p < \infty$, $L_p(\mu)$ shall denote the Banach space of functions f such that $\int |f|^p \, d\mu < \infty$; $L_\infty(\mu)$ denotes the Banach space of μ-essentially bounded functions. We consider the following conditions on a linear contraction operator T defined on $L_p(\mu)$:

(A) For each $f \in L_p(\mu)$, $T^n f$ converges weakly in $L_p(\mu)$.

(B) For each $f \in L_p(\mu)$, $\sum_i a_{ni} T^i f$ converges in $L_p(\mu)$ for each matrix (a_{ni}) satisfying

(UR) $\sup_n \sum_i |a_{ni}| < \infty$; $\lim_n \sum_i a_{ni} = 1$; $\lim_n \max_i |a_{ni}| = 0$.

Condition (A) corresponds to *mixing*, or more generally *stability* in applications to Ergodic Theory (cf. [2, 9, 3, 10]). The matrices satisfying (UR) could be called *uniformly regular*. It is not difficult to see that the last condition in (UR) may be replaced by: $\lim_n a_{ni} = 0$ for each i and $\lim_i \max_n |a_{ni}| = 0$. Lorentz characterized the class of (UR) methods in terms of "summability functions" (see [11] and [12]). In Section 1 we show that (A) and (B) are equivalent for $p = 1$ and 2. This is still true if $1 < p < \infty$, under some additional conditions on T, as shown in Section 2. Passage from L_2 to L_p, $p \neq 2$, is accomplished via an interpolation result (Theorem 1.2) proved by a very simple argument, but including results for point-transformations and Markov operators which seemed fairly difficult to establish (cf. [2]; [9], Theorem 4.1; [10], Theorem 3.2).

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Section 1

Our first theorem generalizes some results in Blum-Hanson [2], Hanson-Pledger [6], Akcoglu-Sucheston [1], and Jones-Kuftinec [8].

Theorem 1.1. *Let T be a contraction operator on L_2 of a σ-finite measure space $(X, \mathscr{A}, \mu)$. Let f be a fixed function in $L_2(\mu)$. Then the following conditions are equivalent:*

(a) *$T^n f$ converges weakly in $L_2(\mu)$.*

(b) *For every (UR)-matrix (a_{ni}), $A_n f = \sum_i a_{ni} T^i f$ converges in $L_2(\mu)$.*

Proof. (b) $\Rightarrow$ (a) In fact, we show that condition (a) is easily implied—not only in L_2, but in any Banach space—by the following weaker condition:

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(b′) There exists a *regular* matrix (a_{ni}) such that for every increasing sequence of positive integers (k_i), $\sum_i a_{ni} T^{k_i} f$ converges weakly.

We recall that a matrix (a_{ni}) is called *regular* if: $\sup_n \sum_i |a_{ni}| < \infty$, $\lim_n \sum_i a_{ni} = 1$, $\lim_n a_{ni} = 0$ for each i. Assume that (b′) holds for some regular matrix (a_{ni}) and that (a) fails. Then there exists $g \in L_2(\mu)$ such that $(T^n f, g) = \int T^n f \cdot g \, d\mu$ diverges. The sequence $c_n = (T^n f, g)$ being bounded, there are sequences of positive integers (r_i) and (t_i) such that $\alpha = \lim_i c_{t_i} \neq \lim_i c_{r_i} = \beta$. We shall form an increasing sequence of positive integers (k_i) such that the sequence $\sum_i a_{ni} c_{k_i}$ does not converge. Since (a_{ni}) is regular, there exist increasing sequences of integers (n_k) and (m_k) such that

$$\lim_{k \to \infty} \Big(\sum_{i \leqq m_{k-1}} |a_{n_k, i}| + \sum_{i > m_k} |a_{n_k, i}| \Big) = 0. \tag{1.1}$$

The sequence (k_i) is formed as follows: take m_1 terms of (t_i), then $(m_2 - m_1)$ larger terms of (r_i), then $(m_3 - m_2)$ larger terms of (t_i), etc. It follows from (1.1) and the regularity of (a_{ni}) that

$$\lim_j \sum_i a_{n_{2j+1}, i} c_{k_i} = \alpha \quad \text{and} \quad \lim_j \sum_i a_{n_{2j}, i} c_{k_i} = \beta.$$

Remark. The above argument establishes the following fact, perhaps already observed: If (c_n) is any bounded sequence, (a_{ni}) is regular and $\sum_i a_{ni} c_{k_i}$ converges for every (k_i), then c_n converges.

(a) ⇒ (b) The following lemma is taken from [1].

Lemma 1.1. *Let S be a contraction operator on $L_2(\mu)$ and assume that $h \in L_2(\mu)$. Then $\|h\|^2 - \|Sh\|^2 \leqq \varepsilon^2$ implies that for each $g \in L_2(\mu)$, $|(h, g) - (Sh, Sg)| \leqq \varepsilon \|g\|$.*

Proof of Lemma. $|(h, g) - (Sh, Sg)| = |(h, g) - (S^* S h, g)| \leqq \|h - S^* S h\| \, \|g\|$. But $\|h - S^* S h\|^2 = \|h\|^2 - 2\|Sh\|^2 + \|S^* S h\|^2 \leqq \|h\|^2 - \|Sh\|^2 \leqq \varepsilon^2$, since S^* is also a contraction.

Assume (a), and let (a_{ni}) be a (UR)-matrix. We note that $\lim_n \|T^n f\|$ exists since T is a contraction. We first make the additional assumption that $T^n f \to 0$ weakly. Hence, for each $\varepsilon > 0$, one can find an integer $K \geqq 0$ such that $k \geqq K$ and $j \geqq 0$ imply that $\|T^k f\|^2 - \|T^{j+k} f\|^2 \leqq \varepsilon^2$ and also $|(T^k f, f)| \leqq \varepsilon$. Applying the lemma with $S = T^j$, $h = T^k f$, $g = f$, we obtain that

$$|(T^{j+k} f, T^j f) - (T^k f, f)| \leqq \varepsilon \cdot \|f\|,$$

and hence

$$|(T^{j+k} f, T^j f)| \leqq \varepsilon \cdot (1 + \|f\|)$$

whenever $j \geqq 0$ and $k \geqq K$. Consequently, we have that

$$|(T^i f, T^j f)| \leqq \varepsilon \cdot (1 + \|f\|) \quad \text{for } |i - j| \geqq K. \tag{1.2}$$

Since $\lim_n \max_i |a_{ni}| = 0$, there exists an integer N such that $n \geqq N$ implies that $\max_i |a_{ni}| < \varepsilon$. Let $m = \sup_n \sum_i |a_{ni}|$.

It follows from (1.2) that for $n \geqq N$

$$\begin{aligned}\|A_n f\|^2 &= (\sum_i a_{ni} T^i f, \sum_i a_{ni} T^i f)\\ &\leqq \sum_{|i-j|<K} |a_{ni}\| a_{nj}\| (T^i f, T^j f)| + \varepsilon(1+\|f\|) \cdot \sum_{|i-j|\geqq K} |a_{ni}|\,|a_{nj}|\\ &\leqq 2K(\max_j |a_{nj}|)\|f\|^2 (\sum_i |a_{ni}|) + \varepsilon(1+\|f\|)\cdot(\sum_i |a_{ni}|)^2\\ &< [2Km\|f\|^2 + (1+\|f\|)m^2]\,\varepsilon.\end{aligned}$$

Thus $\lim_n \|A_n f\| = 0$. In the general case, let $T^n g$ converge weakly to $\bar{g}$. Clearly $\bar{g}$ is T-invariant, hence the above argument applied to $f = g - \bar{g}$ shows that $\|A_n f\| \to 0$. It then follows from

$$\|A_n f\| = \|\sum_i a_{ni} T^i g - \sum_i a_{ni} T^i \bar{g}\| = \|A_n g - (\sum_i a_{ni})\bar{g}\|$$

that $\|A_n g - \bar{g}\| \to 0$ since $\lim_n \sum_i a_{ni} = 1$.

A sequence of operators T_n is said to *converge weakly* in L_p if $T_n f$ converges weakly for each $f \in L_p$.

Theorem 1.2. *Let $(X, \mathscr{A}, \nu)$ be a finite measure space, and let S be a contraction operator on $L_{p_1}(\nu)$ and $L_{p_2}(\nu)$ hence on L_p, $p_1 < p < p_2$, where either* (α) $1 \leqq p_1 \leqq 2 < p_2 \leqq \infty$, *or* ($\beta$) $1 = p_1 < p_2$. *Then the following conditions are equivalent:*

(1) *For some fixed $q_0 \in [p_1, p_2)$, S^n converges weakly in $L_{q_0}(\nu)$.*

(2) *For every $q \in [p_1, p_2)$, S^n converges weakly in $L_q(\nu)$.*

(3) *For every $q \in [p_1, p_2)$ and every $f \in L_q(\nu)$, $\sum_i a_{ni} T^i f$ converges in $L_q(\nu)$ for every (UR)-matrix (a_{ni}).*

If ν is assumed only σ-finite and (α) *holds, then* (1), (2), (3) *are still equivalent, provided that $[p_1, p_2)$ is replaced by (p_1, p_2) when $p_1 = 1$.*

Proof. We only prove the theorem under the assumption that $\nu(X) < \infty$; the proof of the σ-finite case is similar. *Case* (α). The implication (2) $\Rightarrow$ (1) is obvious; the implication (3) $\Rightarrow$ (2) follows from the same argument as in the proof of (b) $\Rightarrow$ (a) in Theorem 1.1; thus we need only to show that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). For $1 \leqq p < \infty$, p' is given by $1/p + 1/p' = 1$.

(1) $\Rightarrow$ (2) Assume that (1) holds for $q_0 \in [p_1, p_2)$, and let $q \in [p_1, p_2)$ be fixed. Since the sequence (S^n) is uniformly bounded, we need only to show that $S^n f$ converges weakly for $f \in L_q(\nu) \cap L_{q_0}(\nu)$ which is dense in $L_q(\nu)$. But it follows from (1) that for each $f \in L_q(\nu) \cap L_{q_0}(\nu)$, $\lim_n \int g \cdot S^n f\, d\nu$ exists for each $g \in L_{q'}(\nu) \cap L_{q_0'}(\nu)$ which is dense in $L_{q'}(\nu)$. Hence S^n converges weakly in $L_q(\nu)$.

(2) $\Rightarrow$ (3) Assume (2); hence S^n converges weakly in $L_2(\nu)$. By Theorem 1.1, for each fixed matrix (a_{ni}) satisfying (UR) and each $f \in L_2(\nu)$, $A_n f = \sum_i a_{ni} S^i f$ converges strongly in $L_2(\nu)$. For a fixed q, $p_1 \leqq q < p_2$, clearly $L_{p_2}(\nu)$ is contained and dense in $L_q(\nu)$. Since $A_n = \sum_i a_{ni} S^i$ is a sequence of uniformly bounded linear operators in $L_q(\nu)$, in order to prove that $A_n f$ converges strongly in $L_q(\nu)$ for

each $f \in L_q(\nu)$, it suffices to verify the convergence for f belonging to a dense subspace, say $L_{p_2}(\nu)$. If $p_1 \leqq q \leqq 2$, set $r = p_1(2-q)/(2-p_1)$, $s = 2(q-p_1)/(2-p_1)$, $p = (2-p_1)/(2-q)$, and $p' = p/(p-1) = (2-p_1)/(q-p_1)$; then $r+s=q$, $rp=p_1$, and $sp'=2$. Let $f \in L_{p_2}(\nu)$; it follows from Hölder's inequality that

$$\begin{aligned}\|A_n f - A_m f\|_q^q &= \int |A_n f - A_m f|^r \, |A_n f - A_m f|^s \, d\nu \\ &\leqq \|(A_n f - A_m f)^r\|_p \, \|(A_n f - A_m f)^s\|_{p'} \\ &= \|A_n f - A_m f\|_{p_1}^{p_1/p} \, \|A_n f - A_m f\|_2^{2/p'} \\ &\leqq \big(2\|f\|_{p_1} \cdot \sup_n \sum_i |a_{ni}|\big)^{p_1/p} \, \|A_n f - A_m f\|_2^{2/p'}\end{aligned}$$

which converges to zero since $\sup_n \sum_i |a_{ni}| < \infty$ and $(A_n f)$ is Cauchy in $L_2(\nu)$. If $2 < q < p_2$, set $r = p_2(q-2)/(p_2-2)$, $s = 2(p_2-q)/(p_2-2)$, $p = (p_2-2)/(q-2)$, $p' = (p_2-2)/(p_2-q)$; a similar argument yields the conclusion. *Case* (β). The proof is the same, with the following Theorem 1.3 applied instead of Theorem 1.1. The proof of Theorem 1.3 depends on (α), but not (β), case of Theorem 1.2.

We next consider the case $p=1$. The following theorem is due to Akcoglu and the second named author [1] in the case when the matrices (a_{ni}) in condition (B) are obtained from the Cesàro matrix by arbitrarily inserting columns of zeros.

Theorem 1.3. *Let T be a contraction operator on $L_1(X, \mathscr{A}, \mu)$. Then the conditions* (A) *and* (B) *are equivalent:*

(A) *T^n converges weakly in $L_1(\mu)$.*

(B) *For each $f \in L_1(\mu)$ and for each (UR)-matrix (a_{ni}), $\sum_i a_{ni} T^i f$ converges in $L_1(\mu)$.*

Proof. We prove that (A) ⇒ (B); the implication (B) ⇒ (A) is valid in general Banach spaces, as remarked in Theorem 1.1. Let τ be the linear modulus of T; τ is a positive linear contraction on $L_1(\mu)$ and $|Tf| \leqq \tau |f|$ for each $f \in L_1(\mu)$ (see Chacon and Krengel [4]). The following lemma is taken from [1]:

Lemma 1.2. *A contraction T on $L_1(X, \mathscr{A}, \mu)$ decomposes the space X into sets G and $F = X - G$ such that*

(i) *If $f \in L_1$ and if $T^n f$ converges weakly in L_1 then $\lim_n \int_F |T^n f| \, d\mu = 0$.*

(ii) *There exists an $h \in L_1^+$ such that the support of h is G and $\tau h = h$, where τ is the linear modulus of T.*

Assume that (A) holds. If $G = \emptyset$, then (B) follows from (i) of the lemma; otherwise, there is an $h \in L_1^+$, $h \not\equiv 0$, satisfying (ii) of the lemma. Let λ be the finite measure defined by $d\lambda = h \, d\mu$. We note that a function $\varphi \in L_1(G, \lambda)$ if and only if $h\varphi \in L_1(G, \mu)$. The operator S defined on $L_1(G, \lambda)$ by

$$S\varphi = \frac{1}{h} T(h\varphi), \qquad \varphi \in L_1(G, \lambda) \tag{1.3}$$

is a contraction since

$$\int |S\varphi| \, d\lambda = \int |T(h\varphi)| \, d\mu \leqq \int |\varphi| \, d\lambda. \tag{1.4}$$

S is also a contraction on $L_\infty(G,\lambda)$ since for each $g\in L_\infty(G,\lambda)$

$$|Sg|=\left|\frac{1}{h}T(hg)\right|\leqq\frac{1}{h}\tau(h|g|)\leqq\|g\|_\infty. \tag{1.5}$$

To prove (A) $\Rightarrow$ (B), we first note that if T^n converges weakly in $L_1(X,\mu)$ then S^n converges weakly in $L_1(G,\lambda)$. Indeed, for $\varphi\in L_1(G,\lambda)$, $g\in L_\infty(G,\lambda)$ we have that $h\varphi\in L_1(G,\mu)$ and hence

$$\int S^n\varphi\cdot g\,d\lambda=\int T^n(h\varphi)\cdot g\,d\mu$$

converges by assumption (A). Applying Theorem 1.2 to S with $p_1=1$ and $p_2=\infty$, we obtain that for each matrix (a_{ni}) satisfying (UR) and for each $\varphi\in L_1(G,\lambda)$, $\lim_n\sum_i a_{ni}S^i\varphi$ exists in $L_1(G,\lambda)$; i.e., the sequence $\sum_i a_{ni}T^i(h\varphi)=h\cdot\sum_i a_{ni}S^i\varphi$ is Cauchy hence convergent in $L_1(G,\mu)$. This proves that $\lim_n\sum_i a_{ni}T^if$ exists for each $f\in L_1(G,\mu)$. For an arbitrary $f\in L_1(X,\mu)$, let $A_nf=\sum_i a_{ni}T^if$. To prove the convergence of A_nf, it suffices to show that for each $\varepsilon>0$ there exists a Cauchy sequence $g_n\in L_1(X,\mu)$ satisfying

$$\limsup_{n\to\infty}\|A_nf-g_n\|_1<\varepsilon.$$

Let $m=\sup_n\sum_i|a_{ni}|$; apply Lemma 1.2 to obtain an integer $i_0>0$ such that

$$\int_F|T^{i_0}f|\,d\mu<\varepsilon/m. \tag{1.6}$$

Let $g=1_G\cdot T^{i_0}f$. It follows from $\lim_n\max_i|a_{ni}|=0$ that

$$\lim_n\sum_{i=1}^{i_0}|a_{ni}|=0. \tag{1.7}$$

Set for $n,i=1,2,\ldots$, $b_{ni}=a_{n,i+i_0}$.

Clearly, (b_{ni}) also satisfies (UR). Since $g\in L_1(G,\mu)$, the first part of the proof shows that the sequence $g_n=\sum_i b_{ni}T^ig$ is Cauchy in $L_1(X,\mu)$. We note that

$$g_n=\sum_i b_{ni}T^{i+i_0}f-\sum_i b_{ni}T^i(1_F\cdot T^{i_0}f)=\sum_i a_{n,i+i_0}T^{i+i_0}f-h_n$$

where by (1.6)

$$\|h_n\|_1=\Big\|\sum_i b_{ni}T^i(1_F\cdot T^{i_0}f)\Big\|_1\leqq\Big(\sum_i|b_{ni}|\Big)(\varepsilon/m)\leqq\varepsilon.$$

Thus

$$\begin{aligned}\|g_n-A_nf\|_1&\leqq\Big\|\sum_i a_{n,i+i_0}T^{i+i_0}f-\sum_i a_{ni}T^if\Big\|_1+\varepsilon\\&\leqq\Big(\sum_{1\leqq i\leqq i_0}|a_{ni}|\Big)\|f\|_1+\varepsilon\end{aligned} \tag{1.8}$$

which implies by (1.7) that $\limsup_{n\to\infty}\|g_n-A_nf\|_1\leqq\varepsilon$.

Section 2

For $p \neq 1, 2$, we require some additional conditions on the operator T. Whether the theorem is true without such conditions we do not know. An operator T on $L_p(\mu)$ is called *positive* if $f \in L_p(\mu)$ and $f \geqq 0$ imply that $Tf \geqq 0$. We first consider the case $1 < p < 2$.

Theorem 2.1. *Let T be a positive contraction on $L_p(X, \mathscr{A}, \mu)$, where p is fixed, $1 < p < 2$. If there exists a positive function $h \in L_p(\mu)$ satisfying $Th \leqq h$ then conditions* (A) *and* (B) *are equivalent:*

(A) *T^n converges weakly in $L_p(\mu)$.*

(B) *For each $f \in L_p(\mu)$ and for each (UR)-matrix (a_{ni}), $\sum_i a_{ni} T^i f$ converges in $L_p(\mu)$.*

Proof. We only prove that (A) $\Rightarrow$ (B). Let ν be the finite measure on $\mathscr{A}$ defined by $d\nu = h^p \, d\mu$, and let S be defined on $L_p(X, \mathscr{A}, \nu)$ by

$$S\varphi = \frac{1}{h} T(h\varphi), \qquad \varphi \in L_p(\nu). \tag{2.1}$$

Clearly S is a positive linear operator on $L_p(\nu)$. Furthermore,

$$\|S\varphi\|_p^p = \int \frac{1}{h^p} |T(h\varphi)|^p \, d\nu = \int |T(h\varphi)|^p \, d\mu \leqq \int |h\varphi|^p \, d\mu = \int |\varphi|^p \, d\nu,$$

which shows that S is a contraction on $L_p(\nu)$. The assumption that $Th \leqq h$ and (2.1) with $\varphi = 1$ imply that $S1 \leqq 1$, and hence S is a contraction on $L_\infty(\nu)$. We note that a function $\varphi \in L_p(\nu)$ if and only if $\varphi \cdot h \in L_p(\mu)$, and $\psi \in L_{p'}(\nu)$ if and only if $\psi \cdot h^{p-1} \in L_{p'}(\mu)$. Furthermore, iteration of (2.1) yields for $n = 1, 2, \ldots$,

$$S^n \varphi = \frac{1}{h} T^n(h\varphi), \qquad \varphi \in L_p(\nu). \tag{2.2}$$

Thus for $\varphi \in L_p(\nu)$ and $\psi \in L_{p'}(\nu)$, we have

$$\int \psi \cdot S^n \varphi \, d\nu = \int (\psi h^{p-1}) \cdot T^n(h\varphi) \, d\mu, \tag{2.3}$$

which shows that S^n converges weakly in $L_p(\nu)$ if and only if T^n converges weakly in $L_p(\mu)$. Similarly, it follows from

$$\begin{aligned} \int \Big|\sum_i a_{ni} S^i \varphi - \sum_i a_{mi} S^i \varphi\Big|^p d\nu &= \int \frac{1}{h^p} \Big|\sum_i a_{ni} T^i(h\varphi) - \sum_i a_{mi} T^i(h\varphi)\Big|^p d\nu \\ &= \int \Big|\sum_i a_{ni} T^i(h\varphi) - \sum_i a_{mi} T^i(h\varphi)\Big|^p d\mu \end{aligned}$$

that S satisfies (B) on $L_p(\nu)$ if and only if T satisfies (B) on $L_p(\mu)$. By assumption (A), T^n converges weakly in $L_p(\mu)$, therefore S^n converges weakly in $L_p(\nu)$. Since S is a contraction on both $L_p(\nu)$ and $L_\infty(\nu)$ of the finite measure space $(X, \mathscr{A}, \nu)$, Theorem 1.2 applied to S with $p_1 = p$ and $p_2 = \infty$ shows that S satisfies (B) on $L_p(\nu)$ and hence T satisfies (B) on $L_p(\mu)$.

Theorem 2.2. *Let T be a positive contraction on $L_p(X, \mathscr{A}, \mu)$, where p is fixed, $2 < p < \infty$. If there exists a positive function $h \in L_p(\mu)$ satisfying $Th \leqq h$ and*

$T^* h^{p-1} \leqq h^{p-1}$ then the following conditions are equivalent:

(A) *T^n converges weakly in $L_p(\mu)$.*

(B) *For each $f \in L_p(\mu)$ and for each (UR)-matrix (a_{ni}), $\sum_i a_{ni} T^i f$ converges in $L_p(\mu)$.*

Proof. Positive functions h satisfying the conditions $Th \leqq h$ and $T^* h^{p-1} \leqq h^{p-1}$ have been introduced and called *semi-fixed points* of T by Chacon-Olsen [5], where it is shown that a positive fixed point is semi-fixed. The proof of Theorem 2.1 shows that the operator S defined on $L_p(\nu)$ of the finite measure space $(X, \mathscr{A}, \nu)$ is a contraction on both $L_p(\nu)$ and $L_\infty(\nu)$. One checks that the adjoint operators S^* is given by

$$S^* \psi = \frac{1}{h^{p-1}} T^*(h^{p-1} \cdot \psi), \qquad \psi \in L_{p'}(\nu). \tag{2.4}$$

Since $\|S^*\|_{p'} = \|S\|_p$, S^* is a contraction on $L_{p'}(\nu)$. Moreover, (2.4) and the assumption that $T^* h^{p-1} \leqq h^{p-1}$ imply that S^* is also a contraction on $L_\infty(\nu)$. Thus S is a contraction on both $L_1(\nu)$ and $L_\infty(\nu)$, and Theorem 1.2 may be applied to S with $p_1 = 1$ and $p_2 = \infty$. The remaining part of the proof is the same as that of Theorem 2.1.

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