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Limiting Behavior of Regular Functionals of Empirical Distributions for Stationary $*$-Mixing Processes ⋆

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1. Introduction

Let $\{X_i, -\infty < i < \infty\}$ be a stationary $*$-mixing stochastic process defined on a probability space $(\Omega, \mathscr{A}, P)$. Thus, if $\mathscr{M}_{-\infty}^{k}$ and $\mathscr{M}_{k+n}^{\infty}$ be respectively the σ-fields generated by $\{X_i, i \leqq k\}$ and $\{X_i, i \geqq k+n\}$, and if, $A \in \mathscr{M}_{-\infty}^{k}$ and $B \in \mathscr{M}_{k+n}^{\infty}$, then for every $k(-\infty < k < \infty)$ and n

$$|P(AB) - P(A)\,P(B)| < \psi_n P(A)\,P(B), \tag{1.1}$$

where $\psi_n \downarrow 0$ as $n \uparrow \infty$. Further conditions on $\{\psi_n\}$ will be stated as and when necessary. We may refer to Blum, Hanson and Koopmans (1963) and Philipp (1969a, b, c) for detailed treatment of $*$-mixing processes in the context of the limiting behavior of sums of the X_i.

We denote the marginal distribution function (d.f.) of X_i by $F(x)$, $x \in R^p$, the $p(\geqq 1)$-dimensional Euclidean space. Consider then a functional

$$\theta(F) = \int_{R^{pm}} \cdots \int g(x_1, \ldots, x_m)\, dF(x_1) \ldots dF(x_m), \tag{1.2}$$

defined over $\mathscr{F} = \{F: |\theta(F)| < \infty\}$, where $g(x_1, \ldots, x_m)$ is symmetric in its $m(\geqq 1)$ arguments. We consider here the following two estimators of $\theta(F)$. For a sample $\{X_1, \ldots, X_n\}$, the empirical d.f. is defined as

$$F_n(x) = n^{-1} \sum_{i=1}^{n} u(x - X_i), \qquad x \in R^p, \tag{1.3}$$

where $u(v)$ is equal to one only when all the p components of v are non-negative; otherwise, $u(v) = 0$. Then, in the same fashion as in von Mises (1947), a differentiable statistical functional $\theta(F_n)$ is defined as

$$\begin{aligned} \theta(F_n) &= \int_{R^{pm}} \cdots \int g(x_1, \ldots, x_m)\, dF_n(x_1) \ldots dF_n(x_m) \\ &= n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} g(X_{i_1}, \ldots, X_{i_m}), \qquad n \geqq 1, \end{aligned} \tag{1.4}$$

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which is the corresponding functional of the empirical d.f. Also, as in Hoeffding (1948), we define a U-statistic

$$U_n = \binom{n}{m}^{-1} \sum_n^* g(X_{i_1}, \ldots, X_{i_m}), \qquad n \geqq m, \tag{1.5}$$

where the summation $\sum_n^*$ extends over all possible $1 \leqq i_1 < \cdots < i_m \leqq n$.

Under suitable moment conditions on g and on $\{\psi_n\}$, to be stated in Section 2, the following three problems are studied here: (i) asymptotic normality of $n^{\frac{1}{2}}[\theta(F_n) - \theta(F)]$ and $n^{\frac{1}{2}}[U_n - \theta(F)]$, (ii) the law of iterated logarithm for $\theta(F_n)$ and U_n, and (iii) weak convergence of continuous sample path versions of the processes $\{n^{-\frac{1}{2}} k[\theta(F_k) - \theta(F)], k \geqq 1\}$ and $\{n^{-\frac{1}{2}} k[U_k - \theta(F)], k \geqq 1\}$ to processes of Brownian motion. It may be noted that (i) is a special case of (iii), and is established under less stringent conditions.

The main results along with the preliminary notions are presented in Section 2. Certain useful lemmas are considered in Section 3, and with the aid of these, the proofs of the main results are outlined in Section 4. The last section deals with a few applications.

We may note that for $m=1$, $\theta(F_n) = U_n = n^{-1} \sum_{i=1}^{n} g(X_i)$, and the corresponding results have already been studied by Ibragimov (1962), Billingsley (1968), Reznik (1968), and Philipp (1969a, b, c), among others. Hence, in the sequel, we shall exclusively consider the case of $m \geqq 2$. We may also remark that the above mentioned authors have considered the general ϕ-mixing processes [where the right hand side of (1.1) is $\phi_n P(A)$ and $\phi_n \downarrow 0$ as $n \to \infty$] which contain $*$-mixing processes as special cases. The simple proof to be considered in the current paper rests on certain basic lemmas on Bernoullian random variables in a $*$-mixing process. These lemmas do not hold for general ϕ-mixing processes, and hence, the same technique of proof is not applicable for the latter processes. Also, the reverse martingale property of U_n [cf. Berk (1966)] or related properties for $\theta(F_n)$ do not hold for ϕ-mixing (or $*$-mixing) processes, so that an alternative approach of Miller and Sen (1972), studied for independent processes, does not seem to be readily adaptable. An altogether different and presumably more involved proof seems to be needed for a general ϕ-mixing process.

2. Statement of the Result

For every c: $0 \leqq c \leqq m$, we let

$$g_c(x_1, \ldots, x_c) = \int_{R^{p(m-c)}} \cdots \int g(x_1, \ldots, x_m)\, dF(x_{c+1}) \ldots dF(x_m), \tag{2.1}$$

so that $g_0 = \theta(F)$ and $g_m = g$. Also, let

$$\zeta_{1,h} = \zeta_{1,h}(F) = E[g_1(X_1)\, g_1(X_{1+h})] - \theta^2(F), \qquad h \geqq 0; \tag{2.2}$$

$$\sigma^2 = \sigma^2(F) = \zeta_{1,0} + 2 \sum_{h=1}^{\infty} \zeta_{1,h}. \tag{2.3}$$

Then, we assume that (i)

$$0 < \sigma^2 < \infty, \tag{2.4}$$

and (ii) for some $r(\geqq 2)$,

$$v_r = \int_{R^{pm}} \cdots \int |g(x_1, \ldots, x_m)|^r \, dF(x_1) \ldots dF(x_m) < \infty. \tag{2.5}$$

Finally, we define for every non-negative d,

$$A_d(\psi) = \sum_{k=0}^{\infty} (k+1)^d \psi_k^{\frac{1}{2}} \quad \text{and} \quad A_d^*(\psi) = \sum_{k=0}^{\infty} (k+1)^{2d} \psi_k. \tag{2.6}$$

Note that $A_d(\psi) < \infty \Rightarrow A_{d'}(\psi) < \infty$ for all $0 \leqq d' \leqq d$, and

$$[A_d(\psi) < \infty] \Rightarrow [A_d^*(\psi) < \infty]. \tag{2.7}$$

Then, we have the following two theorems.

Theorem 1. *If* $A_{m-1}(\psi) < \infty$, (2.4) *and* (2.5) (*with* $r=2$) *hold, then*

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}}[\theta(F_n) - \theta(F)] \leqq x m \sigma\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt, \tag{2.8}$$

for all x: $-\infty < x < \infty$, *and*

$$n^{\frac{1}{2}} |\theta(F_n) - U_n| \to 0, \quad \text{in probability, as } n \to \infty. \tag{2.9}$$

Hence, (2.8) *also holds for* $\theta(F_n)$ *being replaced by* U_n.

Theorem 2. *If for* $m^* = \max[2, m-1]$, $A_{m^*}(\psi) < \infty$, (2.4) *and* (2.5) (*with* $r=4$) *hold, then*

$$P\{\limsup_n n^{\frac{1}{2}}[\theta(F_n) - \theta(F)]/m[2\sigma^2 \log\log n]^{\frac{1}{2}} = 1\} = 1, \tag{2.10}$$

$$P\{\liminf_n n^{\frac{1}{2}}[\theta(F_n) - \theta(F)]/m[2\sigma^2 \log\log n]^{\frac{1}{2}} = -1\} = 1, \tag{2.11}$$

$$P\{\limsup_n n^{\frac{1}{2}} |\theta(F_n) - U_n|/m[2\sigma^2 \log\log n]^{\frac{1}{2}} = 0\} = 1, \tag{2.12}$$

and hence, (2.10) *and* (2.11) *also hold for* U_n.

Consider now the space $C[0,1]$ of all continuous real valued functions $X(t)$, $0 \leqq t \leqq 1$, and associate with it the uniform topology

$$\rho(X, Y) = \sup_{t \in I} |X(t) - Y(t)|, \quad I = \{t: 0 \leqq t \leqq 1\}, \tag{2.13}$$

where both X and Y belong to $C[0,1]$. For every $n \geqq 1$, we define $Y_n(0) = 0$, and let

$$Y_n(t) = Y_n([nt]/n) + (nt - [nt])[Y_n(([nt]+1)/n) - Y_n([nt]/n)], \quad 0 \leqq t \leqq 1, \tag{2.14}$$

where $[s]$ denotes the largest integer contained in $s(\geqq 0)$, and

$$Y_n(k/n) = k[\theta(F_k) - \theta(F)]/(m\sigma n^{\frac{1}{2}}), \quad k = 1, \ldots, n. \tag{2.15}$$

Similarly, replacing $\theta(F_k)$ by U_k for $k \geqq m$ and by $\theta(F)$ for $k \leqq m-1$, we define $Y_n^*(t)$ as in (2.14) and (2.15). Then, $Y_n^*(t) = 0$ for $0 \leqq t \leqq (m-1)/n$. Also, let

$$Y_n = \{Y_n(t), t \in I\}, \quad Y_n^* = \{Y_n^*(t), t \in I\} \quad \text{and} \quad W = \{W(t), t \in I\}, \tag{2.16}$$

where W is a standard Brownian motion. Then, we have the following.

Theorem 3. *If for* $m^*=\max(2,m-1)$, $A_{m^*}(\psi)<\infty$, (2.4) *and* (2.5) (*with* $r=4$) *hold, then both* Y_n *and* Y_n^* *converge weakly in the uniform topology on* $C[0,1]$ *to W, and*

$$\rho(Y_n, Y_n^*)\to 0, \quad \text{in probability, as } n\to\infty. \tag{2.17}$$

In fact, (2.17) *holds even if* (2.5) *holds for* $r=2$ *and* $A_{m-1}(\psi)<\infty$.

The proofs of the theorems are postponed to Section 4.

3. Certain Useful Lemmas

Let $\{X_i, -\infty<i<\infty\}$ be a stationary $*$-mixing process, and for each $j(=1,2,\ldots)$, let $Z_{ji}=h_j(X_i)$, $-\infty<i<\infty$, be zero-one valued random variables, where $h_1(u)$, $h_2(u),\ldots$ are not identical, and

$$P\{Z_{ji}=1\}=1-P\{Z_{ji}=0\}=p_j, \quad j\geqq 1. \tag{3.1}$$

Lemma 3.1. *If for some* $k\geqq 1$, $A_{k-1}(\psi)<\infty$, *then*

$$\left|E\left\{\prod_{j=1}^{2k}\left[\sum_{i=1}^{n}(Z_{ji}-p_j)\right]\right\}\right|\leqq n^k K_\psi\, p_1\ldots p_{2k}, \tag{3.2}$$

where $K_\psi(<\infty)$ *depends only on* $\{\psi_n\}$.

Proof. We sketch the proof only for $k=1$ and 2; for $k\geqq 3$, the same proof (but, evidently, requiring more tedious steps) holds. For $k=1$, we have

$$\left|E\left\{\prod_{j=1}^{2}\left[\sum_{i=1}^{n}(Z_{ji}-p_j)\right]\right\}\right|\leqq \sum_{i=1}^{n}\sum_{j=1}^{n}|E(Z_{1i}-p_1)(Z_{2j}-p_2)|. \tag{3.3}$$

Now, by Lemma 1 of Philipp (1969c), under (1.1) and (3.1),

$$\begin{aligned}|E(Z_{1i}-p_1)(Z_{2j}-p_2)|&\leqq \psi_{|i-j|}\, E|Z_{1i}-p_1|\, E|Z_{2j}-p_2|\\ &=\psi_{|i-j|}\, 4p_1(1-p_1)\,p_2(1-p_2)\leqq 4p_1p_2\,\psi_{|i-j|}.\end{aligned} \tag{3.4}$$

Hence, (3.3) is bounded above by

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{|i-j|}\,4p_1p_2\leqq 8p_1p_2\sum_{i=1}^{n}\sum_{j=i}^{n}\psi_{j-i}\leqq 8n\,p_1p_2\,A_0^*(\psi), \tag{3.5}$$

and therefore, the proof follows by using (2.7).

For $k=2$, we have

$$\begin{aligned}&\left|E\left\{\prod_{j=1}^{4}\left[\sum_{i=1}^{n}(Z_{ji}-p_j)\right]\right\}\right|\\ &\quad\leqq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}|E(Z_{1i}-p_1)(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)| \qquad (3.6)\\ &\quad\leqq \sum{}^*\Big\{\sum_{1\leqq i\leqq j\leqq k\leqq l\leqq n}|E(Z_{\alpha i}-p_\alpha)(Z_{\beta j}-p_\beta)(Z_{\gamma k}-p_\gamma)(Z_{\delta l}-p_\delta)|,\end{aligned}$$

where $(\alpha, \beta, \gamma, \delta)$ is a permutation of $(1, 2, 3, 4)$, and the summation $\sum^*$ extends over all 4! permutations of this type. For simplicity, consider the particular permutation $\alpha=1$, $\beta=2$, $\gamma=3$ and $\delta=4$. Then, we have,

$$\begin{aligned}
&\sum_{1\leqq i\leqq j\leqq k\leqq l\leqq n} |E(Z_{1i}-p_1)(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)| \\
&\quad \leqq \sum_n^{(1)}+\sum_n^{(2)}+\sum_n^{(3)} |E(Z_{1i}-p_1)(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)|,
\end{aligned} \tag{3.7}$$

where the summations $\sum_n^{(1)}, \sum_n^{(2)}$ and $\sum_n^{(3)}$ extend respectively over all $1\leqq i\leqq j\leqq k\leqq l\leqq n$ for which $j-i=\max(j-i, k-j, l-k)$, $k-j=\max(j-i, k-j, l-k)$ and $l-k=\max(j-i, k-j, l-k)$. Again, by Lemma 1 of Philipp (1969c),

$$\begin{aligned}
&\sum_n^{(1)} |E(Z_{1i}-p_1)(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)| \\
&\quad \leqq \sum_n^{(1)} \psi_{j-i}\, E|Z_{1i}-p_1|\, E|(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)| \\
&\quad \leqq 2p_1 \sum_n^{(1)} \psi_{j-i}\, E|(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)|,
\end{aligned} \tag{3.8}$$

as $E|Z_{ji}-p_j|=2p_j(1-p_j)$, $j\geqq 1$. Also, by a few straight forward steps,

$$\begin{aligned}
&E|(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)| \\
&\quad \leqq E\{|(Z_{2j}-p_2)(Z_{3k}-p_3)|\, E[|Z_{4l}-p_4|\,|\mathscr{M}_{-\infty}^k]\} \\
&\quad \leqq E\{|(Z_{2j}-p_2)(Z_{3k}-p_3)|\,[(1-p_4)p_4(1+\psi_{l-k})+p_4(1-p_4)(1+\psi_{l-k})]\} \\
&\quad \leqq 2p_4(1+\psi_{l-k})\, E|(Z_{2j}-p_2)(Z_{3k}-p_3)| \\
&\quad \leqq 8p_2 p_3 p_4 (1+\psi_{k-j})(1+\psi_{l-k}).
\end{aligned} \tag{3.9}$$

Hence, by (3.9), (3.8) is bounded above by

$$\begin{aligned}
&16 p_1 p_2 p_3 p_4 \sum_n^{(1)} \psi_{j-i}(1+\psi_{k-j})(1+\psi_{l-k}) \\
&\quad \leqq 16 p_1 p_2 p_3 p_4 (1+\psi_0)^2 \sum_n^{(1)} \psi_{j-i} \\
&\quad \leqq 16 p_1 p_2 p_3 p_4 (1+\psi_0)^2\, n \sum_{k=0}^{n} (k+1)^2 \psi_k \\
&\quad \leqq 16 p_1 p_2 p_3 p_4\, n (1+\psi_0)^2 A_1^*(\psi),
\end{aligned} \tag{3.10}$$

where by (2.7), $A_1^*(\psi)<\infty$ whenever $A_1(\psi)<\infty$. Similarly,

$$\begin{aligned}
&\sum_n^{(3)} |E(Z_{1i}-p_1)(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)| \\
&\quad \leqq 16 p_1 p_2 p_3 p_4\, n(1+\psi_0)^2 A_1^*(\psi).
\end{aligned} \tag{3.11}$$

Finally, by Lemma 1 of Philipp (1969c), and a few steps similar to those in (3.9) and (3.10), we have

$$\begin{aligned}
&\textstyle\sum_n^{(2)} |E(Z_{1i}-p_1)(Z_{2j}-p_2)(Z_{3k}-p_3)(Z_{4l}-p_4)| \\
&\quad \leqq \textstyle\sum_n^{(2)} |[E(Z_{1i}-p_1)(Z_{2j}-p_2)]\,[E(Z_{3k}-p_3)(Z_{4l}-p_4)]| \\
&\qquad + \textstyle\sum_n^{(2)} \psi_{k-j} E|(Z_{1i}-p_1)(Z_{2j}-p_2)|\, E|(Z_{3k}-p_3)(Z_{4l}-p_4)| \\
&\quad \leqq \textstyle\sum_n^{(2)} \psi_{j-i}\psi_{l-k} |E|Z_{1i}-p_1|\, E|Z_{2j}-p_2|\, E|Z_{3k}-p_3|\, E|Z_{4l}-p_4| \\
&\qquad + \textstyle\sum_n^{(2)} \psi_{k-j}(1+\psi_{j-i})(1+\psi_{l-k}) E|Z_{1i}-p_1|\, p_2\, E|Z_{3k}-p_3|\, p_4 \\
&\quad \leqq 16 p_1 p_2 p_3 p_4 \textstyle\sum_n^{(2)} \psi_{j-i}\psi_{l-k} + 4 p_1 p_2 p_3 p_4 (1+\psi_0)^2 \sum_n^{(2)} \psi_{k-j} \\
&\quad \leqq 16 p_1 p_2 p_3 p_4 n^2 \left(\sum_{k=0}^{\infty} \psi_k\right)^2 + 4 p_1 p_2 p_3 p_4 (1+\psi_0)^2 n \left(\sum_{k=0}^{n} (k+1)^2 \psi_k\right) \\
&\quad \leqq 4 n p_1 p_2 p_3 p_4 [4n\{A_0^*(\psi)\}^2 + (1+\psi_0)^2 A_1^*(\psi)].
\end{aligned} \tag{3.12}$$

Thus, by (2.7), (3.10) and (3.11), whenever $A_1(\psi)<\infty$, (3.7) is bounded above by

$$K_\psi^* n^2 p_1 p_2 p_3 p_4, \quad \text{where } K_\psi^* < \infty. \tag{3.13}$$

Since (3.13) does not depend on the order of the subscript 1, 2, 3, 4 of the p_i, repeating the steps for each permutation $(\alpha, \beta, \gamma, \delta)$ of (1, 2, 3, 4) and choosing $K_\psi = 24 K_\psi^*$, it follows that (3.6) is bounded above by $K_\psi n^2 p_1 p_2 p_3 p_4$, which completes the proof for $k=2$.

Lemma 3.2. *Let $s_j(\geqq 0)$, $j=1, \ldots, r(\geqq 1)$ be such that $\sum_{j=1}^{r} s_j = 2k$, $k \geqq 1$. Then $A_{k-1}(\psi) < \infty$ implies that*

$$\left| E\left\{ \prod_{j=1}^{r} \left[\sum_{i=1}^{n} (Z_{ji}-p_j) \right]^{s_j} \right\} \right| \leqq K_\psi n^k p_1 \ldots p_r, \tag{3.14}$$

where $K_\psi(<\infty)$ depends only on $\{\psi_n\}$.

The proof is similar to that of Lemma 3.1, and hence, is omitted.

Let us now define for every c: $1 \leqq c \leqq m$,

$$V_n^{(c)} = \int_{R^{cp}} \cdots \int g_c(x_1, \ldots, x_c) \prod_{j=1}^{c} d[F_n(x_j) - F(x_j)]. \tag{3.15}$$

Then, upon writing $dF_n = dF + d[F_n - F]$, we have from (1.4) and (3.15) that

$$\theta(F_n) = \theta(F) + \sum_{c=1}^{m} \binom{m}{c} V_n^{(c)}, \quad n \geqq 1. \tag{3.16}$$

Note that, by definition,

$$V_n^{(1)} = n^{-1} \sum_{i=1}^{n} [g_1(X_i) - \theta(F)]. \tag{3.17}$$

Lemma 3.3. *If* (2.5) *holds for* $r=2$, *then for every* c: $1\leqq c\leqq m$, $A_{c-1}(\psi)<\infty$ *implies that*

$$E[V_n^{(c)}]^2\leqq K_\psi n^{-c} v_2, \qquad K_\psi<\infty, \tag{3.18}$$

where K_ψ *depends only on* $\{\psi_n\}$.

Proof. By (3.15) and the Fubini theorem,

$$\begin{aligned} E[V_n^{(c)}]^2 &= \int_{R^{2cp}}\cdots\int g_c(x_1,\ldots,x_c)\,g_c(x_{c+1},\ldots,x_{2c})\,E\left\{\prod_{j=1}^{2c} d[F_n(x_j)-F(x_j)]\right\}\\ &= \int_{R^{2cp}}\cdots\int g_c(x_1,\ldots,x_c)\,g_c(x_{c+1},\ldots,x_{2c})\\ &\quad\cdot n^{-2c}E\left\{\prod_{j=1}^{2c}\left(\sum_{i=1}^{n} d[u(x_j-X_i)-F(x_j)]\right)\right\}. \end{aligned} \tag{3.19}$$

Thus, if we let $Z_{ji}=d[u(x_j-X_i)]$, $i=1,\ldots,n$, $j=1,\ldots,2c$, where $d[u(x-y)]=u(x+dx-y)-u(x-y)$, so that

$$P\{Z_{ji}=1\}=1-P\{Z_{ji}=0\}=dF(x_j), \qquad j=1,\ldots,2c, \tag{3.20}$$

we obtain from Lemmas 3.1 and 3.2 that

$$\left|E\left\{\prod_{j=1}^{2c}\left(\sum_{i=1}^{n} d[u(x_j-X_i)-F(x_j)]\right)\right\}\right|\leqq n^c K_\psi\, dF(x_1)\ldots dF(x_{2c}), \tag{3.21}$$

when $x_1,\ldots,x_{2c}$ are all distinct; otherwise $n^c K_\psi\, dF(x_1)\ldots dF(x_r)$, where $x_1,\ldots,x_r$, $r\geqq 1$, are the distinct set of values of $x_1,\ldots,x_{2c}$. Hence, by (3.19) and (3.21),

$$\begin{aligned} E[V_n^{(c)}]^2 &\leqq K_\psi n^{-c}\int_{R^{2cp}}\cdots\int |g_c(x_1,\ldots,x_c)\,g_c(x_{c+1},\ldots,x_{2c})|\prod_{j=1}^{2c} dF(x_j)\\ &= K_\psi n^{-c}\Big[\int_{R^{2cp}}\cdots\int |g_c(x_1,\ldots,x_c)|\,dF(x_1)\ldots dF(x_c)\Big]^2\\ &\leqq v_2 K_\psi n^{-c}. \quad \text{Q.E.D.} \end{aligned} \tag{3.22}$$

Lemma 3.4. *If* (2.5) *holds for* $r=4$ *and* $A_1(\psi)<\infty$, *then*

$$E|V_n^{(2)}|^4\leqq K_\psi v_4 n^{-4}, \qquad K_\psi<\infty, \tag{3.23}$$

where K_ψ *depends only on* $\{\psi_n\}$.

The proof is similar to that of Lemma 3.3, and hence, is omitted.

We may rewrite (1.5) as

$$\begin{aligned} U_n &= n^{-[m]}\sum_{P_{n,m}}\int_{R^{pm}}\cdots\int g(x_1,\ldots,x_m)\prod_{j=1}^{m} d[u(x_j-X_{ij})]\\ &= n^{-[m]}\sum_{P_{n,m}}\int_{R^{pm}}\cdots\int g(x_1,\ldots,x_m)\prod_{j=1}^{m} d[\{u(x_j-X_{ij})-F(x_j)\}+F(x_j)]\\ &= \theta(F)+\sum_{h=1}^{m}\binom{m}{h} U_n^{(h)}, \end{aligned} \tag{3.24}$$

where $P_{n,m}=\{(i_1,\ldots,i_m)\colon 1\leqq i_1\neq\cdots\neq i_m\leqq n\}$, $n^{-[m]}=\{n\ldots(n-m+1)\}^{-1}$, and

$$U_n^{(h)}=n^{-[h]}\sum_{P_{n,h}}\int_{R^{ph}}\cdots\int g_h(x_1,\ldots,x_h)\prod_{j=1}^{h} d[u(x_j-X_{ij})-F(x_j)], \qquad h=1,\ldots,m. \tag{3.25}$$

Note that $V_n^{(1)} = U_n^{(1)}$, so that by (3.16) and (3.24),

$$\begin{aligned}[\theta(F_n) - U_n] &= \sum_{h=2}^{m} \binom{m}{h} [V_n^{(h)} - U_n^{(h)}] \\ &= \binom{m}{2} n^{-2}[[n^2 V_n^{(2)} - n^{[2]} U_n^{(2)}]] - \binom{m}{2} n^{-1} U_n^{(2)} + \sum_{h=3}^{m} \binom{m}{h} [V_n^{(h)} - U_n^{(h)}].\end{aligned} \tag{3.26}$$

Now, by the same technique as in Lemma 3.3, for every $h \geqq 1$, $A_{h-1}(\psi) < \infty$ implies that

$$E[U_n^{(h)}]^2 \leqq K_\psi n^{-h} v_2, \qquad K_\psi < \infty. \tag{3.27}$$

Also, writing $Q_n = n^2 V_n^{(2)} - n^{[2]} U_n^{(2)}$, and rewriting it as

$$Q_n = \sum_{i=1}^{n} \int \cdots \int_{R^{2p}} g_2(x_1, x_2)\, d[u(x_1 - X_i) - F(x_1)]\, d[u(x_2 - X_i) - F(x_2)], \tag{3.28}$$

we obtain by the same technique as in Lemma 3.3 that $A_1(\psi) < \infty$ implies that

$$n^{-1} E Q_n^2 \leqq K_\psi v_2, \qquad K_\psi < \infty. \tag{3.29}$$

Thus, from (3.18), (3.26), (3.27), (3.28), (3.29) and the c_r-inequality, we obtain that if $v_2 < \infty$ and $A_{m-1}(\psi) < \infty$, then

$$E[\theta(F_n) - U_n]^2 \leqq C_\psi n^{-3}, \qquad C_\psi < \infty. \tag{3.30}$$

These results are used in the next section, in the proof of the theorems.

4. Outline of the Proofs of the Theorems

Let us first consider Theorem 1. By virtue of (3.16) and Lemma 3.3, whenever $A_{m-1}(\psi) < \infty$,

$$\begin{aligned} n E[\theta(F_n) - \theta(F) - m V_n^{(1)}]^2 &= n E \left\{ \sum_{h=2}^{m} \binom{m}{h} V_n^{(h)} \right\}^2 \\ &\leqq n(m-1) \sum_{h=2}^{m} \binom{m}{h}^2 E[V_n^{(h)}]^2 \\ &\leqq n^{-1} K_\psi^* v_2, \qquad \text{where } K_\psi^* (<\infty) \text{ depends only on } \{\psi_n\}. \end{aligned} \tag{4.1}$$

Thus, by (4.1) and the Chebychev inequality

$$n^{\frac{1}{2}} |\theta(F_n) - \theta(F) - m V_n^{(1)}| \xrightarrow{p} 0, \qquad \text{as } n \to \infty, \tag{4.2}$$

which implies that $n^{\frac{1}{2}}[\theta(F_n) - \theta(F)]$ and $m n^{\frac{1}{2}} V_n^{(1)}$ both have the same limiting distribution, if they have one at all. Now, by (3.17) and the central limit theorem for ϕ-mixing (and hence, $*$-mixing) processes [cf. Billingsley (1968, p. 174) and Philipp (1969a)], $m n^{\frac{1}{2}} V_n^{(1)}$ converges in law (whenever $v_2 < \infty$) to a normal distribution with zero mean and variance $m^2 \sigma^2$, where σ^2 is defined by (2.3) and it is assumed that (2.4) holds. This completes the proof of (2.8). By virtue of (3.30) and the Chebychev inequality, (2.9) follows directly, and this, in turn, implies that (2.8) also holds for U_n.

Let us now consider Theorem 2. Here $v_4<\infty$ and $A_{m^*}(\psi)<\infty$ imply, by virtue of (3.23), that for every $\varepsilon>0$,

$$\begin{aligned} &P\{n^{\frac{1}{2}}|V_n^{(2)}|\geqq\varepsilon, \text{ for at least one } n\geqq n_0\}\\ &\quad\leqq\sum_{n\geqq n_0}P\{n^{\frac{1}{2}}|V_n^{(2)}|\leqq\varepsilon\}\leqq(v_4K_\psi/\varepsilon^4)\sum_{n\geqq n_0}\frac{1}{n^2}, \end{aligned}\tag{4.3}$$

which converges to 0 as $n_0\to\infty$. Also, by Lemma 3.3, $v_2<\infty$ and $A_{m^*}(\psi)<\infty$ $(\to A_{m-1}(\psi)<\infty)$ imply that

$$\begin{aligned} &P\left\{n^{\frac{1}{2}}\left|\sum_{h=3}^{m}\binom{m}{h}V_n^{(h)}\right|\geqq\varepsilon, \text{ for at least one } n\geqq n_0\right\}\\ &\quad\leqq\sum_{n\geqq n_0}P\left\{n^{\frac{1}{2}}\left|\sum_{h=3}^{m}\binom{m}{h}V_n^{(h)}\right|>\varepsilon\right\}\\ &\quad\leqq(K_\psi v_2/\varepsilon^2)\sum_{n\geqq n_0}n^{-2}\to 0 \quad \text{as } n_0\to\infty. \end{aligned}\tag{4.4}$$

Thus, for every $\varepsilon>0$

$$P\{n^{\frac{1}{2}}[\theta(F_n)-\theta(F)-m\,V_n^{(1)}]/(2\log\log n)^{\frac{1}{2}}>\varepsilon\sqrt{2} \text{ for at least one } n\geqq n_0\}\to 0 \tag{4.5}$$

as $n_0\to\infty$. Consequently, it suffices to prove (2.10) and (2.11) for $[\theta(F_m)-\theta(F)]$ being replaced by $m\,V_n^{(1)}$. Since $V_n^{(1)}$ involves an average over a stationary *-mixing (and hence, ϕ-mixing) sequence of random variables, by Theorem 1 of Reznik (1968) [see also Philipp (1969c)], under conditions even less stringent than the hypothesized ones, the law of iterated logarithm holds for $\{V_n^{(1)}\}$, i.e., (2.10) and (2.11) hold. Again by (3.30) and the Bonferroni inequality, for every $\varepsilon>0$,

$$\begin{aligned} &P\{n^{\frac{1}{2}}|U_n-\theta(F_n)|>\varepsilon \text{ for at least one } n\geqq n_0\}\\ &\quad\leqq\sum_{n\geqq n_0}P\{n^{\frac{1}{2}}|U_n-\theta(F_n)|>\varepsilon\}\\ &\quad\leqq C_\psi\varepsilon^{-2}\sum_{n\geqq n_0}n^{-2}\to 0 \quad \text{as } n_0\to\infty. \end{aligned}\tag{4.6}$$

Thus, $n^{\frac{1}{2}}|U_n-\theta(F_n)|\to 0$ a.s. as $n\to\infty$, which implies (2.12), and that in turn implies that (2.10) and (2.11) hold for $\{U_n\}$.

We now proceed to the proof of Theorem 3. Let us define on $C[0,1]$ a sequence of processes $\{Y_n^0=[Y_n^0(t), t\in 1], n\geqq 1\}$, by $Y_n^0\left(\frac{k}{n}\right)=k\,V_k^{(1)}/\sigma n^{\frac{1}{2}}$, $k=1,\ldots,n$, $Y_n^0(0)=0$, and

$$Y_n^0(t)=Y_n^0\left(\frac{[nt]}{n}\right)+([nt]-nt)\left[Y_n^0\left(\frac{[nt]+1}{n}\right)-Y_n^0\left(\frac{[nt]}{n}\right)\right],\quad 0\leqq t\leqq 1. \tag{4.7}$$

Since $\{g_1(X_1), -\infty<i<\infty\}$ is stationary *-mixing (and hence, ϕ-mixing), and by (2.4) and (2.5), $0<\sigma^2<\infty$, by Theorem 20.1 of Billingsley (1968, p. 174), it follows that under $A_0(\psi)<\infty$,

$$Y_n^0\xrightarrow{\mathscr{D}}W, \quad \text{as } n\to\infty. \tag{4.8}$$

We complete the proof of the theorem by showing that as $n\to\infty$,

$$\rho(Y_n, Y_n^0)\xrightarrow{P}0 \quad\text{and}\quad \rho(Y_n^*, Y_n^0)\xrightarrow{P}0. \tag{4.9}$$

Now, by (2.14), (2.15), (3.16) and (4.7),

$$\rho(Y_n, Y_n^0) = \max_{1\le k\le n} k|\theta(F_k)-\theta(F)-m V_k^{(1)}|/\{m n^{\frac{1}{2}} \sigma\}$$
$$< \max_{1\le k\le n} \frac{k|V_k^{(2)}|\binom{m}{2}}{m\sigma n^{\frac{1}{2}}} + \max_{1\le k\le n} \frac{k\left|\sum_{h=3}^{m}\binom{m}{h} V_k^{(h)}\right|}{m\sigma n^{\frac{1}{2}}}. \tag{4.10}$$

By Lemma 3.4 and the Bonferonni inequality, under (2.2) for $r=4$, for every $\varepsilon>0$,

$$P\left\{\max_{1\le k\le n} k\binom{m}{2}|V_k^{(2)}| > \varepsilon m\sigma n^{\frac{1}{2}}\right\} \le \sum_{k=1}^{n} P\{k|V_k^{(2)}| > 2\varepsilon\sigma n^{\frac{1}{2}}/(m-1)\}$$
$$\le K_\psi v_4 (m-1)^4/(2\varepsilon^4 n^2)\sum_{k=1}^{n} 1 \qquad (K_\psi<\infty)$$
$$= K_\psi v_4(m-1)^4/(2\varepsilon^4 n) \to 0 \quad \text{as } n\to\infty.$$

Similarly, under (2.2) for $r=2$, by Lemma 3.3 and the c_r-inequality, for every $\varepsilon>0$,

$$P\left\{\max_{1\le k\le n} k\left|\sum_{h=3}^{m}\binom{m}{h} V_k^{(h)}\right| > \varepsilon m\sigma n^{\frac{1}{2}}\right\} \le \sum_{k=1}^{n} k^2 E\left[\sum_{h=3}^{m}\binom{m}{h} V_k^{(h)}\right]^2\Big/[\varepsilon^2 m^2\sigma^2 n]$$
$$\le (C/n\varepsilon^2)\sum_{k=1}^{n} k^{-1} \tag{4.12}$$
$$\le (C\log n)/(n\varepsilon^2) \to 0 \quad \text{as } n\to\infty.$$

Thus, (4.10) converges in probability to zero as $n\to\infty$. Hence, by (4.8),

$$Y_n \xrightarrow{\mathscr{D}} W \quad \text{as } n\to\infty. \tag{4.13}$$

Since W is a standard Brownian motion and $m/n\to 0$ as $n\to\infty$,

$$\sup_{0\le t<m/n} |Y_n(t)| \xrightarrow{p} 0 \quad \text{as } n\to\infty. \tag{4.14}$$

Hence, to show that $\rho(Y_n^*, Y_n^0)\xrightarrow{p} 0$, we use the triangular inequality

$$\rho(Y_n^*, Y_n^0) \le \rho(Y_n^*, Y_n) + \rho(Y_n, Y_n^0), \tag{4.15}$$

and for the first term on the right hand side of (4.15), by virtue of (4.14), it suffices to show that

$$\max_{m\le k\le n} |k(U_k - \theta(F_k))|/[m\sigma\sqrt{n}] \xrightarrow{p} 0 \quad \text{as } n\to\infty. \tag{4.16}$$

By (3.30) and the Bonferroni inequality, for every $\varepsilon>0$,

$$P\{\max_{m\le k\le n} |k(U_k-\theta(F_k))| > \varepsilon m\sigma\sqrt{n}\} \le \sum_{k=m}^{n} P\{|k(U_k-\theta(F_k))| > \varepsilon m\sigma\sqrt{n}\}$$
$$\le \sum_{k=m}^{n} \{C_\psi/(\varepsilon^2 m^2\sigma^2 n k)\} \tag{4.17}$$
$$= (C_\psi/m^2\varepsilon^2\sigma^2)\frac{1}{n}\sum_{k=m}^{n} k^{-1}$$
$$\le (C_\psi/m^2\varepsilon^2\sigma^2)(n^{-1}\log n) \to 0 \quad \text{as } n\to\infty.$$

Thus, $\rho(Y_n^*, Y_n) \xrightarrow{p} 0$ as $n \to \infty$, and hence, by (4.15), $\rho(Y_n^*, Y_n^0) \xrightarrow{p} 0$ as $n \to \infty$, and thereby (4.8) implies that

$$Y_n^* \xrightarrow{\mathscr{D}} W \quad \text{as } n \to \infty, \tag{4.18}$$

which completes the proof. Note that in (4.17), we have made use only of (2.2) with $r=2$ and $A_{m-1}(\psi) < \infty$, which are less restrictive than the hypothesized conditions.

5. A Few Applications

For illustration, we consider the following functional. Let $X_i = (X_i^{(1)}, X_i^{(2)})$ have the d.f. $F(x, y)$, and define

$$\theta(F) = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x, \infty) - \tfrac{1}{2}] [F(\infty, y) - \tfrac{1}{2}] \, dF(x, y), \tag{5.1}$$

which is known as the grade correlation of $X^{(1)}$ and $X^{(2)}$. We have then

$$\begin{aligned} \theta(F_n) &= 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_n(x, \infty) - \tfrac{1}{2}] [F_n(\infty, y) - \tfrac{1}{2}] \, dF_n(x, y) \\ &= 12 n^{-3} \sum_{i=1}^{n} \left(R_i - \frac{n}{2}\right) \left(S_i - \frac{n}{2}\right) \\ &= (1 - n^{-2}) R_g + 3 n^{-2}, \end{aligned} \tag{5.2}$$

where R_g is the classical Spearman rank correlation i.e.,

$$R_g = [12/n(n^2-1)] \sum_{i=1}^{n} \left(R_i - \frac{n+1}{2}\right) \left(S_i - \frac{n+1}{2}\right). \tag{5.3}$$

Thus, for large n, both $\theta(F_n)$ and R_g have the same properties. Now, as in Hoeffding (1948), we have

$$R_g = n^{-[3]} \sum_{\alpha \neq \beta \neq \gamma = 1}^{n} g(X_\alpha, X_\beta, X_\gamma), \tag{5.4}$$

where

$$\begin{aligned} g(x_1, x_2, x_3) = \tfrac{1}{2} [& s(x_1^{(1)} - x_2^{(1)}) \, s(x_1^{(2)} - x_3^{(2)}) + s(x_1^{(1)} - x_3^{(1)}) \, s(x_1^{(2)} - x_2^{(2)}) \\ & + s(x_2^{(1)} - x_1^{(1)}) \, s(x_2^{(2)} - x_3^{(2)}) + s(x_2^{(1)} - x_3^{(1)}) \, s(x_2^{(2)} - x_1^{(2)}) \\ & + s(x_3^{(1)} - x_1^{(1)}) \, s(x_3^{(2)} - x_2^{(2)}) + s(x_3^{(1)} - x_2^{(1)}) \, s(x_3^{(2)} - x_1^{(2)})], \end{aligned} \tag{5.5}$$

and $s(u) = 1, 0$ or -1 according as u is $>, =$ or <0. Since $m=3$ and g is a bounded kernel, (2.5) holds for every $r \geqq 0$. Hence, under (2.4) and the stated conditions on $\{\psi_n\}$, all the three theorems of Section 2 hold. Other examples are easy to construct.

Let us now consider the case of random sample sizes. For every r, let N_r be a positive integer valued random variable such that there is a sequence $\{n_r\}$ of positive numbers for which

$$n_r \to \infty \quad \text{but} \quad N_r/n_r \xrightarrow{p} 1 \quad \text{as} \quad r \to \infty. \tag{5.6}$$

Then, using Lemmas 3.3 and 3.4, it can be shown that under (5.6), (4.2) readily extends to n being replaced by N_r where $r \to \infty$. Also, (4.6) insures that

$N_r^{\frac{1}{2}} |U_{N_r} - \theta(F_{N_r})| \to 0$ a.s. as $r \to \infty$. Further, (4.11), (4.12) and (4.17) can be easily adjusted to random sample sizes. Consequently, using Theorem 20.3 of Billingsley (1968, p. 180) for $\{V_{N_r}^{(1)}\}$, we conclude that both Theorems 1 and 3 remain valid for random sample sizes satisfying (5.6).

The theory developed here is of interest in the developing area of asymptotic sequential inference procedures based on $\{\theta(F_n)\}$ or $\{U_n\}$.

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