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Splitting Times and Shift Functionals

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1. Introduction

Let $X=(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a "nice" Markov process; for definiteness a Borel right process, see [4], or more concretely a Hunt process. A *random time* R is an $\mathscr{F}$ measurable mapping from Ω to $\bar{\mathbb{R}}^+=[0, \infty]$. Associated with R is the σ-algebra $\mathscr{F}_R$ of "events occurring before R" (see Section 2 for the precise definition) and the "post R process" $(X_{R+t}, t>0)$ defined on $\{R<\infty\}$. In recent years there has been considerable interest in finding those times R such that: (i) the post R process is conditionally independent of $\mathscr{F}_R$ given X_R and, perhaps, some additional "auxiliary" variables, and (ii) the post R process is strong Markov. An excellent discussion of these and related questions is given in the recent survey [14] by Millar. Of course, if R is a stopping time, then (i) and (ii) are immediate consequences of the strong Markov property, and the post R process even has the same transition semigroup as X. In addition to stopping times there is another class of times for which (i) and (ii) are known to hold. Let $M\subset \mathbb{R}^+\times\Omega$ be optional and homogeneous. (Definitions are given in Section 2.) Let $L(\omega)=\sup\{t\colon (t,\omega)\in M\}$ be the *end* of M. Again the post L process is defined on $\{L<\infty\}$. It is by now a standard fact (see [3, 5, 8, 13], or [16]) that L satisfies (i) and (ii) above. In this case the post L process $(X_{L+t}; t>0)$ is a temporally homogeneous Markov process but its transition semigroup is *not* the same as that of X. Moreover, it was shown in [13] that the pre-L process $\hat{X}$, $\hat{X}_t=X_t$ if $t<L$, $\hat{X}_t=\Delta$ if $t\geqq L$ is also a temporally homogeneous Markov process in this situation.

Our emphasis in this paper will be on property (i). The post and pre-R processes will only be mentioned in passing. Therefore we shall call a random time R that satisfies (i) a *splitting time* with the given auxiliary variables. Thus stopping times and ends of homogeneous, optional sets are splitting times with no auxiliary variables.

In [15], Millar observed that certain times R which are not the ends of an optional homogeneous set M for X may, nevertheless, be viewed as the ends of

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an optional homogeneous set $\tilde{M}$ relative to an auxiliary process $\tilde{X}$. Thus one may apply the above result to $\tilde{M}$ and $\tilde{X}$ and obtain information about the original R and X. The present paper treats this idea in a systematic way in a general setting. Although the proofs are very easy given the theorem about the end of a homogeneous optional set, it seems to us that the framework of Section 2 is a very natural setting for such results.

In Section 2 we introduce the notion of a *shift functional* which provides the appropriate concept for our basic result. The identity (2.1) defining a shift functional up to regularity assumptions has appeared in numerous places. We mention some of them in Section 4. A particularly simple example which enables one to reduce the properties of $L_t = \sup\{s \leqq t: s \in M\}$ when M is a homogeneous, optional set to the case of the end of such a set is given in Section 2. In Section 3 we discuss two classes of examples. The first deals with the last minimum or the last minimum before a fixed t of a homogeneous functional of X. These are slight generalizations of the particular problems treated by Milnar in [15]. The second class of examples deals with the range of the process; for example, the "last" point of increase of the diameter of $\{X_u: u \leqq s\}$ before t. Such examples seem to be new. In Section 4 we discuss some of the connections between our results and semi-direct products and multiplicative kernels as discussed by Jacod [6], [7], and Meyer [11].

The expert will wonder, and rightly so, why we have restricted ourselves to *Borel* right processes and assumed that shift functionals are jointly Borel measurable (2.4i) rather than, say, $\mathcal{O} \times \mathcal{A}$ measurable where $\mathcal{O}$ is the optional σ-algebra. Especially since this causes some awkwardness in the example of the min-functional in Section 3. The point of this simplification is that it allows us to operate at the level of Borel sets rather than having to introduce $\mathcal{B}^e$ – the σ-algebra generated by excessive functions. See the discussion of branch points in Section 2, for example. It is our hope that this technical simplification will make the paper accessible to a wider class of readers and that it will not disturb the aficionados too much.

2. Shift Functionals

Let Ω be a set and $(\theta_t)_{t \geqq 0}$ a semigroup of mappings from Ω to Ω; that is, $\theta_{t+s} = \theta_t \theta_s$ for $t, s \geqq 0$. We refer to $(\theta_t)_{t \geqq 0}$ as a *shift* on Ω. Let A be another set. Then a family $(I_t)_{t \geqq 0}$ of mappings from $\Omega \times A$ to A is called a (θ_t)-*flow* on A provided.

$$(2.1) \quad I_{t+s}(\omega, a) = I_t(\theta_s \omega, I_s(\omega, a)).$$

Here (2.1) is assumed the hold identically in t, s, ω and a. If $\tilde{\Omega} = \Omega \times A$ and we define

$$(2.2) \quad \tilde{\theta}_t \tilde{\omega} = \tilde{\theta}_t(\omega, a) = (\theta_t \omega, I_t(\omega, a)),$$

then it is immediate from (2.1) that $(\tilde{\theta}_t)_{t\geqq 0}$ is a shift on $\tilde{\Omega}$. If π is the projection of $\tilde{\Omega}=\Omega\times A$ onto Ω, $\pi(\omega,a)=\omega$, then clearly

(2.3) $\theta_t\pi=\pi\tilde{\theta}_t$ for all $t\geqq 0$.

Conversely let $(\tilde{\theta}_t)$ be a shift on $\tilde{\Omega}$ satisfying (2.3). Let $\gamma(\omega,a)=a$ be the projection of $\tilde{\Omega}$ onto A. Then $I_t(\omega,a)=\gamma\,\tilde{\theta}_t(\omega,a)$ satisfies (2.1). Thus giving a (θ_t)-flow on A is equivalent to giving a shift $(\tilde{\theta}_t)$ on $\Omega\times A$ whose projection on Ω is (θ_t).

Now let $X=(\Omega,\mathscr{F},\mathscr{F}_t,X_t,\theta_t,P^x)$ be a Borel right process [4] with state space $(E,\mathscr{E})$. Here Ω is a U-space – Ω is homeomorphic to a universally measurable subset of a compact metric space – and $\mathscr{F}^0=\sigma(X_s\colon s\geqq 0)$ is the Borel σ-algebra of Ω. (The actual topology on Ω is not important, only the Borel structure of $(\Omega,\mathscr{F}^0)$ is relevant.) There exists a distinguished point $\Delta\in E$ which acts as a cemetery and we assume without essential loss of generality that $\zeta=\inf\{t\colon X_t=\Delta\}$ is infinite almost surely. To say that X is a Borel right process means that X is a strong Markov process with right continuous trajectories, that $(E,\mathscr{E})$ is a Lusing space (E is (homeomorphic to) a Borel subset of a compact metric space and that $\mathscr{E}$ is the Borel σ-algebra of E), and that the transition semigroup (P_t) of X maps Borel functions into Borel functions.

Let $(A,\mathscr{A})$ be a Lusin space.

(2.4) *Definition. A shift functional* $(I_t)=(I_t(\omega,a))$ *on* A *is a* (θ_t)*-flow on* A *which satisfies:*

(i) $(t,\omega,a)\to I_t(\omega,a)$ *is* $\mathscr{B}(\mathbb{R}^+)\times\mathscr{F}^0\times\mathscr{A}$ *measurable.*

(ii) $\omega\to I_t(\omega,a)$ *is* $\mathscr{F}_t$ *measurable.*

(iii) $t\to I_t(\omega,a)$ *is right continuous.*

(2.5) *Remarks.* We emphasize again that we are assuming that (2.1) holds identically in the appropriate variables. It follows from (ii) and (iii) of (2.4) that $(t,\omega)\to I_t(\omega,a)$ is optional (i.e. well measurable) relative to the filtration $(\mathscr{F}_t)$.

We adopt the following notational conventions. If $H(\omega,a)$ is defined on $\tilde{\Omega}=\Omega\times A$, then H^a denotes the map $\omega\to H(\omega,a)$. If $H(\omega)$ is defined on Ω, it will sometimes be convenient to identify H with $H\circ\pi$, that is, we will also use H for the map $(\omega,a)\to H(\omega)$ defined on $\tilde{\Omega}$. Thus I_t^a is the map $\omega\to I_t(\omega,a)$. If R and T are random times, then $I_R(\theta_T,a)$ denotes the map $\omega\to I_{R(\omega)}(\theta_{T(\omega)}\,\omega,a)$ defined on the set $\{R<\infty,\ T<\infty\}$. With these examples in mind the reader should have no difficulty deciphering our formulas.

If g is a positive $\mathscr{E}\times\mathscr{A}$ measurable function on $E\times A$, we define

(2.6) $K_t(x,a;g)=E^x[g(X_t,I_t^a)]$.

It is an easy consequence of (2.4) (i) and the fact that (P_t) maps Borel functions into Borel functions that for each $t\geqq 0$, K_t is a Markov kernel on $(E\times A,\ \mathscr{E}\times\mathscr{A})$.

The following elementary result is basic.

(2.7) **Proposition.** (i) $(K_t)_{t\geqq 0}$ *is a semigroup of Markov kernels.*

(ii) *For each initial measure* μ *on* E *and* $a\in A$ *the process* $X_t^a=(X_t,I_t^a)$ *is a right continuous strong Markov process with respect to* P^μ *having* (K_t) *as transition semigroup.*

Proof. From the definition of I, $X^a=(X_t^a)$ is right continuous and $(\mathscr{F}_t)$ adapted. Let T be an $(\mathscr{F}_t^\mu)$ stopping time. We shall need the following form of the strong Markov property for X (see I-(8.16) of [1]). Let $G(\omega, w)$ be a positive $\mathscr{F}^\mu \times \mathscr{F}_T^\mu$ measurable function on $\Omega\times\Omega$ and let $H=G(\theta_T, \cdot)$ – that is, $H(\omega)=G(\theta_{T(\omega)}\,\omega, \omega)$ – on $\{T<\infty\}$. Then on $\{T<\infty\}$,

$$(2.8)\quad E^\mu[H\,|\,\mathscr{F}_T^\mu]=\int G(\cdot, w)\,P^{X(T)}(dw).$$

Now let g be a positive $\mathscr{E}\times\mathscr{A}$ measurable function on $E\times A$. Then for each $t\geqq 0$ one has, making use of (2.8), on $\{T<\infty\}$

$$\begin{aligned}(2.9)\quad E^\mu\{g(X_{t+T}, I_{t+T}^a)\,|\,\mathscr{F}_T^\mu\} &= E^\mu\{g[X_t\circ\theta_T, I_t(\theta_T, I_T^a)]\,|\,\mathscr{F}_T^\mu\}\\ &=\int g[X_t(w), I_t(w, I_T^a)]\,P^{X(T)}(dw)=K_t(X_T, I_T^a; g).\end{aligned}$$

If $T\equiv s$ and $\mu=\varepsilon_x$, then taking expectations with respect to P^x in (2.9) one obtains $K_{t+s}\,g=K_s K_t\, g$ proving (i). But (ii) is now immediate from (2.9).

We now introduce a canonical representation of the family of processes (X^a). Let $\tilde\Omega=\Omega\times A$. On $\tilde\Omega$ define $\tilde X_t(\tilde\omega)=\tilde X_t(\omega, a)=(X_t(\omega), I_t(\omega, a))$ and $\tilde\theta_t\tilde\omega=\tilde\theta_t(\omega, a)=(\theta_t\,\omega, I_t(\omega, a))$. Then $\tilde X_t\circ\tilde\theta_s=\tilde X_{t+s}$. Next define $\tilde P^{x,a}$ on $\mathscr{F}\times\mathscr{A}$ by $\tilde P^{x,a}=P^x\times\varepsilon_a$ where ε_a is unit mass at a. Now consider the process $\tilde X=(\tilde\Omega, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, \tilde X_t, \tilde\theta_t, \tilde P^{x,a})$ with state space $\tilde E=E\times A$. Then $\tilde X$ is a right continuous strong Markov process with transition semigroup (K_t) because $(\tilde X_t)$ under $\tilde P^{x,a}$ is equivalent to (X_t^a) under P^x. Here $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{F}}_t$ are the usual σ-algebras associated with the Markov process $\tilde X$. See [1].

The process $\tilde X$ has all the properties of a Borel right process except that it may have *branch points* because $\tilde X_0=(x, I_0^a)$ almost surely $\tilde P^{x,a}$ and we are *not* assuming that $I_0(\omega, a)=a$. However, these branch points are of a very simple nature; they are *degenerate* branch points in the usual terminology. To see this note that since I_0^a is $\mathscr{F}_0$ measurable the zero-one law for X implies that I_0^a is constant almost surely P^x. Let $b(x,a)$ be the point in A such that $I_0^a=b(x,a)$ almost surely P^x. It is immediate from (2.4)(i) that $(x,a)\to b(x,a)$ is $\mathscr{E}\times\mathscr{A}\,|\,\mathscr{A}$ measurable. Now the set of branch points $\tilde B=\{(x,a)\colon\ b(x,a)\neq a\}$ and the nonbranch points $\tilde D=\{(x,a)\colon\ b(x,a)=a\}$ are Borel subsets of $\tilde E=E\times A$. Clearly from (2.6)

$$\begin{aligned}\varepsilon_{x,b(x,a)}=K_0(x,a;\cdot)&=\int K_0(x,a; d(y,b))\,K_0(y,b;\cdot)\\ &=K_0(x, b(x,a);\cdot),\end{aligned}$$

and so $b(x, b(x,a))=b(x,a)$. It now follows that

$$(2.10)\quad \tilde P^{x,a}=\tilde P^{x,b(x,a)}=P^x\times\varepsilon_{b(x,a)}$$

on $\tilde{\mathscr{F}}^0\subset\mathscr{F}^0\times\mathscr{A}$. If ν is a probability measure on $E\times A$, then, as is customary in Markov processes, one defines

$$\tilde P^\nu(H)=\int \tilde P^{x,a}(H)\,\nu(dx, da)$$

on $\tilde{\mathscr{F}}^0$. If $\tilde{b}$: $(x,a)\to(x,b(x,a))$, then (2.10) extends to $\tilde{P}^\nu=\tilde{P}^\mu$ where $\mu=\tilde{b}(\nu)$. Hence $\tilde{\mathscr{F}}^\nu=\tilde{\mathscr{F}}^\mu$ and $\tilde{\mathscr{F}}_t^\nu=\tilde{\mathscr{F}}_t^\mu$. Finally it is a standard fact that

$$\tilde{P}^{x,a}[\tilde{X}_t\in\tilde{B} \text{ for some } t\geqq 0]=0$$

for all x, a. That is, almost surely the process $\tilde{X}$ never is in the set of branch points. Hence $\tilde{X}$ restricted to set of nonbranch points $\tilde{D}$ is a Borel right process and that if (x,a) is a branch point, then under $\tilde{P}^{x,a}$ the process starts from $(x, b(x,a))\in\tilde{D}$. Therefore the possible presence of branch points causes no difficulty and we may treat $\tilde{X}$ as a Borel right process and apply the theorems about Borel right process to $\tilde{X}$ provided we exercise a modicum of caution. Having said all this we shall ignore the fact that strictly speaking $\tilde{X}$ is not a Borel right process in the sequel.

For our applications we shall need some trivial, but important, measurability facts. Recall that π: $\Omega\times A\to\Omega$ is the projection of $\Omega\times A$ on Ω, and let p: $E\times A\to E$ be the projection of $E\times A$ on E. It is immediate from the definitions of $\tilde{P}^{x,a}$ and $\tilde{P}^\nu$ that $P^{p(\nu)}=\pi(\tilde{P}^\nu)$ for all ν on $E\times A$. This is just a fancy way of saying that $P^{p(\nu)}(\Lambda)=P^\nu(\Lambda\times A)$ for all $\Lambda\in\mathscr{F}$. Recall that a process $Z=(Z_t(\omega))$ is said to be optional (previsible) over $(\mathscr{F}_t)$ if for every initial measure μ on E there exists a process $Z^\mu=(Z_t^\mu(\omega))$ which is optional (previsible) over $(\Omega, \mathscr{F}_t^\mu, P^\mu)$ in the usual sense of general theory [2] such that Z and Z^μ are P^μ indistinguishable. Optionality (previsibility) of a process $\tilde{Z}_t=(\tilde{Z}_t(\omega,a))$ over $(\tilde{\mathscr{F}}_t)$ is defined similarly.

The following lemma lists some trivial, but important, facts that we shall need. Its proof is routine and omitted.

(2.11) **Lemma.** *Let $\tilde{\mathscr{G}}^0=\mathscr{F}^0\times\{\emptyset,A\}$ and $\tilde{\mathscr{G}}_t^0=\mathscr{F}_t^0\times\{\emptyset,A\}$. Then $\tilde{\mathscr{G}}^0\subset\tilde{\mathscr{F}}^0$ and $\tilde{\mathscr{G}}_t^0\subset\tilde{\mathscr{F}}_t^0$. If ν is an initial measure on $E\times A$, let $\tilde{\mathscr{G}}^\nu$ (resp. $\tilde{\mathscr{G}}_t^\nu$) be the σ-algebra on $\tilde{\Omega}=\Omega\times A$ generated by $\tilde{\mathscr{G}}^0$ (resp. $\tilde{\mathscr{G}}_t^0$) and all $\tilde{P}^\nu$ null subsets of $\tilde{\mathscr{F}}^\nu$. Let $\mu=p(\nu)$. Then*

(i) $\mathscr{F}_t^\mu\times\{\emptyset,A\}\subset\tilde{\mathscr{G}}_t^\nu\subset\tilde{\mathscr{F}}_t^\nu$.

(ii) *If $Z=(Z_t)$ is optional over $(\mathscr{F}_t)$, then $Z\circ\pi=(Z_t\circ\pi)$ is optional over $(\tilde{\mathscr{F}}_t)$.*

(iii) *If $\tilde{Z}$ is optional over $(\tilde{\mathscr{F}}_t)$, then $\tilde{Z}^a=(\tilde{Z}_t^a)$ is optional over $(\mathscr{F}_t)$ for each $a\in A$. If $\tilde{H}\in\tilde{\mathscr{F}}$, then $\tilde{H}^a\in\mathscr{F}$ for each $a\in A$.*

Remarks. In fact, if Z is optional over $(\mathscr{F}_t)$, then $Z\circ\pi$ is optional over $(\tilde{\mathscr{G}}_t)$. The process $\tilde{X}$ is a semi-direct product of X in the terminology of Jacod [6].

We are now going to state the basic result on exit systems in form that is convenient for us. Its proof is an immediate corollary of (4.1) in [8] and is omitted. In [8], Maisonneuve assumes the existence of killing operators (k_t), but the following portion of his results is valid without this assumption. For notational simplicity we state the result for the process X, but, of course, the analogous statement is true for any Borel right process. In particular, we shall use it for the process $\tilde{X}$. Let $\mathbb{R}^+=[0,\infty)$. A set $M\subset\mathbb{R}^+\times\Omega$ is *closed* if for each ω the ω-section, $M(\omega)$, of M is closed in $\mathbb{R}^+$. M is *homogeneous* (on $(0,\infty)$) if $t+s\in M(\omega)\Leftrightarrow s\in M(\theta_t\omega)$ for $s>0$, $t\geqq 0$, and ω. It is *important* to note that we do not require this for $s=0$. We say that M is optional provided it is optional over $(\mathscr{F}_t)$. If R: $\Omega\to[0,\infty]$ is $\mathscr{F}$ measurable, then $\mathscr{F}_R$ is defined by saying that an $\mathscr{F}$

measurable H is $\mathscr{F}_R$ measurable provided there exist an optional (over $(\mathscr{F}_t)$) process Z with $H=Z_R$ on $\{R<\infty\}$. For typographical convenience we shall sometimes write $\mathscr{F}(R)$ in place of $\mathscr{F}_R$, and similarly for processes.

Let M be a closed, homogeneous, optional set. Define $R(\omega)=\inf\{t>0: t\in M(\omega)\}$ and $L(\omega)=\sup\{t: t\in M(\omega)\}$. By convention $\inf\emptyset=\infty$ and $\sup\emptyset=0$. Thus $L(\omega)=\sup\{t>0: t\in M(\omega)\}$ also. Since M is homogeneous it is easy to see that R is a terminal time and L is co-optional. We call R the *debut* of M and L the *end* of M. We shall suppose that R is $\mathscr{F}^*$ measurable where $\mathscr{F}^*$ is the σ-algebra of universally measurable sets over $(\Omega, \mathscr{F}^0)$. Meyer [10] has shown that this is not a real restriction. We now state the facts that we need as a theorem. As mentioned at the beginning of the previous paragraph its proof is omitted.

(2.12) **Theorem.** *Let M be a closed, homogeneous, optional set and let L be the end of M. Then there exists a Markov kernel Γ from $(E,\mathscr{E}^*)$ to $(\Omega,\mathscr{F}^*)$ such that*

(i) $E^\mu\{F\circ\theta_L|\mathscr{F}_L\}=\Gamma(X_L,F)$ *on* $\{L<\infty\}$

for all initial measures μ and $F\in b\mathscr{F}^*$. Therefore the post L process $(X_{L+t}, t>0)$ defined on $\{L<\infty\}$ is conditionally independent of $\mathscr{F}_L$ given X_L. Moreover, it is a strong Markov process relative to the fields $(\mathscr{F}_{L+t})$ with transition semigroup (Q_t^L) given by $Q_t^L f(x)=Q_t(f\psi)(x)/\psi(x)$ if $\psi(x)>0$ and $Q_t^L f(x)=0$ if $\psi(x)=0$ where $\psi(x)=P^x(R=\infty)=P^x(L=0)$, and where (Q_t) is the semigroup of X killed at R; that is, $Q_t f(x)=E^x\{f(X_t); t<R\}$. Also the pre-L process $\hat{X}$, $\hat{X}_t=X_t$ if $t<L$ and $X_t=\Delta$ if $t\geqq L$, is strong Markov with transition semigroup $(\hat{Q}_t)$ given by $\hat{Q}_t f(x)=P_t(fc)(x)/c(x)$ if $c(x)>0$ and $\hat{Q}_t f(x)=0$ if $c(x)=0$ where $c(x)=P^x(L>0)=P^x(R<\infty)$.

Remark. If x is not regular for R, that is, if $P^x(R=0)=0$, then $\Gamma(x,F)=E^x(F; R=\infty)/\psi(x)$. This is why (2.12)(i) holds on $\{L=0\}$. The assertion about the pre-L process is proved in [13].

Our main purpose is to apply this theorem to sets $\tilde{M}$ which are homogeneous for $(\tilde{\theta}_t)$. Let us establish the necessarily somewhat cumbersome notation. If $\tilde{H}$ is a function defined on $\tilde{\Omega}=\Omega\times A$, then we shall write H^a for $\tilde{H}^a$ for typographical simplicity. Let $\tilde{M}\subset\mathbb{R}^+\times\tilde{\Omega}=\mathbb{R}^+\times\Omega\times A$ be closed, homogeneous with respect to $(\tilde{\theta}_t)$, and optional over $(\tilde{\mathscr{F}}_t)$. Let $\tilde{R}$ and $\tilde{L}$ be the debut and end of $\tilde{M}$. It follows from I-T32 of [2], that $\tilde{L}$ is $\tilde{\mathscr{F}}$ measurable, and consequently by (2.11)(iii), L^a is $\mathscr{F}$ measurable for each $a\in A$. It is easy to see using (2.11)(ii) that if $H\in\mathscr{F}(L^a)$ for a fixed a, then there exists $\tilde{H}\in\tilde{\mathscr{F}}(\tilde{L})$ such that $H^a=H$. Let $\tilde{\Gamma}(x,a;\cdot)$ be the Markov kernel from $(E\times A, (\mathscr{E}\times\mathscr{A})^*)$ to $(\tilde{\Omega}, \tilde{\mathscr{F}}^*)$ appearing in (2.12)(i), and let $\Gamma(x,a;\cdot)=\pi[\tilde{\Gamma}(x,a;\cdot)]$. Thus if $\Lambda\in\mathscr{F}^*$, $\Gamma(x,a;\Lambda)=\Gamma^*(x,a;\Lambda\times A)$, and note that for each $a\in A$, $x\to\Gamma(x,a;\Lambda)$ is $\mathscr{E}^*$ measurable. Now if $F\in b\mathscr{F}^*$, then from (2.12)(i) we obtain for each fixed $a\in A$ and μ on E (let $\tilde{F}=F\circ\pi$, $\nu=\mu\times\varepsilon_a$)

$$
\begin{aligned}
(2.13)\quad E^\mu\{F\circ\theta_{L^a}|\mathscr{F}(L^a)\}&=\tilde{E}^\nu\{\tilde{F}\circ\tilde{\theta}_{\tilde{L}}|\tilde{\mathscr{F}}(\tilde{L})\}\\
&=\Gamma(X(L^a), I^a(L^a); F)\quad \text{on}\quad \{L^a<\infty\}.
\end{aligned}
$$

where we have written $I^a(t)$ for I_t^a for typographical reasons. It follows from (2.13) that L^a is a splitting time with auxiliary variable $I^a(L^a)$. Also the second part of (2.12) implies that

$$E^\mu[f\circ X(L^a+s+t)\,|\,\mathscr{F}(L^a+s)]=\Gamma_s(X(L^a+t), I^a(L^a+t); f) \tag{2.14}$$

on $\{L^a<\infty\}$, where

$$\Gamma_s(x,a;f)=\frac{\tilde{E}^{x,a}\{f(X_s)\,\varphi(X_s,I_s): s<\tilde{R}\}}{\varphi(x,a)} \tag{2.15}$$

with $\varphi(x,a)=\tilde{P}^{x,a}(\tilde{R}=\infty)$. In general $(X(L^a+t); t>0)$ is *not* a temporally homogeneous strong Markov process on $\{L^a<\infty\}$ relative to P^μ, but of course, $(X(L^a+t), I^a(L^a+t))$ is.

Let us give a very simple example in order to illustrate the ideas before passing to more interesting examples in the next section. Let $A=\mathbb{R}$ and $I_t(\omega,a)=a-t$. Then $\tilde{X}_t(\omega,a)=(X_t(\omega),a-t)$ is just the familiar space-time process. Of course, $\tilde{\theta}_t(\omega,a)=(\theta_t\,\omega,a-t)$. Now suppose that M is a closed, homogeneous, optional subset of $\mathbb{R}^+\times\Omega$ for the *original* process X. Define $\tilde{M}$ by $\tilde{M}(\omega,a)=M(\omega)\cap[0,a]$. Thus $t\in\tilde{M}(\omega,a)$ if and only if $t\in M(\omega)$ and $t\leqq a$. It is evident that $\tilde{M}$ is $(\tilde{\mathscr{F}}_t)$ optional and $(\tilde{\theta}_t)$ homogeneous. Moreover for each $a>0$

$$L^a(\omega)=\sup\{t\leqq a: t\in M(\omega)\}, \tag{2.16}$$

and applying (2.13) we obtain

$$E^\mu[F\circ\theta_{L^a}|\mathscr{F}(L^a)]=\Gamma(X(L^a), a-L^a; F) \tag{2.17}$$

which is essentially the result given in [5]. In particular, L^a is a splitting time with auxiliary variable L^a.

Given a general shift functional $I_t(\omega,a)$ on A, we may apply (2.17) to a $(\tilde{\theta}_t)$ homogeneous set $\tilde{M}$. If $t>0$, let

$$L_t(\omega,a)=\sup\{s\leqq t: s\in\tilde{M}(\omega,a)\}. \tag{2.18}$$

Then combining (2.17) and (2.13), there exists a kernel $\Gamma(x,s,a:\cdot)$ from $(\Omega,\mathscr{F}^*)$ to $(E\times\mathbb{R}\times A, (\mathscr{E}\times\mathscr{B}(\mathbb{R})\times\mathscr{A})^*)$ such that

$$E^\mu[F\circ\theta_{L^a_t}|\mathscr{F}(L^a_t)]=\Gamma(X(L^a_t), t-L^a_t, I^a(L^a_t); F). \tag{2.19}$$

Therefore L^a_t is a splitting time with auxiliary variables $(L^a_t, I^a(L^a_t))$.

One can also write down the analog of (2.14) and (2.15), but we shall not bother. The reader should just keep in mind that the process

$$Y_s=[X(L^a_t+s), t-L^a_t-s, I^a(L^a_t+s)], \qquad s>0$$

is strong Markov and that its transition semigroup may be obtained by deciphering (2.14) and (2.15) in the present case, or, equivalently, from (2.19). There are analogous results about the pre-L^a or L^a_t processes which may be obtained from (2.12). For example, $\tilde{Y}_s=(X(s), t-s, I^a(s))$ for $s<L^a_t$ and $\tilde{Y}_s=\Delta$ for $s\geqq L^a_t$ is a temporally homogeneous strong Markov process whose transition semigroup is obtained from the last sentence in (2.12).

In many examples L^a will be infinite with probability one in which case (2.13) and (2.14) give no information. But $L_t^a \leqq t$, and so (2.19) always contains useful information.

3. Examples

In this section we shall give some examples of shift functionals and some corresponding splitting times. We begin by describing a general class of shift functionals. All of the examples of this section are of this type.

Let X be a Borel right process with state space $(E, \mathscr{E})$ as in Section 2 and let A be a Lusin space with Borel σ-algebra $\mathscr{A}$. Let " $*$ " be an associative operation on A such that $a*b$ is separately continuous in a and b. Then $(a,b) \to a*b$ is jointly Borel measurable. Let $J=(J_t(\omega))$ be a map from $\mathbb{R}^+ \times \Omega \to A$ such that

(3.1) (i) J_t is right continuous, $(\mathscr{F}_t)$ adapted, and $(t,\omega) \to J_t(\omega)$ is $(\mathscr{B}(\mathbb{R}^+) \times \mathscr{F}^0)|\mathscr{A}$ measurable.
(ii) $J_{t+s}(\omega) = J_s(\theta_t \omega) * J_t(\omega)$ identically.

Let us call such a functional J a $*$-*functional*. Now define

(3.2) $I_t(\omega, a) = J_t(\omega) * a$.

It is immediate that I is a shift functional as defined in (2.4). Thus to each $*$-functional, (3.2) associates a shift functional

If $A=\mathbb{R}$ and $*$ is addition (resp. multiplication), then a $*$-functional is a perfect additive (resp. multiplicative) functional in the usual sense – perfect since we are assuming (3.1) (ii) holds identically. Also (3.1) (i) is stronger than the usual regularity assumptions on such functionals, but that does not concern us here. We shall say no more about this class of examples except to point out that if $*$ is addition and $J_t(\omega) = -t$ we obtain the example which led to (2.16) and (2.17).

Suppose that $A=\mathbb{R}$ and $a*b=a \wedge b$. Then we obtain the examples discussed by Millar in [15]. Let us sketch his results. Let $H=(H_t)$ be a real valued, optional, homogeneous (that is, $H_s \circ \theta_t = H_{s+t}$ for $s>0$ nad $t \geqq 0$) process which is $\mathscr{B}(\mathbb{R}^+) \times \mathscr{F}^0$ measurable. For example, $H_t = f(X_t)$, or, if left limits exist, $H_t = g(X_{t-}, X_t)$, where f (resp. g) is a continuous function on E (resp. $E \times E$). (The expert will realize that by passing to the Ray-Knight compactification the assumption of left limits here is a mere convenience.) For $t>0$ define $J_t^0 = \inf_{0<s<t} H_s$ and suppose that $J_t^0 > -\infty$ for all $t>0$. Clearly J^0 is left continuous and decreasing. We *assume* that J^0 is $\mathscr{B}(\mathbb{R}^+) \times \mathscr{F}^0$ measurable and $(\mathscr{F}_t)$ adapted – this is certainly the case in the two particular examples mentioned above. Finally for $t \geqq 0$, define

$$J_t = \lim_{s \downarrow \downarrow t} J_s^0 = \sup\{J_q^0;\, q>t,\, q \text{ rational}\}.$$

Then J_t is right continuous, decreasing, adapted to $(\mathscr{F}_t)$, and measurable relative to $\mathscr{B}(\mathbb{R}^+)\times\mathscr{F}^0$. If $t\geqq 0$, $s\geqq 0$, then using the homogeneity of H

$$\begin{aligned} J_{t+s}(\omega) &= \lim_{q\downarrow\downarrow 0} J^0_{t+s+q}(\omega) \\ &= \lim_{q\downarrow\downarrow 0} [(\inf_{0<r<t+q} H_r(\omega))\wedge(\inf_{t<r<t+s+q} H_r(\omega))] \\ &= \lim_{q\downarrow\downarrow 0} J^0_{t+q}(\omega)\wedge J^0_{s+q}(\theta_t\,\omega) = J_t(\omega)\wedge J_s(\theta_t\,\omega). \end{aligned}$$

Therefore J is a min-functional, and, hence, $I_t(\omega,a)=J_t(\omega)\wedge a$ is a shift functional.

We shall give several examples of splitting times associated with J. Suppose first of all that $t\to H_t(\omega)$ has right limits, and let $H_t^+=H_{t+}$. This assumption is certainly valid in the two specific examples mentioned in the fourth sentence of the previous paragraph. Then H_t^+ is a right continuous, homogeneous, $(\mathscr{F}_t)$-optional process. Let

(3.3) $$\tilde{M}^0(\omega,a)=\{t: H_t(\omega)=J_t(\omega)\wedge a \quad \text{or} \quad H_t^+=J_t(\omega)\wedge a\},$$

and let $\tilde{M}$ be the closure of $\tilde{M}^0$. Clearly $\tilde{M}^0$ is $(\tilde{\mathscr{F}}_t)$ optional and $(\tilde{\theta}_t)$ homogeneous. It is easy to see that the closure of a homogeneous set is homogenous, and it is well known [2] that the closure of an optional set is optional. Thus we can apply (2.13) to $\tilde{M}$. Let

(3.4) $$\begin{aligned} &L=\sup\{t: H_t=J_t \quad \text{or} \quad H_t^+=J_t\} \\ &L^a=\sup\{t: t\in\tilde{M}(\omega,a)\}=\sup\{t: t\in\tilde{M}^0(\omega,a)\}. \end{aligned}$$

If $r>t>0$, then $J_r^0\leqq H_t$ and so $J_t\leqq H_t$ and $J_t\leqq H_t^+$. Consequently $L^a\leqq L$ and $L^a=L$ for a sufficiently large, for example, $a\geqq J_0$. If $F\in b\mathscr{F}^*$ and Z is a bounded $(\mathscr{F}_t)$ optional process, then (2.13) gives

$$E^\mu[F\circ\theta_{L^a}Z_{L^a};L^a<\infty]=E^\mu[\Gamma(X_{L^a},J(L^a)\wedge a;F)Z_{L^a};L^a<\infty],$$

and letting $a\uparrow\infty$ we obtain on $\{L<\infty\}$

(3.5) $$E^\mu[F\circ\theta_L|\mathscr{F}_L]=\Gamma(X_L,J_L,F).$$

Let $J=\inf_{s>0} H_s$ be the ultimate infimum of H. Clearly $J_L\geqq J$. We claim that $J_L=J$ on $\{L<\infty\}$. To see this suppose $L<\infty$. Then $J_L>-\infty$. Let $u=\sup\{t: J_t>J_L-\varepsilon\}$ where $\varepsilon>0$ and suppose u is finite. Then for every n there exists $s_n\in\left[u,u+\frac{1}{n}\right)$ with $H_{s_n}\leqq J_u$. But $J_{u+1/n}\leqq J_{s_n}\leqq J_u$, and letting $n\to\infty$, the right continuity of J implies that either $H_u=J_u$ or $H_u^+=J_u$. Since $u>L$ this contradicts the definition of L. Combining this with (3.5) we see that L is a splitting time with auxiliary variable J. We may also apply (2.19). For that define

(3.6) $$\begin{aligned} &L_t=\sup\{s\leqq t: H_s=J_s \quad \text{or} \quad H_s^+=J_s\} \\ &L_t^a=\sup\{s\leqq t: s\in\tilde{M}(\omega,a)\}, \end{aligned}$$

and observe that $L_t^a = \sup\{s \leqq t; s \in \tilde{M}^0(\omega, a)\}$ since $\tilde{M}^0$ is closed from the right. Hence $L_t^a = L_t$ for large a, and so applying (2.19) and letting $a \uparrow \infty$ we find

$$E^\mu[F \circ \theta_{L_t} | \mathscr{F}(L_t)] = \Gamma(X_{L_t}, t - L_t, J_{L_t}; F).$$

Arguing as before one sees that $J_{L_t} = J_t$, and so L_t is a splitting time with auxiliary variables (L_t, J_t).

If we drop the assumption that H has right limits and define $L = \sup\{t: H_t = J_t\}$, then the same argument shows that L is a splitting time with auxiliary variable J_L. In this situation J_L may be strictly larger than $J = \inf_{s>0} H_s$. Similarly $L_t = \sup\{s \leqq t: H_s = J_s\}$ is a splitting time with auxiliary variables (L_t, J_{L_t}).

We turn now to another class of examples. For these we need some facts about the space of compact subsets of a locally compact space with a countable base (LCCB). Let E be an LCCB space and let d be a metric on E compatible with the given topology. Let $\mathscr{K}_0$ be the class of all *non-empty* compact subsets of E and define for $K, L \in \mathscr{K}_0$

$$\text{(3.7)} \quad \rho(K, L) = \max[\sup_{x \in L} d(x, K), \sup_{x \in K} d(x, L)],$$

where $d(x, B) = \inf_{y \in B} d(x, y)$ is the *distance* from x to $B \subset E$. Then ρ is a metric on $\mathscr{K}_0$ and the corresponding topology is called the *Hausdorff topology* on $\mathscr{K}_0$. Let $\mathscr{K}$ be all compact subsets of E. Then $\mathscr{K} = \mathscr{K}_0 \cup \{\emptyset\}$. Define a topology on $\mathscr{K}$ by saying that a set $\mathscr{L} \subset \mathscr{K}$ is open if $\mathscr{L} \cap \mathscr{K}_0$ is open in $\mathscr{K}_0$. Hence $\emptyset$ is an isolated point in $\mathscr{K}$ and the topology on $\mathscr{K} - \{\emptyset\} = \mathscr{K}_0$ is just the Hausdorff topology. Following [9], we shall call this the *myope* topology of $\mathscr{K}$. In the myope topology $\mathscr{K}$ is a LCCB space. This fact and all other facts cited below may be found in [9]. If (K_n) is a sequence in $\mathscr{K}$ and $K_n \downarrow K$, then $K_n \to K$ in $\mathscr{K}$; if $K_n \uparrow A$ then $K_n \to \bar{A}$ in $\mathscr{K}$ if and only if $\bar{A}$ is compact. The map $x \to \{x\}$ is an isometry from (E, d) into $(\mathscr{K}_0, \rho)$. The map $(K, L) \to K \cup L$ from $\mathscr{K} \times \mathscr{K}$ to $\mathscr{K}$ is continuous. If $E = \mathbb{R}^d$ and $c(K)$ denotes the closed convex hull of K, then the map $K \to c(K)$ from $\mathscr{K}$ to $\mathscr{K}$ is continuous.

We now fix a Borel right process X with state space $(E, \mathscr{E})$ which is a LCCB space and we assume that $t \to X_t$ has left limits in E on $(0, \infty)$ for all ω. This implies that (the bar "$-$" denotes closure in E)

$$\text{(3.8)} \quad J_t^0(\omega) = \{X_s(\omega): s < t\}^-$$

is compact for each $t \in \mathbb{R}^+$ and ω. Hence $J_t^0: \Omega \to \mathscr{K}$. From the properties of the maps $x \to \{x\}$ and $(K, L) \to K \cup L$ cited above, it is clear that if F is a *finite* subset of $[0, t)$ then the map $\omega \to \{X_s(\omega): s \in F\}$ from Ω to $\mathscr{K}$ is $\mathscr{F}_t^0 | \mathscr{B}(\mathscr{K})$ measurable where $\mathscr{B}(\mathscr{K})$ is σ-algebra of Borel sets in $\mathscr{K}$. If $\{F_n\}$ is a sequence of finite subsets of $[0, t)$ which increase to a dense subset of $[0, t)$, then

$$(\bigcup_n \{X_s: s \in F_n\})^- = J_t^0,$$

and hence $\{X_s: s\in F_n\} \to J_t^0$ in $\mathscr{K}$ as $n\to\infty$, Therefore J_t^0 is $\mathscr{F}_t^{\,0}|\mathscr{B}(\mathscr{K})$ measurable for each $t\geqq 0$. Clearly $t\to J_t^0$ is increasing and so defining

(3.9) $\quad J_t(\omega)=\bigcap_{s>t} J_s^0(\omega),$

it follows that $t\to J_t(\omega)$ is right continuous and adapted to $(\mathscr{F}_{t+}^0)$. In particular, $(t,\omega)\to J_t(\omega)$ is $(\mathscr{B}(\mathbb{R}^+)\times\mathscr{F}^0)|\mathscr{B}(\mathscr{K})$ measurable. It is immediate that if $r>0$, then $J_{t+s+r}^0(\omega)=J_{t+r}^0(\omega)\cup J_{s+r}^0(\theta_t\omega)$, and so letting $r\downarrow 0$ we obtain

(3.10) $\quad J_{t+s}=J_t\cup J_s\circ\theta_t \qquad \text{for } s,t\geqq 0.$

Consequently (J_t) is a "union" functional with values in $\mathscr{K}$.

We shall now give some examples of splitting times based on this functional. Let $\varphi:\mathscr{K}\to\mathbb{R}$ be increasing $(K\subset L\Rightarrow\varphi(\mathscr{K})\leqq\varphi(L))$, and descending $(K_n\downarrow K\Rightarrow\varphi(K_n)\downarrow\varphi(K))$. For example, $\varphi(K)$ might be the diameter of K, or any Choquet capacity relative to the paving $\mathscr{K}$. Then according to (1.4.2) of [9], the map $K\to\varphi(K)$ is usc (upper-semicontinuous). Consequently $\varphi(J_t)$ is right continuous, increasing, and adapted to $(\mathscr{F}_{t+}^0)$. In our present notation $\tilde\Omega=\Omega\times\mathscr{K}$ and $\tilde\theta_t(\omega,K)=(\theta_t\omega, J_t(\omega)\cup K)$. Let

(3.11) $\quad \tilde m_t(\omega,K)=\varphi(J_t\cup K).$

Then $\tilde m_t$ is increasing, right continuous, $(\tilde{\mathscr{F}}_t)$ adapted, and $(\tilde\theta_t)$ homogeneous. Let $\tilde M$ be the support of the measure $d\tilde m_t$. Then the results of Section 2 may be applied. If $\tilde L=\sup\{t\in\tilde M\}$, then $L^K=\tilde L(\cdot,K)$ is a splitting time with auxiliary variable $J(L^K)\cup K$. If $K=\emptyset$, we see that $L=L^\emptyset$, the "last" point of increase of $\varphi(J_t)$, is a splitting time with auxiliary variable J_L. Of course, J_L is essentially the range of the process on $[0,L]$. We may also apply (2.19). For example, if $L_t=\sup\{s\leqq t: s\in\tilde M(\cdot,\emptyset)\}$ is the "last" point of increase of $\varphi(J_s)$ before t, then L_t is a splitting time with auxiliary variables (L_t, J_{L_t}).

If $E=\mathbb{R}^d$, then $\varphi(K)=\psi(c(K))$ where c is the convex hull of K is increasing and descending whenever ψ is. For example, if ψ is Lebesgue measure in $\mathbb{R}^d$, then L_t, the "last" time the volume of $c(J_s)$ increases before t is a splitting time with auxiliary variables (L_t, J_{L_t}).

4. Concluding Remarks

Let X be a Borel right process with state space $(E,\mathscr{E})$ and let $(A,\mathscr{A})$ be a Lusin space. A *raw multiplicative kernel* (abbreviated RMK) on A is a family $q=(q_{t,\omega};\, t\in\mathbb{R}^+, \omega\in\Omega)$ of sub-Markov kernels on $(A,\mathscr{A})$ satisfying

(4.1) $\quad q_{t+s,\omega}(a,B)=\int q_{t,\omega}(a,db)\, q_{s,\theta_t\omega}(b,B)$

identically in $t,s,\omega,a\in A$, and $B\in\mathscr{A}$. A *multiplicative kernel* (MK) is a RMK subject to appropriate measurability and regularity conditions. See [6] and [11]. In [6] and [11], (4.1) is only assumed to hold almost surely where the exceptional set may depend on s,t,a and B, and a good portion of these papers is devoted to showing that (4.1) may be assumed to hold in a stronger sense under various supplementary hypotheses. Since we want to emphasize shift properties and point out the relationship with (θ_t)-flows, the present definition is convenient for us.

The relation (4.1) may be written, using the usual composition of kernels, in the form

$$(4.2)\quad q_{t+s,\omega}=q_{t,\omega}\,q_{s,\theta_t\omega}.$$

Let Λ be the collection of all positive measures λ on $(A,\mathscr{A})$ with $\lambda(A)\leqq 1$. If d is a totally bounded metric on A compatible with its topology, then the topology on Λ generated by the functions $\lambda\to\lambda(f)$ as f ranges over all bounded d-uniformly continuous functions on A is a Lusin topology on Λ. Let $\mathscr{B}(\Lambda)$ be the Borel σ-algebra in Λ. It is known that $\mathscr{B}(\Lambda)$ does not depend on the choice of d, and is, in fact, the σ-algebra generated by the maps $\lambda\to\lambda(B)$, $B\in\mathscr{A}$. Define for $\lambda\in\Lambda$

$$(4.3)\quad I_t(\omega,\lambda)=\lambda\, q_{t,\omega}.$$

Obviously I is a (θ_t)-flow on Λ for which $\lambda\to I_t(\omega,\lambda)$ is $\mathscr{B}(\Lambda)$ measurable and satisfies

$$(4.4)\quad I_t(\omega,\lambda)=\int I_t(\omega,\varepsilon_a)\,\lambda(da).$$

Observe also that I is right continuous if and only if $t\to q_{t,\omega}(a,\cdot)$ is right continuous. Conversely if $I_t(\omega,\lambda)$ is a (θ_t)-flow on Λ satisfying (4.4) – in particular with $\lambda\to I_t(\omega,\lambda)$ being $\mathscr{B}(\Lambda)$ measurable – then $q_{t,\omega}(a,\cdot)=I_t(\omega,\varepsilon_a)$ defines a RMK on A. Thus there is a one to one correspondence between (θ_t)-flows on Λ satisfying (4.4) and RMK's on A given by (4.3).

Let q be a RMK on A such that for each $a\in A$, $q_{t,\omega}(a,\cdot)$ is unit mass at a point $J_t(\omega,a)$ in A. One easily checks that $J=(J_t(\omega,a))$ is a (θ_t)-flow on A. Conversely if J is a (θ_t)-flow on A, then $q_{t,\omega}(a,\cdot)$ defined to be unit mass at $J_t(\omega,a)$ is a RMK on A.

Suppose we are given a *shift functional* on I on A. Then we have already pointed out that the process $\tilde{X}$ is a semi-direct product as defined in [6] by Jacod. However, it is a very special type of semi-direct product since I_t is $\mathscr{F}_t$ adapted. In [6], Jacod associates a MK with any semi-direct product. In present case $q_{t,\omega}(a,\cdot)$ is just unit mass at $I_t(\omega,a)$.

There are many possible generalizations of the theory we have presented. What we have developed might be called an algebraic theory since we have assumed that (2.1) holds identically. An important generalization is to relax this assumption and to allow exceptional sets. What needs to be done is clear in outline following the course of [6] and [11]. In fact passing to the associated MK, most likely one can apply the results of these papers directly to the problem at hand. An important example of a (θ_t)-flow with exceptional sets is the prediction process $Z_t^\mu(\omega)$ as defined by Meyer in [12].

We have based our main theorem in Section 2 on the product spaces $\tilde{\Omega}=\Omega\times A$ and $\tilde{E}=E\times A$. Following Meyer [11] one can build a theory based on spaces $\tilde{\Omega}$ and $\tilde{E}$ and given surjections π and p of $\tilde{\Omega}$ and $\tilde{E}$ onto Ω and E

respectively subject to appropriate conditions. The interested reader should consult [12].

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