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Self-Decomposability of the Generalized Inverse Gaussian and Hyperbolic Distributions

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1. Introduction and Summary

In Barndorff-Nielsen and Halgreen (1977) infinite divisibility was proved for the generalized inverse Gaussian distribution, GIGD(λ, χ, ψ), whose probability density function is

$$\frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1} e^{-\frac{1}{2}(\chi x^{-1}+\psi x)} \qquad (x>0) \tag{1}$$

where K_λ is the modified Bessel function of the third kind with index λ. In this note it will, firstly, be shown that a GIGD is self-decomposable, or belongs to class L (Feller, 1971). In fact, any GIGD is a generalized Γ-convolution. This latter term was introduced by Thorin (1977) to denote the class of self-decomposable distributions F, concentrated on $[0, \infty)$, whose Laplace-transform

$$f(s)=\int_0^\infty e^{-sx} F(dx)$$

can be written as

$$f(s)=\exp\left\{-as-\int_0^\infty \ln(1+s/y)\, U(dy)\right\} \tag{2}$$

where $a \geqq 0$ and the measure U is concentrated on $(0, \infty)$ and

$$\begin{aligned} &\int_0^1 |\ln y|\, U(dy)<\infty \\ &\int_1^\infty y^{-1}\, U(dy)<\infty. \end{aligned} \tag{3}$$

Note that for U concentrated on a finite set, F is a convolution of gamma-distributions, translated by a.

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Extending a wellknown result on variance mixtures of normal distributions $N(\xi, \sigma^2)$, Barndorff-Nielsen and Halgreen (1977) also proved that any mixture of $N(\xi, \sigma^2)$ determined by the relation

$$\xi = \mu + \beta\,\sigma^2, \tag{4}$$

where $\mu \in R$ and $\beta \in R$ are constant parameters and σ^2 is endowed with an infinitely divisible distribution, is infinitely divisible. As the second result of the present note it is shown that for a variance mixture, i.e. for $\beta = 0$, self-decomposability of the distribution of σ^2 implies self-decomposability of the mixture. It is an open question whether this conclusion holds in general, i.e. for any $\beta \in R$. However, we shall show that the conclusion is true under the stricter assumption that σ^2 follows a generalized Γ-convolution. As a direct consequence of this and of the above-mentioned result that the GIGD distribution is a generalized Γ-convolution we obtain that the generalized hyperbolic distribution with density

$$\{\sqrt{2\pi}(\chi/\psi)^{\lambda/2}(\beta^2+\psi)^{(\lambda-1/2)/2} K_\lambda(\sqrt{\chi\psi})\}^{-1}\{\chi+(x-\mu)^2\}^{(\lambda-1/2)/2} \cdot K_{\lambda-1/2}(\sqrt{(\beta^2+\psi)(\chi+(x-\mu)^2)})\, e^{\beta(x-\mu)}$$

is self-decomposable. (The generalized hyperbolic distributions were introduced in Barndorff-Nielsen (1977a); see also his (1977b). The "Student"-distribution with f degrees of freedom is as a special case of these distributions, obtained by setting $\mu = \beta = \psi = 0$ and $-2\lambda = \chi = f$.)

2. The GIGDs are Generalized Γ-Convolutions

The variation domain of the parameters (λ, χ, ψ) in (1) is given by

$$\begin{aligned} &\chi > 0, \quad \psi \geqq 0 \text{ for } \lambda < 0\\ &\chi > 0, \quad \psi > 0 \text{ for } \lambda = 0\\ &\chi \geqq 0, \quad \psi > 0 \text{ for } \lambda > 0. \end{aligned}$$

In the boundary cases $\chi = 0$ or $\psi = 0$ the norming constant in (1) is to be interpreted as the limiting value. For $\lambda > 0$ and $\chi = 0$ we have the gamma-distributions, for which the statement that a GIGD is a generalized Γ-convolution is obvious. For $\lambda < 0$ and $\psi = 0$, where we get the distribution of the inverse of a gamma variate, the statement was proved by Bondesson (1977) by use of the same technique as is employed below.

Let ξ denote the Laplace-transform of (1), and suppose $\chi > 0$, $\psi \geqq 0$ and $-\lambda = \nu \geqq 0$. Using the formulas

$$\begin{aligned} &K_\nu(x) = K_{-\nu}(x),\\ &K_{\nu+1}(x) = 2(\nu/x)\,K_\nu(x) + K_{\nu-1}(x),\\ &K_{\nu-1}(x) + K_{\nu+1}(x) = -2K'_\nu(x) \end{aligned}$$

(see for instance Erdélyi et al. (1953)) we find

$$\frac{d}{ds}\ln\xi(s) = -\chi\,\Phi_\nu(\chi(\psi+2s))$$

where

$$\Phi_\nu(x) = K_{\nu-1}(\sqrt{x})/(\sqrt{x}\,K_\nu(\sqrt{x})).$$

If we use the integral-representation (Grosswald, 1976)

$$\Phi_\nu(t) = \int_0^\infty (t+x)^{-1}\,g_\nu(x)\,dx,$$

where

$$g_\nu(x) = 2\{\pi^2 x(J_\nu^2(\sqrt{x}) + Y_\nu^2(\sqrt{x}))\}^{-1} > 0 \qquad (x>0),$$

and integrate in s we obtain

$$\ln\xi(s) = -\int_{\psi/2}^\infty \chi\,g_\nu(2t\chi-\chi\psi)\ln\left(1+\frac{s}{t}\right)dt. \tag{5}$$

Formulas 9.1.7–9 in Abramowitz and Stegun (1964) tell that as x tends to 0, $g_\nu(x)$ is asymptotically proportional to $x^{\nu-1}$ for $\nu>0$ and to $(x\ln^2(x))^{-1}$ for $\nu=0$. For x tending to infinity formulas 9.2.1 and 9.2.2 tell that $g_\nu(x)$ is asymptotically proportional to $x^{-1/2}$. If we define U as the measure concentrated on $\left(\frac{\psi}{2},\infty\right)$ with density function

$$u(y) = \chi\,g_\nu(2y\chi-\chi\psi), \qquad 2y-\psi>0$$

we see that for $\nu>0$ and for $\nu=0$ and $\psi>0$, U fulfills the conditions (3). Comparing (5) and (2) we see, that the GIGDs with $\lambda\leqq 0$ are generalized Γ-convolutions. To complete the proof we only have to note the convolution formula

$$\mathrm{GIGD}(-\lambda,\chi,\psi) * \mathrm{GIGD}(\lambda,0,\psi) = \mathrm{GIGD}(\lambda,\chi,\psi).$$

3. Self-Decomposability of Normal Mixtures

We now turn to the decomposability of mixtures of normal distributions. If the mean value ξ and variance σ^2 are related as in (4), and if σ^2 is endowed with a distribution, whose Laplace-transform will be denoted by ζ, then the characteristic function of the mixture is

$$\varphi(t) = e^{i\mu t}\,\zeta\left(\frac{t^2}{2}-i\beta t\right). \tag{6}$$

In the case $\beta=0$ the self-decomposability of φ follows simply from the self-decomposability of ζ. Letting $0<a<1$ we have

$$\frac{\varphi(t)}{\varphi(a t)} = e^{i\mu(1-a)t}\cdot\frac{\zeta(t^2/2)}{\zeta(a^2 t^2/2)}.$$

The function $\zeta(\cdot)/\zeta(a^2\cdot)$ is a Laplace transform due to the self-decomposability of ζ, and hence $\varphi(\cdot)/\varphi(a\cdot)$ is the characteristic function of a variance-mixture where σ^2 is endowed with the distribution corresponding to $\zeta(\cdot)/\zeta(a^2\cdot)$.

For unconstrained $\beta\in R$ we shall prove the self-decomposability of the mixture under the condition that the distribution of σ^2 is a generalized Γ-convolution. Here we use the characterisation of the self-decomposable distributions in terms of the canonical measure M in the canonical representation

$$\ln\varphi(t)=\int_{-\infty}^{\infty}\frac{e^{ixt}-1-i\,t\sin x}{x^2}M(dx)+i\,b\,t$$

of the characteristic function. The self-decomposable distributions are those, for which

$$\int_{e^x}^{\infty}y^{-2}M(dy)\quad\text{and}\quad\int_{-\infty}^{-e^x}y^{-2}M(dy)$$

are convex functions of x. Note that, if M has density m with respect to the Lebesgue-measure on $R\setminus\{0\}$, then this condition is fulfilled if $x^{-1}m(x)$ is nonincreasing on R_+ and R_-. These results are given in Feller (1971) Chapter XVII.8.

Firstly, we restrict ourselves to the case of $\mu=0$ and of ζ being the Laplace transform of a gamma variate with scale and index parameters both equal to 1, i.e.

$$\zeta(t)=(1+t)^{-1}.$$

From (6) we get

$$\psi(\beta,t)=\ln\varphi(\beta,t)=-\ln\left(1+\frac{t^2}{2}-i\,\beta\,t\right).$$

By formula 2.4.1 in Erdélyi et al. (1954) it follows that

$$-\psi'(\beta,t)=\frac{2(t-i\,\beta)}{2+\beta^2+(t-i\,\beta)^2}=2\int_0^{\infty}e^{-\sqrt{2+\beta^2}\,x}\sin(x(t-i\,\beta))\,dx.$$

Integrating this expression yields

$$\psi(\beta,t)=\int_{-\infty}^{\infty}\frac{e^{ixt}-1}{x^2}\,|x|\,e^{-\sqrt{2+\beta^2}|x|+\beta x}\,dx$$

so the canonical measure of the mixture has density function

$$f(\beta,x)=|x|\,e^{-\sqrt{2+\beta^2}|x|+\beta x}$$

with respect to the Lebesgue-measure. We may now conclude, that the mixture is self-decomposable by observing that $x^{-1}f(\beta,x)$ is nonincreasing in x on R_+ and R_-.

In the general case, where $\mu\in R$ and σ^2 follows an arbitrary generalized Γ-convolution, the density function of the canonical measure for the mixture is of the form

$$g(x)=\int_0^\infty (1/\sqrt{y})\, f(\beta/\sqrt{y}, x/\sqrt{y})\, U(dy)$$

and it follows that the mixture is self-decomposable.

It should be noted that the mixtures are contained in the class of self-decomposable distributions treated in Thorin (1978) under the name "extended generalized gamma convolutions". I am indebted to Dr. Olof Thorin for pointing out this fact.

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