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Infinite Divisibility of the Hyperbolic and Generalized Inverse Gaussian Distributions

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In this note we show that the generalized inverse Gaussian distribution, whose probability (density) function is

$$\frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})}\, x^{\lambda-1}\, e^{-\frac{1}{2}(\chi x^{-1}+\psi x)}, \qquad (x>0), \tag{1}$$

has the property of infinite divisibility. It follows simply from this that any mixture of the r-dimensional normal distributions $N_r(\xi,\Sigma)$ determined by setting

$$\xi=\mu+\sigma^2\beta\Delta \quad \text{and} \quad \Sigma=\sigma^2\Delta \tag{2}$$

and letting σ^2 follow the distribution (1) is infinitely divisible; here μ, β and Δ are new parameters, μ and β being r-dimensional vectors while Δ is a positive definite $r\times r$ matrix with determinant $|\Delta|=1$. This class of mixtures includes the r-dimensional hyperbolic distribution

$$a(\alpha,\beta,\delta,\Delta)\, e^{-\alpha\sqrt{\delta^2+(v-\mu)\Delta^{-1}(v-\mu)'}+\beta\cdot(v-\mu)} \tag{3}$$

where, setting $\kappa=\sqrt{\alpha^2-\beta\Delta\beta'}$, the norming constant is given by

$$a(\alpha,\beta,\delta,\Delta)=\frac{1}{(2\pi)^{(r-1)/2}}\,\frac{\kappa^{(r+1)/2}}{2\alpha\,\delta^{(r+1)/2}K_{(r+1)/2}(\delta\kappa)},$$

which was introduced in Barndorff-Nielsen (1976). It also includes the 'Student' distribution, and the proof we shall give is, in fact, an application of the key result in Grosswald (1976), where the infinite divisibility of the 'Student' distribution was established.

The variation domain of the parameter (λ,χ,ψ) in (1) is given by

$$\begin{aligned}
&\chi>0, \quad \psi\geqq 0 \quad \text{for } \lambda<0,\\
&\chi>0, \quad \psi>0 \quad \text{for } \lambda=0,\\
&\chi\geqq 0, \quad \psi>0 \quad \text{for } \lambda>0,
\end{aligned}$$

and K_λ is the modified Bessel function of the third kind and with index λ. For $\lambda > 0$ and $\chi = 0$ or $\lambda < 0$ and $\psi = 0$ the norming constant in (1) is to be interpreted as the limit value, the distribution being, respectively, that of a gamma variate or the reciprocal of such a variate. The inverse Gaussian distribution is obtained for $\lambda = -1/2$ and $\psi > 0$, while for $\lambda = -1/2$ and $\psi = 0$ one has the stable distribution with characteristic exponent 1/2.

Note that the normalization constant in (1) can be derived from the integral formula

$$K_\lambda(z) = 1/2 \int_{-\infty}^{\infty} \cosh(\lambda u)\, e^{-z\cosh u}\, du,$$

by using the transformation

$$u = \ln((\psi/\chi)^{1/2} x).$$

Let ζ denote the Laplace transform of (1) and suppose $\chi > 0$ and $\psi > 0$. Due to the exponential character of (1), $\zeta(s)$ is simply the ratio of the norming constants of the probability function corresponding to the parameter values $(\lambda, \chi, \psi + 2s)$ and (λ, χ, ψ), i.e.

$$\zeta(s) = \left(\frac{\psi}{\psi + 2s}\right)^{\lambda/2} \frac{K_\lambda(\sqrt{\chi(\psi + 2s)})}{K_\lambda(\sqrt{\chi\psi})}.$$

By Theorem 1, p. 425 in Feller (1966), the distribution (1) is infinitely divisible if and only if $-\ln\zeta$ has a completely monotone derivative. Using the formulas

$$K_\lambda(x) = K_{-\lambda}(x),$$

$$K_{\lambda+1}(x) = 2(\lambda/x) K_\lambda(x) + K_{\lambda-1}(x),$$

$$K_{\lambda-1}(x) + K_{\lambda+1}(x) = -2K'_\lambda(x)$$

(see for instance Erdélyi et al. (1953)) we find

$$-\frac{d\ln\zeta(s)}{ds} = \begin{cases} \dfrac{2\lambda}{\psi + 2s} + \chi Q_\lambda(\chi(\psi + 2s)) & \text{for } \lambda \geqq 0 \\[2ex] \chi Q_{-\lambda}(\chi(\psi + 2s)) & \text{for } \lambda \leqq 0 \end{cases}$$

where

$$Q_\nu(x) = \frac{K_{\nu-1}(\sqrt{x})}{\sqrt{x}\, K_\nu(\sqrt{x})} \qquad (\nu \geqq 0,\ x > 0).$$

It was shown by Grosswald (1976) that the function Q_ν is completely monotone for every $\nu \geqq 0$ and this result yields the infinite divisibility of the generalized inverse Gaussian distribution (including the cases where χ or ψ equals 0).

Scale mixtures of normal distributions have received some attention in recent years, see Kelker (1971), Andrews and Mallows (1974), Keilson and Steutel (1974) and Kent (1976). As is wellknown in the one-dimensional case such a mixture is infinitely divisible if the mixing measure is infinitely divisible. More generally, if the normal distribution is r-dimensional ($r=1,2,\ldots$) with the mean ξ and the variance Σ related as in (2) and if σ^2 is endowed with an arbitrary distribution, whose Laplace transform will be denoted by ζ, then the characteristic function of the mixture is

$$\varphi(t)=e^{i\mu\cdot t}\,\zeta(\tfrac{1}{2}t\,\Delta t'-i\beta\,\Delta t')$$

and this is infinitely divisible provided the distribution of σ^2 is infinitely divisible, cf. Feller (1966) p. 538.

In particular, then, taking the generalized inverse Gaussian distribution (1) as the distribution of σ^2 one obtains an infinitely divisible mixture. In the case $\lambda=(r+1)/2$ this mixture equals the hyperbolic distribution (3) (cf. Barndorff-Nielsen (1976)). For arbitrary λ and $r=1$ the probability function of the mixture is

$$\begin{aligned}&\{\sqrt{2\pi}\,(\chi/\psi)^{\lambda/2}(\beta^2+\psi)^{(\lambda-1/2)/2}K_\lambda(\sqrt{\chi\psi})\}^{-1}\{\chi+(x-\mu)^2\}^{(\lambda-1/2)/2}\\&\quad\cdot K_{\lambda-1/2}(\sqrt{(\beta^2+\psi)(\chi+(x-\mu)^2)})\,e^{\beta(x-\mu)}.\end{aligned}\tag{4}$$

The 'Student' distribution with f degrees of freedom emerges from (4) by setting $\mu=0$, $\beta=0$, $\lambda=-f/2$, $\chi=f$ and $\psi=0$.

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