Z. Wahrscheinlichkeitstheorie verw. Gebiete
40, 317–337 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Limit Theory for Multivariate Sample Extremes

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Summary. Let $\{(X_n^{(1)}, \ldots, X_n^{(k)}), n \geqq 1\}$ be k-dimensional iid random vectors. Necessary and sufficient conditions are found for the weak convergence of the maxima $\left\{\bigvee_{j=1}^{n} X_j^{(1)}, \ldots, \bigvee_{j=1}^{n} X_j^{(k)}\right\}$ suitably normed to a non-degenerate limit df. The class of such limits is specified and conditions stated for the limit joint df to be a product of marginal df's. Some results are presented concerning extremal processes generated by multivariate df's.

1. Introduction

Suppose $\{\mathbf{X}_n, n \geqq 1\} = \{(X_n^{(1)}, \ldots, X_n^{(k)}), n \geqq 1\}$ are independent, identically distributed (iid) random vectors with k-dimensional distribution function (df) F. Define the sample maxima as

$$\mathbf{Y}_n = (Y_n^{(1)}, \ldots, Y_n^{(k)}) = \left(\bigvee_{j=1}^{n} X_j^{(1)}, \ldots, \bigvee_{j=1}^{n} X_j^{(k)}\right).$$

We seek conditions under which $\exists$ normalizing constants $a_n^{(j)} > 0$, $b_n^{(j)}$, $n \geqq 1$, $1 \leqq j \leqq k$ such that

$$\left(\frac{Y_n^{(1)} - b_n^{(1)}}{a_n^{(1)}}, \ldots, \frac{Y_n^{(k)} - b_n^{(k)}}{a_n^{(k)}}\right) \tag{1}$$

* Research supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.) and by CSIRO, Division of Mathematics and Statistics, Canberra while on leave from Erasmus University, Rotterdam

** Supported provided by National Science Foundation Grant OIP75-14513 while on leave from Stanford University. The hospitality of CSIRO, Division of Mathematics and Statistics, Canberra and the Department of Statistics, SGS, Australian National University is gratefully acknowledged

converges weakly to a non-degenerate limit df and we seek specifications of the class of such limits. To avoid trivialities we assume each marginal sequence $(Y_n^{(j)}-b_n^{(j)})/a_n^{(j)}$ in (1) converges weakly to a non-degenerate limit. This problem has also been considered by Geffroy (1958), Galambos (1975), Tiago de Oliveira (1959), Pickands (1976) and Sibuya (1960).

A multivariate convergence of types argument (see Geffroy (1958)) quickly shows that the class of limit df's for (1) is the class of *max-stable* distributions where we define a df G in R^k to be max-stable iff for every n, $\exists\, \alpha_n^{(j)}>0, \beta_n^{(j)}, 1\leqq j\leqq k$ such that

$$G^n(\alpha_n^{(1)}x_1+\beta_n^{(1)},\dots,\alpha_n^{(k)}x_k+\beta_n^{(k)})=G(x_1,\dots,x_k). \tag{2}$$

Note that each marginal of G must be one of the three classical extreme value df's studied by Gnedenko (1943) and de Haan (1970, 1971). Max-stable df's form a subclass of the max-infinitely divisible (max-id) df's introduced and characterized in Balkema and Resnick (1977).

We begin in section 1 by deriving the form of max-stable df's in R^k which have specified marginals. Several representations are given. The restriction on the marginals is next removed after which we take up domain of attraction and asymptotic independence questions. Finally we close with some observations about the extremal processes generated by the max-stable and max id df's.

The max-id df's as discussed in Balkema and Resnick (1977) are a proper subclass of the df's on R^k which can be defined as follows: $F(x_1,\dots,x_k)$ is max id iff for every $t>0$, $F^t(x_1,\dots,x_k)$ is a df or equivalently iff $\forall n\, F^{1/n}$ is a df. It is then immediate from (2) that max-stable df's are max id.

The following is a criterion for F to be max id: Let $A(x_1,\dots,x_k)=[-\infty,x_1]\times\dots\times[-\infty,x_k]$. Then there must exist a measure ν on $[-\infty,\infty)^k$, called the *exponent measure*, such that

$$\nu(\mathbb{R}_1\times\dots\times\mathbb{R}_{i-1}\times[-\infty,\infty)\times\mathbb{R}_{i+1}\times\dots\times\mathbb{R}_k)=\infty \qquad \text{for all } i=1,\dots,k,$$

$$\nu(A^c(x_1,\dots,x_k))<\infty \qquad \text{for some } (x_1,\dots,x_k)$$

and

$$F(x_1,\dots,x_k)=\exp\{-\nu(A^c(x_1,\dots,x_k))\}$$

where

$$A^c(x_1,\dots,x_k)=[-\infty,\infty)^k-A(x_1,\dots,x_k).$$

From a process point of view the max id df's are precisely the class of df's F which can be used to define a multivariate extremal process $\mathbf{Y}(t)=(Y_1(t),\dots,Y_k(t))$. Such a process is defined to have marginals: $\forall n, \forall\, 0<t_1<\dots<t_n$

$$\begin{aligned}P[Y_1(t_i)\leqq x_1^{(i)},\dots,Y_k(t_i)\leqq x_k^{(i)}, i=1,\dots,n]\\ =F^{t_1}\left(\bigwedge_{i=1}^n x_1^{(i)},\bigwedge_{i=1}^n x_2^{(i)},\dots,\bigwedge_{i=1}^n x_k^{(i)}\right)\\ \cdot F^{t_2-t_1}\left(\bigwedge_{i=2}^n x_1^{(i)},\dots,\bigwedge_{i=2}^n x_k^{(i)}\right)\cdots\\ \cdot F^{t_n-t_{n-1}}(x_1^{(n)},\dots,x_k^{(n)}).\end{aligned} \tag{3}$$

A related viewpoint is that F is max id iff there exists a measure ν on $[-\infty, \infty)^k$ such that if we construct a Poisson random measure on $R_+ \times [-\infty, \infty)^k$ with points $\{(T_n; J_n^{(1)}, \ldots, J_n^{(k)})\}$ and mean measure $dt \times \nu(dx_1, \ldots, dx_k)$ then defining the extremal process $\mathbf{Y}(t)$ by

$$Y_i(t) = \sup_n \{J_n^{(i)} \mid T_n \leqq t\} \tag{4}$$

we have

$$F^t(x_1, \ldots, x_k) = P[Y_i(t) \leqq x_i, i=1, \ldots, k] = \exp\{-t\,\nu(A^c(x_1, \ldots, x_k))\}.$$

Our methods differ from those of previous authors because of our reliance on the concept of max infinite divisibility and judicious use of polar coordinates. Also insight is gained by comparing the multivariate stable Lévy processes with certain of our extremal processes $\mathbf{Y}$ which satisfy $\{\mathbf{Y}(at), t>0\} = \{a^\alpha \mathbf{Y}(t), t>0\}$ $\forall a>0$ where α is a positive parameter.

2. Max-Stable df's with Prescribed Marginals

Call a max-stable df G in R^k *simple* if each marginal is equal to the extreme value df $\Phi_1(x) = e^{-x^{-1}}, x>0$; i.e.

$$G(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty) = e^{-x_i^{-1}}, x_i > 0.$$

We begin by deriving the form of a simple G. The reason why it is sensible to start with a simple G becomes clear in section 4 where we remove this restriction on the marginals.

The exponent measure of a max id df need not be unique. For instance consider the simple stable df $G(x_1, x_2) = \exp\{-(x_1^{-1} + x_2^{-1})\}$, $x_1 \geqq 0$, $x_2 \geqq 0$. One possible choice of exponent measure ν is

$$\nu(dx_1, dx_2) = x_1^{-2} dx_1 1_{\{x_1>0, x_2=0\}}(x_1, x_2) + x_2^{-2} dx_2 1_{\{x_2>0, x_1=0\}}(x_1, x_2),$$

but another perfectly good choice is

$$\nu(dx_1, dx_2) = x_1^{-2} dx_1 1_{\{x_1>0, x_2=-\infty\}}(x_1, x_2) + x_2^{-2} dx_2 1_{\{x_1=-\infty, x_2>0\}}(x_1, x_2).$$

What is important for our future work is that the exponent measure of a simple stable df can always be chosen so that $\nu\{([0, \infty)^k)^c\} = 0$. This is easy to check and henceforth when dealing with exponent measures of simple stable df's we suppose this property is in force.

Suppose now that G is simple stable. Consideration of properties exhibited by $\Phi_1(x)$ shows that (2) can be written as

$$G^n(nx_1, \ldots, nx_k) = G(x_1, \ldots, x_k)$$

$\forall n$ and it is easy to switch to a continuous variable s in place of n so that $\forall s>0$

$$G^s(sx_1, \ldots, sx_k) = G(x_1, \ldots, x_k). \tag{5}$$

From (5) it follows that $G(x_1, \ldots, x_k) > 0$ if $x_i > 0$ for $i = 1, 2, \ldots, k$.

Letting ν be the exponent measure of G (5) becomes

$$s\nu(A^c(sx_1, \ldots, sx_k)) = \nu(A^c(x_1, \ldots, x_k)), \tag{6}$$

where recall $A(x_1, \ldots, x_k) = [-\infty, x_1] \times \cdots \times [-\infty, x_k]$ so that (6) entails

$$s\nu(sA^c(x_1, \ldots, x_k)) = \nu(A^c(x_1, \ldots, x_k)).$$

Note $\nu(A^c(x_1, \ldots, x_k))$ is finite if $x_i > 0$ for $i = 1, 2, \ldots, k$.

For fixed s the measure $s\nu(s\,\cdot)$ agrees with ν on a generating class closed under intersections and hence we conclude $\forall\, B \in \mathscr{B}([0, \infty)^k)$

$$s\nu(sB) = \nu(B). \tag{7}$$

Let $\Xi = [0, \frac{\pi}{2}]^{k-1}$ and let $T\colon R^k \to R_+ \times \Xi$ be the transformation to polar coordinates: $T(x_1, \ldots, x_k) = (r, \boldsymbol{\theta})$ where $r^2 = \sum_{i=1}^{k} x_i^2$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{k-1})$ and $\sin^2 \theta_i = (x_{i+1}^2 + \cdots + x_k^2)/(x_i^2 + \cdots + x_k^2)$ for $i = 1, \ldots, k-1$. Fix a Borel set $C \subset \Xi$ and set $D(r, C) = \{(s, \boldsymbol{\theta}) \mid s > r, \boldsymbol{\theta} \in C\}$. Note that for $r > 0$, $\nu(T^{-1}(D(r, C))) < \infty$ because for some $x_1, \ldots, x_k$ with $x_i > 0$, $i = 1, \ldots, k$ we have $T^{-1}(D(r, C)) \subset A^c(x_1, \ldots, x_k)$ and $\nu(A^c(x_1, \ldots, x_k)) < \infty$. Referring back to (7) we have

$$s\nu(sT^{-1}(D(r, C))) = s\nu(T^{-1}(D(rs, C))) = \nu(T^{-1}(D(r, C))),$$

i.e. if $M(r) = \nu(T^{-1}(D(r, C)))$ we have

$$M(r) = sM(rs).$$

Setting $s = r^{-1}$ and $S(C) = M(1)$ gives $M(r) = r^{-1} S(C)$ where S is a finite measure on Ξ. Thus we have

Theorem 1. *G is simple stable with exponent measure ν iff there exists a finite measure S on Ξ such that*

$$\nu \circ T^{-1}(dr, d\boldsymbol{\theta}) = r^{-2}\, dr\, S(d\boldsymbol{\theta})$$

and

$$\int_{\Xi} \sin\theta_1, \ldots, \sin\theta_{i-1} \cos\theta_i\, S(d\boldsymbol{\theta}) = 1$$

for $i = 1, \ldots, k$ with the convention that $\theta_k = 0$ and for $i = 1$ the integrand is just $\cos\theta_1$. Recall T is the transformation to polar coordinates.

The integral condition in Theorem 1 arises because of the requirement that G be simple (cf. Theorem 2) and disappears when this requirement is waived. To check that the integral must equal 1 note that for $i = 1, \ldots, k$

$$\begin{aligned} x_i^{-1} &= \nu(A^c(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty)) \\ &= \int_{TA^c} r^{-2}\, dr\, S(d\boldsymbol{\theta}) \end{aligned}$$

where $TA^c = \{(r, \boldsymbol{\theta}) \mid r \sin\theta_1, \ldots, \sin\theta_{i-1} \cos\theta_i > x_i\}$. Integrating on r gives the result.

Further understanding of the meaning of Theorem 1 is obtained from the following considerations: For a real function $x(t)$ which is right continuous with finite left limits $\forall t>0$ define the functional h via

$$(hx)(t)=\sup_{0<s\leq t}((x(s)-x(s-))\vee 0).$$

Corollary 1. *Let* $\mathbf{X}(t)=(X_1(t),\ldots,X_k(t))$ *be a k-variate stable Lévy process of index* 1; i.e. *a process with stationary independent increments and the property* $\forall a>0$ $\{\mathbf{X}(at),t\geq 0\}=\{a\mathbf{X}(t)+\mathbf{C}(a),t\geq 0\}$ *where* $\mathbf{C}(a)$ *is a nonrandom vector* (Lévy, 1937). *Suppose further that for* $i=1,\ldots,k$ *the Lévy measure* ν_i *of* X_i *has the property that* $\nu_i(x,\infty)=x^{-1}$ *for* $x>0$. *The class of extremal processes generated by the simple stable df's described in Theorem 1 is precisely the class of extremal processes realized through the scheme* $\mathbf{Y}(t)=\{Y_1(t),\ldots,Y_k(t)\}=\{(hX_1)(t),\ldots,(hX_k)(t)\}$.

Proof. That $\mathbf{Y}$ is an extremal process follows as in the 1-dimensional case (cf. Dwass 1964, Resnick and Rubinovitch 1973) from the fact that $\mathbf{X}$ induces Poisson random measure with points $\{T_n; J_n^{(1)},\ldots,J_n^{(k)}\}$ where T_n is the time of a jump and $(J_n^{(1)},\ldots,J_n^{(k)})=\mathbf{X}(T_n)-\mathbf{X}(T_n-)$. The mean measure is $dt\times\nu(dx_1,\ldots,dx_k)$ where ν is the Lévy measure of $\mathbf{X}$. However, if $\mathbf{X}$ is stable with index 1, it is well known (Lévy, 1937) that $\nu\circ T^{-1}(dr,d\boldsymbol{\theta})=r^{-2}\,dr\,S(d\boldsymbol{\theta})$ where S is a finite measure on Ξ.

In case $k=2$, the criteria obtained in terms of $\nu\circ T^{-1}$ for G to be max stable can be rephrased in terms of ν:

Corollary 2. *$G(x,y)$ is simple stable with exponent measure ν iff*

$$G(x,y)=\exp-\Big\{x^{-1}\int_{[0,\arctan y/x]}\cos\theta\,S(d\theta)+y^{-1}\int_{(\arctan y/x,\pi/2]}\sin\theta\,S(d\theta)\Big\}$$

where $S(\cdot)$ is a finite measure on $[0,\frac{\pi}{2}]$ such that

$$\int_0^{\pi/2}\cos\theta\,S(d\theta)=\int_0^{\pi/2}\sin\theta\,S(d\theta)=1.$$

Proof. The last two conditions arise because we require $G(x,\infty)=\exp\{-x^{-1}\}=G(\infty,x)$. For the rest note that by Theorem 1 $\nu\circ T^{-1}(dr,d\theta)=r^{-2}\,dr\,S(d\theta)$ so that

$$\begin{aligned}\nu(A^c(x,y))&=\iint_{T(A^c(x,y))}r^{-2}\,dr\,S(d\theta)\\&=\iint_{\{(r,\theta)\mid r\cos\theta\leq x,\,r\sin\theta\leq y\}^c}r^{-2}\,dr\,S(d\theta)\\&=\iint_{\left\{(r,\theta)\mid r>\frac{x}{\cos\theta}\wedge\frac{y}{\sin\theta}\right\}}r^{-2}\,dr\,S(d\theta)\end{aligned}$$

and evaluating the integral on r for fixed θ gives

$$x^{-1}\int_{[0,\arctan y/x]}\cos\theta\,S(d\theta)+y^{-1}\int_{(\arctan y/x,\pi/2]}\sin\theta\,S(d\theta)$$

as asserted.

Corollary 3. *If G is as in Corollary 2 and $P(X \leqq x, Y \leqq y) = G(x, y)$ then*

(i) *X, Y are independent iff $S\{0\} = S\{\frac{\pi}{2}\} = 1$ and S places no mass elsewhere. This can be seen either from Corollary 2 or by checking directly from $G(x, y) = \exp\left\{-\left(\frac{1}{x} + \frac{1}{y}\right)\right\}$ that $\nu\{(t, s) | t > x, s > y\} = 0$ for all $x, y > 0$.*

(ii) *$P(X = Y) = 1$ iff $S\{\frac{\pi}{4}\} = \sqrt{2}$ and S places no mass elsewhere.*

Remark. If the measure S concentrates on some point $\theta_0 \in [0, \frac{\pi}{2}]$ with $\theta_0 \neq \frac{\pi}{4}$ we have $Y = (\tan \theta_0) X$ a.s. and hence the marginals are both of type $\Phi_1(x)$, but are not equal. This means that G is not simple according to our definition.

Remark. We can connect our results with those of Sibuya (1960) (see also Geffroy (1958)) as follows: In Corollary 2 when $k = 2$ set

$$W(t) = \int_{[0, \arctan t]} \cos \theta \, S(d\theta) = \int_{[0, t]} \cos(\arctan y) \, dS(\arctan y) \tag{8}$$

and

$$G(x, y) = \exp\left\{-\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{y} \chi\left(\frac{y}{x}\right)\right)\right\}, \tag{9}$$

so that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{y} \chi\left(\frac{y}{x}\right) = \frac{1}{x} \int_{[0, \arctan y/x]} \cos \theta \, S(d\theta) + \frac{1}{y} \int_{(\arctan y/x, \pi/2]} \sin \theta \, S(d\theta)$$

i.e.

$$\begin{aligned}
t + 1 + \chi(t) &= t \int_{[0, \arctan t]} \cos \theta \, S(d\theta) + \int_{(\arctan t, \pi/2]} \sin \theta \, S(d\theta) \\
&= t\, W(t) + \int_{[\arctan t, \pi/2)} \tan \theta \cos \theta \, S(d\theta) + S(\{\tfrac{\pi}{2}\}) \\
&= t\, W(t) + \int_t^\infty y\, W(dy) + S(\{\tfrac{\pi}{2}\}) \\
&= \int_t^\infty (y - t)\, W(dy) + t(1 - W(t)) + t\, W(t) + S(\{\tfrac{\pi}{2}\}) \\
&= t + \int_t^\infty (1 - W(s))\, ds + S(\{\tfrac{\pi}{2}\}).
\end{aligned}$$

Therefore we conclude

$$\chi(t) = \int_t^\infty (1 - W(s))\, ds + S(\{\tfrac{\pi}{2}\}) - 1.$$

Note χ has the properties specified by Sibuya:

χ is continuous and convex since it is the integral of a monotone function (10)

$$\max(-t, -1) \leqq \chi(t) \leqq 0, \qquad \forall\, t \geqq 0. \tag{11}$$

Conversely if G is of form (9) where χ satisfies (10) and (11) then one defines S via W; S then satisfies the conditions of Theorem 1.

Example 1 (cf. Geffroy, 1958, p. 71). Let $S[0,\theta]=\theta$ for $0\leqq\theta\leqq\frac{\pi}{2}$ so that

$$\int_0^{\pi/2}\cos\theta\, S(d\theta)=\int_0^{\pi/2}\sin\theta\, S(d\theta)=1.$$

Then $\chi(t)=(1+t^2)^{\frac{1}{2}}-(1+t)$ and $G(x,y)=\exp\{-(x^{-2}+y^{-2})^{\frac{1}{2}}\}$ for $x\geqq 0$, $y\geqq 0$.

Example 2. Take $S[0,\theta]=3\int_0^\theta \cos t \sin t\, dt$, $0\leqq\theta\leqq\frac{\pi}{2}$. Then $\chi(t)=-t(1+t^2)^{-\frac{1}{2}}$ and for $x\geqq 0$, $y\geqq 0$ $G(x,y)=\exp\{-(x^{-1}+y^{-1}-(x^2+y^2)^{-\frac{1}{2}})\}$.

Example 3 (Sibuya, 1960, p. 208). $\chi(t)=-kt(1+t)^{-1}$ for $0\leqq k\leqq 1$ corresponds to

$$S\{0\}=S\{\tfrac{\pi}{2}\}=1-k,\ S(0,\theta)=\int_0^\theta 2k(\cos y+\sin y)^{-3}\,dy$$

and

$$G(x,y)=\exp\{-(x^{-1}+y^{-1}-k(x+y)^{-1})\}.$$

A constructive Approach. Next we follow a constructive approach which leads to a representation of the simple stable df's in Cartesian coordinates. Recalling that the required marginals are $\Phi_1(x)=e^{-x^{-1}}$, $x>0$ observe that in R^2 the Frechet df $G(x,y)$ $=\Phi_1(x)\wedge\Phi_1(y)=\exp\{-x^{-1}\vee y^{-1}\}$ for $x,y>0$ is a simple stable df. This df is concentrated on the line $x=y$. Let U_1 and U_2 be independent random variables both with distribution function Φ_1. Take

$$(X,Y)=(\max(r_1U_1\cos\varphi_1, r_2U_2\cos\varphi_2), \max(r_1U_1\sin\varphi_1, r_2U_2\sin\varphi_2)),$$

then (X,Y) has df

$$G(x,y)=\exp-\left\{r_1\left(\frac{\cos\varphi_1}{x}\vee\frac{\sin\varphi_1}{y}\right)+r_2\left(\frac{\cos\varphi_2}{x}\vee\frac{\sin\varphi_2}{y}\right)\right\}.$$

G satisfies (5) and is simple stable provide $r_1\cos\varphi_1+r_2\cos\varphi_2=r_1\sin\varphi_1+r_2\sin\varphi_2$ $=1$. Its S-measure concentrates on the points φ_1 and φ_2. Generalizing this procedure we get the most general simple stable df in R^k. Let

$$\Omega=\left\{(x_1,\ldots,x_k)\,|\,x_i\geqq 0,\ i=1,\ldots,k,\ \sum_1^k x_i^2=1\right\}.$$

Theorem 2. *$G(x_1,\ldots,x_k)$ is simple stable iff there exists a finite measure U on Ω with*

$$\int_\Omega a_i\,U(da_1,\ldots,da_k)=1 \quad \text{for } i=1,\ldots,k$$

and such that

$$G(x_1,\ldots,x_k)=\exp\{-\int_\Omega \max(a_1x_1^{-1},\ldots,a_kx_k^{-1})\,U(da_1,\ldots,da_k)\}.$$

Proof. That any G of the given form is simple can be verified easily. To prove the converse we use Theorem 1. We have

$$-\log G(x_1, \dots, x_k) = \int_{\substack{\{(r,\theta)\,|\, r\sin\theta_1 \dots \\ \sin\theta_{i-1}\cos\theta_i \leqq x_i, \\ i=1,\dots,k\}^c}} r^{-2}\, dr\, S(d\theta)$$

and integrating on r gives

$$= \int_{\theta\in\Xi} \max\left\{\frac{\sin\theta_1 \dots \sin\theta_{i-1}\cos\theta_i}{x_i},\ i=1,\dots,k\right\} S(d\theta)$$

which completes the proof.

Remark. Independence of the k-marginals of G corresponds to a measure U concentrated on the k extreme points of Ω.

Here are some examples in R^3:

Example 4. Suppose

$$U\{(1/\sqrt{2}, 1/\sqrt{2}, 0)\} = U\{(1/\sqrt{2}, 0, 1/\sqrt{2})\} = U\{(0, 1/\sqrt{2}, 1/\sqrt{2})\} = 1/\sqrt{2}$$

with U placing no mass elsewhere. Then

$$G(x, y, z) = \exp\{-\tfrac{1}{2}(x^{-1} \vee y^{-1} + y^{-1} \vee z^{-1} + x^{-1} \vee z^{-1})\}$$

for $x, y, z \geqq 0$.

Example 5. Let U concentrate on $\Omega \cap \{(x, y, z)\,|\, x=0 \text{ or } y=0 \text{ or } z=0\}$ and have density $\frac{1}{2}$ there. Then

$$G(x, y, z) = \exp -\tfrac{1}{2}\{(x^{-2}+y^{-2})^{\frac{1}{2}} + (x^{-2}+z^{-2})^{\frac{1}{2}} + (y^{-2}+z^{-2})^{\frac{1}{2}}\}.$$

Remark. Examples 4 and 5 are based on the observation that if Ω is partitioned into n measurable sets $\Omega_1, \dots, \Omega_n$, the stable df can be written as the product of n stable df's with angular measures concentrated on $\Omega_i (i=1, \dots, n)$.

3. Domains of Symmetric Attraction of Simple Max-Stable Distributions

Here we study the domain of attraction of a simple stable df G and again we recall that each marginal of G equals $\Phi_1(x) = e^{-x^{-1}}$, $x > 0$.

Suppose F is in the domain of attraction of a simple stable df G; i.e. $\exists\, a_n^{(j)} > 0, b_n^{(j)}$, $n \geqq 1, j=1, \dots, k$ such that

$$F^n(a_n^{(1)} x_1 + b_n^{(1)}, \dots, a_n^{(k)} x_k + b_n^{(k)}) \to G(x_1, \dots, x_k) \tag{12}$$

for $(x_1, \dots, x_k)$ a continuity point of G, $x_i \geqq 0$, $i=1, \dots, k$. Consideration of the marginals shows that (12) still holds if $b_n^{(j)} = 0$, $n \geqq 1, j=1, \dots, k$ (cf. Gnedenko (1943), de Haan (1970)). Suppose for the moment $a_n = a_n^{(1)} = \dots = a_n^{(k)}$. When this is the case we say F is in the *domain of symmetric attraction* of G. Recall the notation $A(x_1, \ldots, x_k) = [-\infty, x_1] \times \cdots \times [-\infty, x_k]$. Note (12) holds iff for $x_i \geqq 0$, $i = 1, \ldots, k$:

$$\lim_{n\to\infty} n(1 - F(a_n x_1, \ldots, a_n x_k)) = -\log G(x_1, \ldots, x_k)$$

so that if v is the exponent measure of G we have

$$\lim_{n\to\infty} nP[\mathbf{X} \in a_n A^c(x_1, \ldots, x_k)] = v(A^c(x_1, \ldots, x_k))$$

for all A with $v(\partial A) = 0$, where we suppose $\mathbf{X}$ is a random vector with df F. Hence for all $B \in \mathscr{B}([0, \infty)^k - \{\mathbf{0}\})$ with $v(\partial B) = 0$ we have

$$\lim_{n\to\infty} nP[\mathbf{X} \in a_n B] = v(B). \tag{13}$$

Now we switch to polar coordinates. Let C be a Borel subset of Ξ and set for $r > 0$

$$B(r, C) = \left\{ (x_1, \ldots, x_k) \,\middle|\, \sum_1^k x_i^2 > r^2, \boldsymbol{\theta} \in C \right\}.$$

Then

$$\begin{aligned} \lim_{n\to\infty} nP[\mathbf{X} \in B(a_n r, C)] &= \lim_{n\to\infty} nP[\mathbf{X} \in a_n B(r, C)] \\ &= v(B(r, C)) = r^{-1} S(C) \end{aligned}$$

by Theorem 1, i.e.

$$\lim_{n\to\infty} nP[\|\mathbf{X}\| > a_n r, \Theta(\mathbf{X}) \in C] = r^{-1} S(C) \tag{14}$$

where $\|\mathbf{X}\|$, Θ are the polar coordinates of $\mathbf{X}$. Setting $r = 1$ and $C = \Xi$ we obtain

$$\lim_{n\to\infty} \frac{P[\|\mathbf{X}\| > a_n r, \Theta(\mathbf{X}) \in \Xi]}{P[\|\mathbf{X}\| > a_n, \Theta(\mathbf{X}) \in \Xi]} = r^{-1} \tag{15}$$

and furthermore it follows from (14) and (15) that

$$\lim_{n\to\infty} P[\Theta(\mathbf{X}) \in C \mid \|X\| > a_n, \Theta(X) \in \Xi] = \lim_{n\to\infty} \frac{P[\|\mathbf{X}\| > a_n, \Theta(\mathbf{X}) \in C]}{P[\|\mathbf{X}\| > a_n, \Theta(X) \in \Xi]} = \frac{S(C)}{S(\Xi)}.$$

It is not hard to see that a_n may be replaced by a continuous variable t.

Theorem 3. *The random vector X with df F is in the domain of symmetric attraction of the simple stable df G with exponent measure v and $v \circ T^{-1}(dr, d\boldsymbol{\theta}) = r^{-2}\, dr\, S(d\boldsymbol{\theta})$ iff*

$$\lim_{t\to\infty} \frac{P[\|\mathbf{X}\| > \mathrm{tr}, \Theta(X) \in \Xi]}{P[\|\mathbf{X}\| > t, \Theta(\mathbf{X}) \in \Xi]} = r^{-1} \quad \textit{for all } r > 0 \tag{16}$$

and

$$\lim_{t\to\infty} P[\Theta(\mathbf{X}) \in C \mid \|\mathbf{X}\| > t, \Theta(\mathbf{X}) \in \Xi] = S(C)/S(\Xi) \tag{17}$$

for all Borel sets $C \subset \Xi$ with $S(\partial C) = 0$.

To check that (16) and (17) are sufficient observe that these conditions are equivalent to (13) where we choose a_n to satisfy

$$P[\|\mathbf{X}\| > a_n, \Theta(\mathbf{X}) \in \Xi] = n^{-1} S(\Xi).$$

In (13) put $B = A^c(x_1, \dots, x_k) \cap [0, \infty)^k$ for $x_i > 0$, $i = 1, \dots, k$ and the result of this substitution is equivalent to

$$\lim_{n\to\infty} P^n[a_n^{-1}\mathbf{X} \in A(x_1, \dots, x_k) \cup ([0, \infty)^k)^c] = \exp\{-\nu(A^c(x_1, \dots, x_k))\}$$

and since $a_n \to \infty$ we have $P[a_n^{-1}\mathbf{X} \in ([0, \infty)^k)^c] \to 0$ so that (12) follows as desired.

Remark. The criteria for convergence of *sums* of iid vectors are similar. See Rvačeva (1962, Theorem 4.2; set $\alpha = 1$).

The situation of non-symmetric attraction is discussed in the next section.

Example 6. Let (X_1, X_2) have a 2-dimensional Cauchy distribution i.e. its density is $(2\pi)^{-1}(1 + x^2 + y^2)^{-3/2}$. Then $\|\mathbf{X}\|$ and $\theta(\mathbf{X})$ are independent, $\|\mathbf{X}\|$ has density $r(1 + r^2)^{-3/2}$, $r > 0$ and $\Theta(X)$ has uniform density on $[0, 2\pi)$. It follows that this distribution is in the domain of symmetric attraction of a simple stable law and $S[0, \theta] = \theta$, $0 < \theta \leq \frac{\pi}{2}$. Cf. Example 1.

Sufficient conditions for convergence can be given in terms of the density of F when this density exists.

Corollary 4. *Suppose G is simple stable and the measure S appearing in the representation of Theorem 1 has density $s(\boldsymbol{\theta})$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{k-1}) \in \Xi$. Suppose F has density f. Then F is in the domain of symmetric attraction of G if for all $r > 0$*

$$\lim_{t\to\infty} \frac{\int_\Xi f(tr\cos\theta_1, tr\sin\theta_1\cos\theta_2, \dots, tr\sin\theta_1 \dots \sin\theta_{k-2}\cos\theta_{k-1}, tr\sin\theta_1 \dots \sin\theta_{k-1})\, d\theta}{\int_\Xi f(t\cos\theta_1, t\sin\theta_1\cos\theta_2, \dots, t\sin\theta_1 \dots \sin\theta_{k-2}\cos\theta_{k-1}, t\sin\theta_1 \dots \sin\theta_{k-1})\, d\theta}$$
$$= r^{-(k+1)} \tag{18}$$

and (with $\boldsymbol{\frac{\pi}{4}} = (\frac{\pi}{4}, \dots, \frac{\pi}{4})$)

$$\lim_{t\to\infty} \frac{f(t\cos\theta_1, t\sin\theta_1\cos\theta_2, \dots, t\sin\theta_1 \dots \sin\theta_{k-2}\cos\theta_{k-1}, t\sin\theta_1 \dots \sin\theta_{k-1})}{f(t/\sqrt{2}, t/(\sqrt{2})^2, \dots, t/(\sqrt{2})^k)}$$
$$= s(\boldsymbol{\theta})/s(\boldsymbol{\tfrac{\pi}{4}}). \tag{19}$$

Proof. Let $f_*(r, \boldsymbol{\theta})$ be the density of $\|\mathbf{X}\| = \left(\sum_1^k X_i^2\right)^{\frac{1}{2}}$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{k-1})$ where $\Theta_i = \arcsin\left(\sum_{l=i+1}^k X_l^2 / \sum_{l=i}^k X_l^2\right)^{\frac{1}{2}}$ and suppose

$$\lim_{t\to\infty} \frac{\int_{\boldsymbol{\theta}\in\Xi} f_*(rt, \boldsymbol{\theta})\, d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta}\in\Xi} f_*(t, \boldsymbol{\theta})\, d\boldsymbol{\theta}} = r^{-2} \tag{20}$$

and

$$\lim_{t\to\infty} f_*(t,\boldsymbol{\theta})/f_*(t,\tfrac{\pi}{4}) = s(\boldsymbol{\theta})/s(\tfrac{\pi}{4}). \tag{21}$$

Note that (21) and de l'Hospital's rule give

$$\lim_{t\to\infty} \frac{\int_{r=t}^{\infty} f_*(r,\boldsymbol{\theta})\,dr}{\int_{r=t}^{\infty} f_*(r,\tfrac{\pi}{4})\,dr} = \frac{s(\boldsymbol{\theta})}{s(\tfrac{\pi}{4})}$$

and therefore

$$\lim_{t\to\infty} \frac{P[\|\mathbf{X}\|>t, \boldsymbol{\theta}\in C]}{\int_t^{\infty} f_*(r,\tfrac{\pi}{4})\,dr} = \lim_{t\to\infty} \frac{\int_t^{\infty}\int_{\boldsymbol{\theta}\in C} f_*(r,\boldsymbol{\theta})\,d\boldsymbol{\theta}\,dr}{\int_t^{\infty} f_*(r,\tfrac{\pi}{4})\,dr}$$

$$= \frac{\int_{\boldsymbol{\theta}\in C} s(\boldsymbol{\theta})\,d\boldsymbol{\theta}}{s(\tfrac{\pi}{4})} = \frac{S(C)}{s(\tfrac{\pi}{4})}$$

from which (17) follows and (16) follows directly from (20) thus implying F is symmetrically attracted to G. The conditions (20) and (21) readily translate into (18) and (19) and the proof is complete.

Example 7. On R^2 suppose $S[0,\theta]=\theta$, $0\leqq\theta\leqq\frac{\pi}{2}$ for G. Then (19) means

$$f(t\cos\theta_1, t\sin\theta_1) \sim f(t\cos\theta_2, t\sin\theta_2)$$

as $t\to\infty$ $\forall\,\theta_1,\theta_2\in[0,\frac{\pi}{2}]$.

4. Stable df's that are not Simple; Domains of Attraction

We again suppose that (12) holds but now make no assumption about the marginals of the limit G except that they be non-degenerate. Denote the marginals of F by F_i, $i=1,\ldots,k$ and let $U_i(x)$ be an inverse of the monotone function $1/(1-F_i(x))$. Then U_i satisfies

$$\lim_{t\to\infty} \frac{U_i(tx)-U_i(t)}{U_i(te)-U_i(t)} = \Psi_i(x)$$

where

$$\Psi_i(x) = \frac{x^\rho-1}{e^\rho-1}, \frac{1-x^{-\rho}}{1-e^{-\rho}} \text{ or } \log x \tag{22}$$

for $x>0$ where ρ is a positive parameter (de Haan, 1970); in particular

$$\lim_{n\to\infty}(U_i(nx)-b_n^{(i)})/a_n^{(i)}=\frac{\psi_i(x)-B_i}{A_i}=\tilde{\psi}_i(x) \tag{23}$$

where $a_n^{(i)}>0$, $b_n^{(i)}$, $i=1,\dots,k$, $n\geqq 1$ are the normalizing constants appearing in (12) and $A_i>0$ and B_i real constants. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} P\left[\frac{1}{1-F_i}(Y_n^{(i)})\leqq n x_i,\ i=1,\dots,k\right]\\
&\quad=\lim_{n\to\infty} P[(Y_n^{(i)}-b_n^{(i)})/a_n^{(i)}\leqq(U_i(nx_i)-b_n^{(i)})/a_n^{(i)},\ i=1,\dots,k]\\
&\quad=G(\tilde{\Psi}_1(x_1),\dots,\tilde{\Psi}_k(x_k))
\end{aligned}$$

from (12) and (23). We can choose the $a_n^{(i)}$ and $b_n^{(i)}$ such that the marginals of $G(\tilde{\psi}_1(x_1),\dots,\tilde{\psi}_k(x_k))$ are $e^{-x^{-1}}$, $x>0$, then we have symmetric convergence of the maxima of $\left\{\frac{1}{1-F_1}(X_n^{(1)}),\dots,\frac{1}{1-F_k}(X_n^k);n\geqq 1\right\}$ to the simple stable df $G(\tilde{\Psi}_1(x_1),\dots,\tilde{\Psi}_k(x_k))$. Using this we can generalize the results in the previous two sections to the general case:

Theorem 4. *The type of the most general max-stable df with non-degenerate marginals is of the form $G_*(\tilde{\Psi}_1^{-1}(x_1),\dots,\tilde{\Psi}_k^{-1}(x_k))$ with G_* a simple stable df and $\tilde{\Psi}_i$ one of the functions given in (22) and (23), $i=1,\dots,k$. A df F is in the domain of attraction of G iff $F(U_1(x_1),\dots,U_k(x_k))$ is in the domain of symmetric attraction of $G(\tilde{\Psi}_1(x_1),\dots,\tilde{\Psi}_k(x_k))$.*

We end this section with a remark concerning our definition of the simple stable df's. We chose the approach used in Section 2 because of the links described in Corollary 1 with the stable Lévy processes. However an alternative approach would be to start with df's whose marginals are double exponential df's. The transformation to polar coordinates is then replaced by the transformation

$$z_1=x_1+\dots+x_k,\ z_2=x_1-x_2,\dots,z_k=x_{k-1}-x_k$$

and all results can then be derived in an analogous fashion to the one given in Section 2. For example in R^2 if

$$B(w,z)=\{(x,y)\mid x+y>2w,\ x-y>2z\}$$

then (6) is replaced by

$$s+\log v(B(w+s,z))=\log v(B(w,z))$$

which entails

$$v(B(w,z))=e^{-w}\rho(z)$$

where ρ is decreasing. There is a problem here however. In the previous case we had measures on the closed set $[0,\frac{\pi}{2}]^{k-1}$ so that here we have to consider measures on the closed set $[-\infty,\infty]^{k-1}$.

In R^2 the approach using marginals equal to $\exp\{-e^{-x}\}$ could be linked to the approach using marginals equal to $e^{-x^{-1}}$ directly if in the latter approach we had used the transformation $z=xy$, $w=\arctan y/x$ instead of the conventional transformation $(x, y) \to (r, \theta)$ to polar coordinates. Similar remarks hold in higher dimensions.

5. Asymptotic Independence

For completeness we derive by our methods two results of Sibuya (1960) concerning asymptotic independence and asymptotic full dependence of the components of the vector of maxima. We suppose that the vector of maxima converges to a limit df and for ease of writing we assume symmetric convergence to a simple stable df. Asymptotic independence then carries over to the general case.

Theorem 5 (Sibuya). *Suppose* (X, Y) *has* df $F(x, y)$. F *is in the domain of symmetric attraction of* $\exp\{-x^{-1}+y^{-1})\}$, $x \geqq 0$, $y \geqq 0$ *(the simple stable df which is a product of its marginals) iff*

(i) $P[X>x] \sim P[Y>x] \sim x^{-1} L(x)$ *where* $L(x)$ *is slowly varying as* $x \to \infty$ *and*

(ii) $\lim_{x \to \infty} P[Y>x \mid X>x]=0$.

Remarks. (1) In the case of convergence to a non-simple stable df, (ii) remains the same provided the marginals of F are equal (replace "$x \to \infty$" by $x \uparrow x_0$, the right endpoint of the common marginal df).

(2) For asymptotic independence of k-dimensional extremes one has to require for each subset $\{i_1, i_2, \ldots, i_r\}$ of $\{1, 2, \ldots, k\}$

$$\lim_{x \to \infty} P\{X_{i_1}>x, \ldots, X_{i_r}>x \mid X_1>x\}=0. \tag{24}$$

This becomes apparent from the multidimensional analogue of (26).

Proof. Suppose F is symmetrically attracted to $\exp\{-(x^{-1}+y^{-1})\}$. From marginal convergences, there exist $a_n \uparrow \infty$ such that

$$\lim_{n \to \infty} n\{1-F(a_n x, \infty)\} = \lim_{n \to \infty} n\{1-F(\infty, a_n x)=x^{-1} \tag{25}$$

and moreover

$$\lim_{n \to \infty} n\{1-F(a_n x, a_n x)\}=2x^{-1}.$$

From

$$P[X>x, Y>x]=(1-F(x, \infty))+(1-F(\infty, x))-(1-F(x, x)) \tag{26}$$

we immediately get (ii). It is well known that (i) is necessary and sufficient for marginal converges with $a_n^{(1)}=a_n^{(2)}=a_n$.

Conversely suppose (i) and (ii) hold. From (i) we have $P[X>ta]\sim a^{-1}P[X>t]$, as $t\to\infty$ $\forall a>0$ so by (ii)

$$\lim_{t\to\infty}\frac{P[X>tx,\, Y>ty]}{P[X>t]}=0$$

and picking a_n to satisfy $P[X>a_n]=n^{-1}$ we have

$$\lim_{n\to\infty} nP[X>a_n x,\, Y>a_n y]=0.$$

Therefore

$$\lim_{n\to\infty} n(1-F(a_n x, a_n y))=\lim_{n\to\infty} n\{(1-F(a_n x,\infty))+(1-F(\infty, a_n y)) - P[X>a_n x,\, Y>a_n y]\}=x^{-1}+y^{-1}$$

and this is equivalent to F being symmetrically attracted to $\exp\{-(x^{-1}+y^{-1})\}$.

Example 8. Sibuya (1960) showed that partial maxima of the components of a bivariate normal distribution with correlation coefficient ρ, $|\rho|<1$, are asymptotically independent. Asymptotic independence also holds for the two-dimensional df

$$F(x,y)=\exp\{-(x^{-1}+y^{-1}+ax^{-1}y^{-1})\} \qquad \text{for } x,y>0 \ (a>0).$$

It is easy to check the conditions of Theorem 5 are satisfied or alternatively verify directly that

$$\{1-F(t,t)\}^{-1}\{1-F(tx,ty)\}\to\frac{x^{-1}+y^{-1}}{2} \qquad \text{as } t\to\infty.$$

Example 9. Suppose F is the joint df of $(X,-X)$ and that for $x\to\infty$

$$P[X>x]\sim P[X<-x]\sim x^{-1}L(x).$$

Since $P[X>x,\,-X>x]=0$ for $x>0$ we have (ii) of Theorem 5 is satisfied. Thus if $\{X_n, n>1\}$ are iid copies of X there exist $a_n\to\infty$ such that

$$P\left[\frac{\bigvee_{i=1}^n X_i}{a_n}\leqq x,\ \frac{-\bigwedge_{i=1}^n X_i}{a_n}\leqq y\right]\to\Phi_1(x)\,\Phi_1(y)$$

and consequently a limit law for the range ensues:

$$P\left[\frac{\bigvee_{i=1}^n X_i-\bigwedge_{i=1}^n X_i}{a_n}\leqq x\right]\to\Phi_1 * \Phi_2(x).$$

Cf. de Haan, 1974.

We have the following counterpart of Theorem 5.

Theorem 6 (Sibuya). *Suppose* (X, Y) *has* df $F(x, y)$. *F is in the domain of symmetric attraction of* $G(x, y)=\exp\{-(x^{-1} \vee y^{-1})\}$, $x \geqq 0$, $y \geqq 0$ *(the simple stable df concentrating on the* 45° *line in the positive quadrant) iff*

(i) $P[X>x] \sim P[Y>x] \sim x^{-1} L(x)$ as $x \to \infty$ *where L is slowly varying and*

(ii) $\lim_{x \to \infty} P[Y>x \mid X>x]=1$.

Remark. For k-dimensional extremes the conditions are similar. In particular (ii) is replaced by

$$\lim_{x \to \infty} P[X_1>x, \ldots, X_k>x \mid X_1>x]=1. \tag{27}$$

This follows from the k-dimensional version of (28).

Proof. To see that (i) and (ii) are necessary, proceed in a manner analogous to the previous proof. For the converse suppose (i) and (ii) hold and note for $t, x, y>0$ with $y>x$:

$$\frac{P[X>ty, Y>ty]}{P[X>t]} \leqq \frac{P[X>tx, Y>ty]}{P[X>t]} \leqq \frac{P\{Y>ty\}}{P\{X>t\}}. \tag{28}$$

Hence by (i) and (ii):

$$\lim_{t \to \infty} P[X>tx, Y>ty]/P[X>t]=y^{-1}$$

and replacing t by a_n we see

$$\begin{aligned}\lim_{n \to \infty} n(1-F(a_n x, a_n y)) &= \lim_{n \to \infty} n\,(1-F(a_n x, \infty)+1-F(\infty, a_n y) \\ &\quad -P[X>a_n x, Y>a_n y]) \\ &= x^{-1}+y^{-1}-y^{-1}=x^{-1}=x^{-1} \vee y^{-1}\end{aligned}$$

as required.

6. Multidimensional Extremal Processes

Here we collect some results about multidimensional extremal processes in R^k. Let $\mathbf{Y}(t)=(Y_1(t), \ldots, Y_k(t))$ be an extremal process generated by the max-id df F according to (3). From the form of the joint distribution of $\mathbf{Y}(t_1), \ldots, \mathbf{Y}(t_n)$ given by (3) it is clear that $\mathbf{Y}$ is a Markov process in R^k with stationary transition probabilities. Again from (3) it follows that regular versions of the transition probabilities are

$$\begin{aligned}&P_{x_1, \ldots, x_k}[Y_i(t) \leqq y_i, i=1, \ldots, k] \\ &\quad := P[Y_i(t+s) \leqq y_i, i=1, \ldots, k \mid Y_i(s)=x_i, i=1, \ldots, k] \\ &\quad = F^t(y_1, \ldots, y_k)\, 1_{[y_i \geqq x_i, i=1, \ldots, k]}.\end{aligned}$$

The process $\mathbf{Y}$ is in fact a Markov jump process and we will compute the parameters governing holding times and jumps. To facilitate this we compute the generator w. The computation is conducted for $k=2$. For f a bounded and continuous function $R^2 \to R$ we have for wf:

$$\begin{aligned} wf(x_1, x_2) &= \lim_{t \downarrow 0} t^{-1} E_{x_1, x_2}(f(Y_1(t), Y_2(t)) - f(x_1, x_2)) \\ &= \lim_{t \downarrow 0} t^{-1} \iint (f(y_1, y_2) - f(x_1, x_2))\, P_{x_1, x_2}[Y_1(t) \in dy_1, Y_2(t) \in dy_2]. \end{aligned}$$

Since

$$\begin{aligned} P_{x_1,x_2}&[Y_1(t) \in dy_1, Y_2(t) \in dy_2] \\ &= P[Y_1(t) \in dy_1, Y_2(t) \in dy_2]\, 1_{[y_1 > x_1, y_2 > x_2]} \\ &\quad + P[Y_1(t) \leqq x_1, Y_2(t) \in dy_2]\, 1_{[y_1 = x_1, y_2 > x_2]} \\ &\quad + P[Y_1(t) \in dy_1, Y_2(t) \leqq x_2]\, 1_{[y_1 > x_1, y_2 = x_2]} \end{aligned}$$

and recalling $t^{-1} P[Y_1(t), Y_2(t) \in \cdot] \Rightarrow \nu(\cdot)$ as $t \downarrow 0$ where ν is the exponent measure of F (Balkema and Resnick, 1977, Th. 6) we have:

$$\begin{aligned} wf(x_1, x_2) = \iint (f(y_1, y_2) - f(x_1, x_2)) \{ &1_{[y_1 > x_1, y_2 > x_2]} \nu(dy_1, dy_2) \\ &+ 1_{[y_1 > x_1, y_2 = x_2]} \nu(dy_1, [-\infty, x_2]) \\ &+ 1_{[y_1 = x_1, y_2 > x_2]} \nu([-\infty, x_1], dy_2)\}. \end{aligned}$$

Comparing the form just obtained with the canonical form of the generator for a Markov jump process (cf. Breiman, 1969, p. 331) we obtain the mean $\alpha^{-1}(x_1, x_2)$ of the holding time in state (x_1, x_2) and the conditional probability $\Pi((x_1, x_2); B)$ that starting from (x_1, x_2) the process jumps into B. For arbitrary k these quantities are given by

$$\begin{aligned} &\alpha((x_1, \ldots, x_k)) = \nu(A^c(x_1, \ldots, x_k)) \\ &\Pi((x_1, \ldots, x_k); A^c(y_1, \ldots, y_k)) = \frac{\nu(A^c(y_1, \ldots, y_k))}{\nu(A^c(x_1, \ldots, x_k))} \end{aligned} \tag{29}$$

for $y_i \geqq x_i$, $i = 1, \ldots, k$ where as usual

$$A(y_1, \ldots, y_k) = [-\infty, y_1] \times \cdots \times [-\infty, y_k].$$

More specifically:

Let τ be the time of the first jump after $t=1$.
Then

$$\begin{aligned} &P[\mathbf{Y}(\tau) \in A^c(y_1, \ldots, y_k) \mid Y_i(1) = x_i, i = 1, \ldots, k] \\ &\quad =: P_{(x_1, \ldots, x_k)}[\mathbf{Y}(\tau) \in A^c(y_1, \ldots, y_k)] = \frac{\nu(A^c(y_1, \ldots, y_k))}{\nu(A^c(x_1, \ldots, x_k))} \end{aligned}$$

for $y_i \geqq x_i$, $i = 1, \ldots, k$.

If $B=\{(t_1,\dots,t_k)|t_i>x_i, i=1,\dots,k\}$ then

$$P_{(x_1,\dots,x_k)}[\mathbf{Y}(\tau)\in A\cap B]=\frac{\nu(A\cap B)}{\nu(A^c(x_1,\dots,x_k))}$$

for any $A\in\mathscr{B}(R^k)$.

If $C=\{(t_1,\dots,t_k)|t_i=x_i$ for some $i=1,2,\dots,k$ and $t_j\geqq x_j$ for all $j=1,2,\dots,k\}$ then

$$P_{(x_1,\dots,x_k)}[\mathbf{Y}(\tau)\in A\cap C]=\frac{\nu((A\cap C)^*)}{\nu(A^c(x_1,\dots,x_k))}$$

with $(A\cap C)^*=\{(t_1,\dots,t_k)|$ there exists $y\in A\cap C$ with $t_i=y_i$ if $y_i>x_i$ and $t_j\leqq y_j$ otherwise$\}$ for any $A\in\mathscr{B}(R^k)$.

For processes generated by simple stable df's these formulas can be interpreted as follows: If T is the transformation to polar coordinates and $T\mathbf{Y}(t)=(\|\mathbf{Y}\|,\boldsymbol{\Theta})$ we have on sets A' such that $T^{-1}A'\subset B$:

$$P_{(x_1,\dots,x_k)}[(\|\mathbf{Y}\|,\boldsymbol{\Theta})\in A']=\int\int_{A'} r^{-2}\,dr\,S(d\boldsymbol{\theta})/\nu(A^c(x_1,\dots,x_k))$$

so that with respect to $P_{(x_1,\dots,x_k)}(\cdot)$ we have $\|\mathbf{Y}\|$ and $\boldsymbol{\Theta}$ independent when restricted to B.

Another independence result is given below which describes when the jumps of $\mathbf{Y}$ are iid random vectors. Preparatory to this discussion we discuss the range $\mathscr{R}(\mathbf{Y})$ which we define as

$$\mathscr{R}(\mathbf{Y})=\{\mathbf{x}|\forall \text{ open sets } 0\ni\mathbf{x},\ P[\mathbf{Y}(t)\in 0 \text{ for some } t]>0]\}.$$

For what follows we denote the support of a measure ν by supp ν.

To characterize $\mathscr{R}(Y)$ we need hitting probabilities for rectangles. This computation is done for $k=2$ and we seek $P[Y$ hits $(x_1,x_2]\times(y_1,y_2]]$ where $x_1<x_2, y_1<y_2$. Assume Y is related to a Poisson random measure as described in the introduction. Define $\sigma(A)=\inf\{T_k|(J_k^{(1)},J_k^{(2)})\in A\}$ to be the first time there is a point in $A\in\mathscr{B}(R^2)$. Then

$$\begin{aligned}&P[Y(t)\in(x_1,x_2]\times(y_1,y_2] \text{ for some } t]\\&\quad=P[\sigma((-\infty,x_2]\times(y_1,y_2])\vee\sigma((x_1,x_2]\times(-\infty,y_2])\\&\qquad<\sigma(A^c(x_2,y_2))].\end{aligned}$$

Note

$$\begin{aligned}&\sigma((-\infty,x_2]\times(y_1,y_2])\\&\quad=\sigma((-\infty,x_1]\times(y_1,y_2])\wedge\sigma((x_1,x_2]\times(y_1,y_2])=U\wedge V\end{aligned}$$

and

$$\sigma((x_1,x_2]\times(-\infty,y_2])=\sigma((x_1,x_2]\times(-\infty,y_1])\wedge V=W\wedge V.$$

Set $Z=\sigma(A^c(x_2,y_2))$ and the required probability is

$$P[(U\wedge V)\vee(W\wedge V)<Z]$$

where U, V, W, Z are independent and for any $A\in\mathscr{B}(R^2)$ $P[\sigma(A)>t]=e^{-t\nu(A)}$. Set $\lambda_1=\nu((x_1,x_2]\times(y_1,y_2])$, $\lambda_2=\nu((-\infty,x_1]\times(y_1,y_2])$, $\lambda_3=\nu((x_1,x_2]\times(-\infty,y_1])$, $\lambda_4=\nu(A^c(x_2,y_2))$. Performing the required calculation by capitalizing on independence gives

$$P[Y \text{ hits } (x_1,x_2]\times(y_1,y_2]]$$
$$=\lambda_4\left\{\frac{1}{\lambda_4}+\frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}-\frac{1}{\lambda_1+\lambda_3+\lambda_4}-\frac{1}{\lambda_1+\lambda_2+\lambda_4}\right\}$$

provided $\lambda_4>0$ while if $\lambda_4=0$ the required probability is $P[U\wedge V)\vee(W\wedge V)<\infty]=1$ since U, V, W are each exponentially distributed. If $\lambda_4>0$ we observe that the hitting probability is positive iff $\lambda_1+\lambda_2>0$ and $\lambda_1+\lambda_3>0$. This leads to:

Theorem 7. Let $\mathbf{Y}$ be extremal in R^k with df G and exponent measure ν. Then $(x_1,\ldots,x_k)\in\mathscr{R}(\mathbf{Y})$ iff for all $\varepsilon>0$

$$\nu\{(-\infty,x_1+\varepsilon]\times\cdots\times(-\infty,x_{i-1}+\varepsilon]\times(x_i-\varepsilon,x_i+\varepsilon]\times(-\infty,x_{i+1}+\varepsilon]$$
$$\times\cdots\times(-\infty,x_k+\varepsilon]\}>0 \quad \text{for } i=1,\ldots,k.$$

Equivalently we have

$$\mathscr{R}(\mathbf{Y})=\{(x_1,\ldots,x_k)\,|\,x_i=\sup\{y_i\,|\,\mathbf{y}\in A\},\ i=1,\ldots,k \text{ for some } A\subset\operatorname{supp}\nu\}.$$
$$=\operatorname{supp}\nu\vee\operatorname{supp}\nu\vee\cdots\vee\operatorname{supp}\nu(k\text{-times})=\operatorname{supp}G$$

(cf. Balkema and Resnick, 1977).

Notice when G is simple stable the support of ν has the form $\operatorname{supp}\nu=\{\mathbf{x}\,|\,\boldsymbol{\theta}\in\operatorname{supp}S\}$ where $T\mathbf{x}=(r,\boldsymbol{\theta})$ and $\nu\circ T^{-1}(dr,d\boldsymbol{\theta})=r^{-2}dr\,S(d\boldsymbol{\theta})$.

We now consider the following problem: Let $1<T_1<T_2<\cdots$ be the times $\mathbf{Y}$ jumps past $t=1$. For convenience set $T_0=1$. When is $\{\mathbf{Y}(T_n)-\mathbf{Y}(T_{n-1}), n\geqq 1\}$ a sequence of iid random vectors? We begin by reviewing and completing the situation for $k=1$ (cf. Resnick and Rubinovitch, 1973).

If Y is extremal in one dimension generated by $F(x)$ set $Q(x)=-\log F(x)=\nu(x,\infty)$. Suppose $a=\inf\{x\,|\,F(x)>0\}$. If the jumps of Y are iid then

$$\{Y(T_n), n\geqq 0\}\overset{d}{=}\left\{Z_0+\sum_1^n Z_j, n\geqq 0\right\} \tag{30}$$

where $\{Z_n, n\geqq 1\}$ are iid rv's with common df $H(x)$. Note (30) holds iff $\forall\, x\in\mathscr{R}(Y)$

$$1-Q(y)/Q(x)=H(y-x) \tag{31}$$

for $y\geqq x$ (cf. 29). The following facts are evident

(i) $\mathscr{R}(Y)=\operatorname{supp} \nu$,

(ii) $t \in \operatorname{supp} H$ iff $t \geqq 0$ and $t=x_2-x_1$ where $x_1, x_2 \in \operatorname{supp} \nu$. This follows from (31).

(iii) If $x_1, x_2 \in \mathscr{R}(Y)$ and $x_1<x_2$ then $\forall z \in \mathscr{R}(Y)$

$z+(x_2-x_1) \in \mathscr{R}(y)$.

This is clear since $x_2-x_1 \in \operatorname{supp} H$.

(iv) Either $\mathscr{R}(y)=(a, \infty)$

or $\mathscr{R}(y)=\{x_0+nd, -\infty<n<\infty$ and $x_0+nd \geqq a\}, d>0$.

This is easily seen once one defines

$$d=\inf\{y-x \mid y>x, x, y \in \mathscr{R}(Y)\}.$$

Thus one is led to the possible structure of $\mathscr{R}(Y)$ when independent jumps are present. Analyzing (31) leads to functional equations which Q must satisfy. These equations are easily solved and the result is: Y has iid jumps iff

(i) $\mathscr{R}(Y)=(a, \infty)$, $-\infty \leqq a$ and F is of type

$$F(x)=\begin{cases} e^{-e^{-x}}, & x \geqq a \\ 0 & x<a \end{cases}$$

or (ii) $\mathscr{R}(Y)=\{x_0+nd, \forall n$ such that $x_0+nd \geqq a\}$
and F concentrates on $\{x_0+nd\}$ and is of the form

$$F(x_0+nd)=\begin{cases} e^{-p^n} & \text{for } x_0+nd \geqq a \\ 0 & \text{otherwise} \end{cases}$$

where $0<p<1$.

We now consider the problem in R^k so suppose the jumps of $\mathbf{Y}(\cdot)=(Y_1(\cdot), \ldots, Y_k(\cdot))$ are iid vectors. We are going to prove that the process is then one-dimensional; i.e. that $\mathscr{R}(\mathbf{y})$ is contained in a straight line. Obviously the slope of this line has to be positive. Pick two arbitrary components of Y. These components constitute an extremal process in R^2 and the jumps are iid pairs. The desired result will be proved if we prove the result for any two components of $\mathbf{Y}$; i.e. it suffices to suppose $k=2$.

Suppose in order to get a contradiction the process is not concentrated on a line. Then there are points $(x_1, x_2), (y_1, y_2) \in \mathscr{R}(\mathbf{Y})$ with (say) $x_1 \leqq y_1, x_2>y_2$. It is evident that the following points must be in $\mathscr{R}(\mathbf{Y})$:

$$\{z(n, m)\}:=\{(y_1+n(y_1-x_1), x_2+m(x_2-y_2))\}$$

where $n \geqq -1$, $m \geqq -1$, n, m integers but we exclude $n=m=-1$. Define $g(n, m) = \nu\{A^c(z(n, m))\}$. Referring to (29) and using the asumption of iid jumps we have that $g(n+r, m+s)/g(n, m)$ does not depend on n or m for $r, s=0, 1, \ldots$. Call this ratio $f(r, s)$ so that

$$g(n+r, m)=g(n, m)\, f(r, 0).$$

From this we deduce

$$f(r+s,0)=f(r,0)\,f(s,0)$$

and thus $f(r,0)=e^{\alpha r}$ for some constant α which entails

$$g(n,m)=e^{\alpha(n-1)}g(1,m).$$

Similar analysis on the second variable shows

$$\begin{aligned} g(n,m)&=e^{\alpha(n-1)}e^{\beta(m-1)}g(1,1)\\ &=c\,e^{\alpha n}e^{\beta m} \end{aligned}$$

where c, α, β are constants and $c>0$.

Since g must be decreasing in n and m we must have $\alpha<0$, $\beta<0$.

Define sets

$$\begin{aligned} B_{n,m}=\{(z_1,z_2)|&y_1+(n-1)(y_1-x_1)<z_1\leqq y_1+n(y_1-x_1),\\ &x_2+(m-1)(x_2-y_2)<z_2\leqq x_2+m(x_2-y_2)\} \end{aligned}$$

for $n, m=1,2,\dots$ say and note

$$\begin{aligned} v(B_{n,m})&=-g(n-1,m-1)+g(n-1,m)+g(n,m-1)-g(n,m)\\ &=-c\,e^{\alpha n}e^{\beta m}(e^{-\alpha}-1)(e^{-\beta}-1)<0 \end{aligned}$$

which gives the desired contradiction.

Thus if $\mathbf{Y}$ has iid jumps then $\mathbf{Y}$ is one-dimensional. The structure of $\mathscr{R}(\mathbf{Y})$ and the possible distributions of the process are then obtained from the one-dimensional results.

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Received January 26, 1976