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Multiple Points for the Sample Paths of the Symmetric Stable Process

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1. Introduction

Properties of the sample path for the symmetric stable process of order α, $0 < \alpha \leqq 2$ in Euclidean n-space, R^n have been studied by many authors (see for example [*1, 2, 13, 16*]). It is well known that for $1 < \alpha \leqq 2$, $n = 1$, the process is point recurrent so that in fact for any given $x \in R^1$ there is probability 1 that x will be visited $\boldsymbol{c}$ times ($\boldsymbol{c}$ is the cardinal of the continuum) and there is therefore no problem about the existence of multiple points. For other values of α, n the process is not point recurrent but there may be *some* points which the path visits more than once. A point visited at least twice is called a double point, while if it is visited at not less than k different time instants we call it a k-multiple point. In the present paper we completely settle the question of the existence of k-multiple points for $0 < \alpha < 2$ and go on to consider the extent of the set of k-multiple points when this is known to be non-void. The case $\alpha = 2$ corresponds to Brownian motion and has previously been completely settled in the series of papers [*6 — 9*] mainly due to DVORETZKY, ERDÖS and KAKUTANI. For $k = 2$ and other values of α, TAKEUCKI [*16*] has shown that double points exist with probability 1 if $2 \geqq \alpha > \frac{1}{2} n$, but that for $0 < \alpha \leqq \frac{1}{2} n$ the path enters no point twice with probability 1. We will make use of the estimates in [*16*], though our method has to be somewhat different.

We now summarize our main results: $X(t)$, $t \geqq 0$ will denote the symmetric stable process of order α in R^n.

Theorem 1. *For each positive integer k,*

(i) *if $2 \geqq \alpha > n(k-1)/k$, with probability 1 there is a point $\zeta \in R^n$ and k distinct times $0 \leqq t_1 < t_2 < \cdots < t_k$ such that*

$$X(t_i) = \zeta, \qquad i = 1, 2, \ldots, k:$$

(ii) *if $0 < \alpha \leqq n(k-1)/k$, with probability 1 there is no point $\zeta \in R^n$ which is entered k times by $X(t)$.*

The case $\alpha = 1 = n$ corresponds to the Cauchy process. It was proved by MCKEAN [*13*] that the path set of the Cauchy process has zero Lebesgue measure and that the hitting probability of any fixed point is zero. Theorem 1 (i) says, however, that the Cauchy process has points of arbitrarily high finite multiplicity with probability 1. This result was obtained for Brownian motion in the plane ($\alpha = 2 = n$) by DVORETZKY, ERDÖS and KAKUTANI [7] who went on to show in

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[9] that in this case there are $\boldsymbol{c}$-multiple points. We can use the estimates of this paper to adapt the arguments of [9], and prove:

Theorem 2. *Almost all Cauchy processes on the line have* $\boldsymbol{c}$*-multiple points.*

These two theorems together mean that for each allowable α, n there is a $k = k(\alpha, n)$ such that the path set of $X(t)$ has maximum multiplicity k (here k is either a positive integer or $\boldsymbol{c}$). Thus

if $n \geqq 4$, $k = 1$ for $0 < \alpha \leqq 2$;
if $n = 3$, $k = 1$ for $0 < \alpha \leqq \frac{3}{2}$, $k = 2$ for $\frac{3}{2} < \alpha \leqq 2$;
if $n = 2$, $k = 1$ for $0 < \alpha \leqq 1$, $k = r$ for $\frac{r-1}{r} < \frac{1}{2}\alpha \leqq \frac{r}{r+1}$,
$r = 2, 3, \ldots$, $k = \boldsymbol{c}$ for $\alpha = 2$;
if $n = 1$, $k = 1$ for $0 < \alpha \leqq \frac{1}{2}$, $k = r$ for $\frac{r-1}{r} < \alpha \leqq \frac{r}{r+1}$,
$r = 2, 3, \ldots$, $k = \boldsymbol{c}$ for $2 \geqq \alpha \geqq 1$.

Whenever the set of k-multiple points is known to be nonvoid it is of interest to ask questions about its nature. It is easy to see that it is a Borel set and that it has cardinal $\boldsymbol{c}$ with probability 1, so it becomes interesting to ask for its dimensional number in the sense of Hausdorff-Besicovitch. This question was settled for the whole path set by McKean [13] and Blumenthal and Getoor [1]. For Brownian motion it was conjectured in [8] that in the plane ($\alpha = 2 = n$) the set of k-multiple points has dimension 2 for each k, while in R^3 the set of double points for the Brownian path has dimension 1 with probability 1. The upper bounds in each case are easy but previously used techniques do not seem to yield any lower bound for $k \geqq 2$. In the present paper we use the technique of trying to 'hit' the set with an independent symmetric stable process of order β for different values of β; and solve the problem completely for $n = 1$ or 2. Unfortunately this method will not work for $n \geqq 3$ (because we cannot use values of $\beta > 2$). We obtain:

Theorem 3. *For each positive integer* k:

(i) *if* $n = 2$, $2 \geqq \alpha > 2\,\frac{k-1}{k}$, *the set* E_k *of* k*-multiple points of the process* $X(t)$ *has dimension* $k\alpha - 2(k-1)$ *with probability* 1;

(ii) *if* $n = 1$, $1 \geqq \alpha > \frac{k-1}{k}$, *the set* E_k *of* k*-multiple points of the process* $X(t)$ *has dimension* $k\alpha - (k-1)$ *with probability* 1.

This theorem includes the conjectured result ($\alpha = 2 = n$) that the set of k-multiple points for Brownian motion in the plane has dimension 2, and a similar result for the Cauchy process on the line. It is clear that for $\alpha > 1 = n$, the set of $\boldsymbol{c}$-multiple points has dimension 1 (it actually has positive Lebesgue measure). By theorem 2 we can examine $\boldsymbol{c}$-multiple points for the Cauchy process in R^1 as well as Brownian motion in R^2. It turns out that both these sets have maximum dimension.

(iii) *The set of* $\boldsymbol{c}$*-multiple points of a Brownian path in the plane has dimension* 2 *with probability* 1.

(iv) *The set of* $\boldsymbol{c}$*-multiple points of a Cauchy process on the line has dimension* 1 *with probability* 1.

We have been unable to prove that the set of double points for the process of order $\alpha (> \frac{3}{2})$ in R^3 has dimension $(2\alpha - 3)$, though this seems very likely.

The paper is arranged as follows. Section 2 summarizes the main properties of the symmetric stable process which we will require, and goes on to give a sequence of estimates of various probabilities needed for later computations. Section 3 gives the relationship between Hausdorff measure and capacity and establishes the method of finding the dimension of any Borel or analytic set in R^2 or R^1 by using stable processes of different orders. In section 4 we establish the existence of k-multiple points for the right values of $\alpha < n$ and in section 5 obtain a restriction on the Hausdorff measure of this set which enables us to complete the proof of theorem 1 (ii). Section 6 deals with the remaining case ($\alpha = 1 = n$) of the Cauchy process, indicating how theorem 2 may be proved by modifying the arguments of [*9*]. Finally in the last section we examine the dimensional number of the set of k-multiple points and prove theorem 3.

2. Preliminary Results and Notation

The symmetric stable process in R^n of order α, $0 < \alpha < 2$ will be denoted $X_{\alpha, n}(t, \omega)$. When α, n are fixed they will usually be omitted and we will normally suppress ω as well and simply talk of the process $X(t)$. Thus $X_{\alpha, n}(t, \omega)$ is a Markov process with stationary independent increments, whose transition density $f_{\alpha, n}(t, x - y)$ relative to Lebesgue measure in R^n is uniquely determined by its Fourier transform

$$e^{-t|\xi|^\alpha} = \int_{R^n} e^{i(x, \xi)} f_{\alpha, n}(t, x)\, dx$$

where ξ, $x \in R^n$, (x, ξ) is the ordinary inner product in R^n and $|x| = (x, x)^{1/2}$ is the usual Euclidean norm. It is worth remarking that the transition densities, and therefore the process, are invariant under a change of scale in which t is replaced by λt and x by $\lambda^{-1/\alpha} x$. Only for $\alpha = 1 = n$ (the Cauchy process), and $\alpha = 2 = n$ (Brownian motion in the plane) can the transition density be written down in simple form. We write P_x and E_x for the conditional probability and expectation given $X(0) = x$. Unless otherwise stated we will assume that $X(0) = 0$ with probability 1 and suppress the suffix so that $P = P_0$, $E = E_0$. For all statements that follow, other than formal theorems, we will suppress the words 'with probability 1'; so that for any given statement there may be a set of zero probability in the underlying probability space on which the statement is false.

Using the terminology introduced in [*3*], we will assume that we have a version of process which is a *Hunt process*, that is we assume hypothesis (A) of Hunt [*11*]. The fact that such a version exists can be checked by showing that our transition functions satisfy the conditions of paragraph 1.7 of Getoor [*10*]. This paper also contains an admirable summary of the meaning of the statement that $X(t)$ is a Hunt process. We state only the more important properties which are needed in the sequel.

P_1. The function $X(t)$ is continuous on the right and has left hand limits everywhere.

P_2. The strong Markov property is satisfied. This loosely means that, if $T(\omega)$ is a stopping time, the behavior of the process for $t \geq T(\omega)$ given $X(T(\omega))$ is independent of the path up to $T(\omega)$.

P_3. The process is quasi-left continuous. This means that if $\{T_n\}$ is an increasing sequence of stopping times with limit T, then $X(T_n) \to X(T)$ almost surely on $\{T < +\infty\}$.

For any analytic set $A \subset R^n$,

$$T_A(\omega) = \inf\{t > 0 : X(t, \omega) \in A\}$$

is called the hitting time (or entry time) of A: it is understood that if the set in braces is empty then $T_A(\omega) = \infty$. We use the fact that $T_A(\omega)$ is a stopping time for any such A.

We need a notation for the part of the sample path corresponding to the time interval $[a, b]$. Thus

$$\begin{aligned} L(a, b; \omega) &= L_{\alpha, n}(a, b; \omega) \\ &= \{x \in R^n : X_{\alpha, n}(t, \omega) = x \text{ for some } t, \quad a \leqq t \leqq b\}. \end{aligned}$$

Similarly we denote the set of k-multiple points by

$$L^k(a, b; \omega) = \{x \in R^n : X(t, \omega) = x \text{ for } k \text{ different } t,\ a \leqq t \leqq b\}.$$

Fortunately the probability of certain events which are important in our argument has been estimated by previous authors — or can be easily deduced from known estimates. For convenience we now list a number of such results as lemmas. In the present section we give results only for the transient case $0 < \alpha \leqq 2$, $\alpha < n$. Estimates will not be needed for $\alpha > 1 = n$, and they take a completely different form for $\alpha = 1 = n$ and $\alpha = 2 = n$. Where possible, we use probabilistic arguments rather than analytic ones. Constants whose value is not important occur frequently. We will use $c_1, c_2, \ldots, c_{32}$ to denote positive real numbers whose value may depend on α and n but is independent of all other parameters.

If a transient process starts inside a sphere in R^n it is not certain to return to it for values of $t \geqq T_0$; similarly if it starts outside the sphere it need never hit it. The first two lemmas give estimates for these probabilities.

Lemma 1. *If $|x| > r > 0$, then*

$$c_1\left(\frac{r}{|x| + r}\right)^{n-\alpha} \leqq P_x\{|X(t)| \leqq r \text{ for some } t > 0\} \leqq c_1\left(\frac{r}{|x| - r}\right)^{n-\alpha}.$$

This is lemma 2 of [*16*].

Lemma 2. *If $T^{1/\alpha} \geqq r > 0$ and $|x| \leqq r$, put*

$$Q(x, r, T) = P_x\{|X(t)| \leqq r \text{ for some } t \geqq T\}.$$

Then

$$Q(x, r, T) \geqq c_2\left(\frac{r}{T^{1/\alpha}}\right)^{n-\alpha}.$$

Proof. It is clear that

$$Q(x, r, T) \geqq \int_{|y| > r} P_y\{|X(t)| \leqq r \text{ for some } t > 0\}\, p(y - x, T)\, dy$$

where $p(y, T)$ is the transition density of the process. Using the lower estimate in lemma 1 and making a standard change of scale gives, on using $r\,T^{-1/\alpha} \leqq 1$,

$$Q(x, r, T) \geqq c_1 r^{n-\alpha} \int_{|y|>r} (|y| + r)^{-n+\alpha} p(y - x, T)\, dy$$

$$\geqq c_1 (\tfrac{1}{2} r)^{n-\alpha} \int_{|y|>r} |y|^{-n+\alpha} p(y - x, T)\, dy$$

$$= c_1 \left(\frac{r}{2\,T^{1/\alpha}}\right)^{n-\alpha} \int_{|y|>r T^{-1/\alpha}} |y|^{-n+\alpha} p(y - x\,T^{-1/\alpha}, 1)\, dy$$

$$\geqq c_1 \left(\frac{r}{2\,T^{1/\alpha}}\right)^{n-\alpha} \int_{|y|>1} |y|^{-n+\alpha} p(y - x\,T^{-1/\alpha}, 1)\, dy$$

$$\geqq c_2 \left(\frac{r}{T^{1/\alpha}}\right)^{n-\alpha},$$

since $p(y, 1)$ is bounded below by a positive constant in the region $0 \leqq |y| \leqq 3$ and this lower bound will be a lower bound for $p(y - x\,T^{-1/\alpha}, 1)$ in $1 < |y| \leqq 2$, since $|x|\,T^{-1/\alpha} \leqq 1$.

Lemma 3. *For $T > 0$, $r > 0$ and all x*

$$Q(x, r, T) \leqq c_3 \left(\frac{r}{T^{1/\alpha}}\right)^{n-\alpha}.$$

This is proved in [*16*] for $x = 0$. Only obvious modifications are needed to give the result for general x.

Lemma 4. *If $|x| > r > 0$ and $T^{1/\alpha} > c_4 |x|$, then*

$$P_x\{|X(t)| \leqq r \text{ for some } t \in [0, T]\} > c_5 \left(\frac{r}{|x| + r}\right)^{n-\alpha}.$$

Proof. Clearly the required probability is at least

$$P_x\{|X(t)| \leqq r \text{ for some } t > 0\} - P_x\{|X(t)| \leqq r \text{ for some } t \geqq T\}$$

$$> c_1 \left(\frac{r}{|x| + r}\right)^{n-\alpha} - c_3 \left(\frac{r}{T^{1/\alpha}}\right)^{n-\alpha} > \tfrac{1}{2} c_1 \left(\frac{r}{|x| + r}\right)^{n-\alpha}, \text{ provided } T^{1/\alpha} > c_4 |x|.$$

This lemma shows that, provided $|x|$ is not large compared to $T^{1/\alpha}$, the probability of entering a sphere in $[0, T]$ is of the same order as the probability of entering at all. We need to show also that the probability of returning to a sphere in $[T_1, T_2]$ is of the same order as the probability of returning in $[T_1, \infty]$. This is

Lemma 5. *For $r > 0$, $T^{1/\alpha} \geqq r \geqq |x| \geqq 0$, provided $T_2 > c_6 T_1$,*

$$P_x\{|X(t)| \leqq r \text{ for some } t \in [T_1, T_2]\} > c_7 \left(\frac{r}{T_1^{1/\alpha}}\right)^{n-\alpha}.$$

This can be deduced from our lemmas 2 and 3 in the same way that lemma 5 is deduced from lemmas 3 and 4 in [*16*].

Finally we need a lemma which strengthens the result that the process $X(t)$ is continuous in probability. It follows from $P\{|x(1)| > 1\} > 0$ and the fact that we have independent increments by a standard argument (see, for example, Stein [*15*]).

Lemma 6. *For $\lambda > 8$,*

$$P\{\sup_{0 \leqq \tau \leqq t} |x_{\alpha, n}(\tau)| \geqq \lambda t^{1/\alpha}\} < e^{-\lambda c_8}.$$

3. Riesz Capacity and Hausdorff Dimension

Since we are considering a Hunt process, the probability

$$\Phi_{\alpha,n}(x, A) = P_x\{X_{\alpha,n}(t) \in A \text{ for some } t > 0\}$$

that a symmetric stable process starting at x will hit the analytic set $A \subset R^n$ is defined. This function is the 'natural' potential of A associated with the process. Many authors have observed that the general theory of Hunt [*11*] implies that when A is a compact set K of positive $(n - \alpha)$ Riesz capacity, then $\Phi_{\alpha,n}(x, K)$ is the value of the equilibrium Riesz potential of order $(n - \alpha)$ of K at x: that is

$$\Phi_{\alpha,n}(x, K) = \lambda(K) \int_K |x - y|^{-n+\alpha} \mu_K(dy)$$

where μ_K is the equilibrium distribution on K and $\lambda(K)$ is a constant. Further, when the Riesz capacity of K is zero, $\Phi_{\alpha,n}(x, K)$ is zero for all x. There are several different definitions of β-capacity, but this will not worry us as the numerical value will play no role and the class of compact sets of zero β-capacity is the same for all of them (see for example, [*18*]). The definition of β-capacity can be extended (see, Brelot [*4*]) to the class of capacitable sets which includes the analytic subsets of R^n. Since $\Phi_{\alpha,n}(x, A)$ was also defined for analytic A we can state ($C_\beta(A)$ denotes some definition of the β-capacity of A):

Lemma 7. *For $n > \alpha$, $x \in R^n$ and any analytic $A \subset R^n$,*

$$\Phi_{\alpha,n}(x, A) = P_x\{X_{\alpha,n}(t) \in A \text{ for some } t > 0\}$$

is positive or zero according as $C_{n-\alpha}(A)$ is positive or zero.

There is also a version of this result for the special cases of the Cauchy process $(\alpha = 1 = n)$ and Brownian motion is the plane $(\alpha = 2 = n)$ which involves the use of logarithmic potential and capacity. This will not be required explicitly.

It is clear from any of the definitions of $C_\alpha(A)$ that if $\alpha < \beta$,

$$C_\alpha(A) = 0 \Rightarrow C_\beta(A) = 0 \quad \text{and} \quad C_\beta(A) > 0 \Rightarrow C_\alpha(A) > 0.$$

We can therefore define the capacity dimension of A by

$$C - \dim(A) = \inf\{\beta : C_\beta(A) = 0\}.$$

If $A \subset R^n$, $C_\beta(A) = 0$ for $\beta > n$ so that $C-\dim(A) \leq n$. Now the Hausdorff dimension of any subset of R^n can be defined in terms of the Hausdorff measures Λ^α with respect to the measure function h^α. Since it is known (see [*18*] for references) that

$$C_\alpha(A) > 0 \Rightarrow \Lambda^\alpha(A) = +\infty,$$
$$C_\alpha(A) = 0 \Rightarrow \Lambda^\beta(A) = 0,\ \beta > \alpha;$$

it follows that the Hausdorff dimension

$$\dim(A) = \inf\{\beta : \Lambda^\beta(A) = 0\}$$

is the same as the capacity dimension for any analytic set A. It is worth remarking that, though this correspondence is exact for these simple power functions h^α it does not apply to finer distinctions like $h^\alpha((\log 1/h))^\beta$ — see [*18*].

We now state as a theorem a technique which utilizes the symmetric stable processes of varying order to determine the Hausdorff dimension of an analytic set.

Theorem 4. *Suppose A is an analytic subset of the line or the plane. Then, for any point x,*

$$\text{if } A \subset R^1, \dim(A) = 1 - \inf\{\alpha: \Phi_{\alpha,1}(x, A) > 0\};$$
$$\text{if } A \subset R^2, \dim(A) = 2 - \inf\{\alpha: \Phi_{\alpha,2}(x, A) > 0\}.$$

Proof. We write out the argument only for $A \subset R^1$. Suppose first that α is such that $\Phi_{\alpha,1}(x, A) > 0$. Then, by lemma 7, $C_{1-\alpha}(A) > 0$ which implies that $\Lambda^{1-\alpha}(A) = +\infty$. This in turn gives $\dim(A) \geqq 1 - \alpha$, so that

$$\dim(A) \geqq 1 - \inf\{\alpha: \Phi_{\alpha,1}(x, A) > 0\}.$$

Conversely, if $\beta < \dim(A)$ there is a compact $K \subset A$ such that $\beta < \dim(K) = C - \dim(K)$ so that $C_\beta(K) > 0$. By lemma 7 we now have $\Phi_{\gamma,1}(x, A) > 0$ for $\gamma = 1 - \beta$. Hence

$$\inf\{\alpha: \Phi_{\alpha,1}(x, A) > 0\} \leqq 1 - \beta$$

for every $\beta < \dim(A)$, and this completes the proof.

Remark. It is clear that one can state a corresponding theorem for subsets of R^n, $n \geqq 3$ but it is incomplete since the order α of a stable process is restricted by $0 < \alpha \leqq 2$. The technique will therefore only work in R^n, $n \geqq 3$ for analyzing sets whose dimension is known to be greater than $(n - 2)$.

Previous workers have used the result of lemma 7 by first obtaining information about the capacity of a set and making a deduction about the hitting probability. We will use it in the reverse direction in section 7 — making deductions about the β-capacity from information about the hitting probability by a process of order $(n - \beta)$.

4. Existence of Points of Multiplicity k

In the present section we restrict our attention completely to the transient processes $(n > \alpha)$. It is clear that for $\alpha > 1 = n$ there is no problem since every point is entered arbitrarily often — in fact any given point is a point of multiplicity $\boldsymbol{c}$ (see [*2*], where Blumenthal and Getoor actually determine the Hausdorff dimension of the set of time instants at which a fixed point is visited). Of the remaining cases, $\alpha = 2 = n$ is Brownian motion in the plane which was settled in [*7*] and [*9*], and we will deal with the Cauchy process ($\alpha = 1 = n$) in section 6.

The first step is to show that there is a constant $c_9 > 0$ such that, for any $\varrho > 0$, there is a probability of at least c_9 that there are k time instants $0 \leqq t_1 < t_2 < \cdots < t_k \leqq c_{10}$ with $t_j - t_{j-1} \geqq 1$ $(j = 2, \ldots, k)$ such that $X(t_i)$ $(i = 1, 2, \ldots, k)$ are all within some sphere of radius ϱ. In fact this will follow if we can show that there is a fixed finite set, depending on ϱ, of spheres of radius ϱ for which the probability that at least one of them contains k entries by the process in $[0, c_{10}]$ — the successive entries all separated by at least unit time — is not less than c_9. This is the essential content of lemma 9. In order to save writing we will state and prove lemmas 8 and 9 for processes in the plane $(n = 2)$. It is clear that dimensionality changes are all that is needed to cover the cases $n = 1,3$. Values of $n \geqq 4$ are not relevant to the present section since we know that even double points cannot exist in this case.

Now put $U = \max(c_6, c_4)$, $c_{10} = kU$. This means that, if $|x| < 1$ in lemma 4 and $T < 1$ then the estimates of lemma 4 for entry in $[0, U]$ and lemma 5 for reentry in $[1, U]$ are valid. This allows us to state

Lemma 8. *Suppose k is a positive integer and $\alpha > 2(k-1)/k$. Let S_1, S_2 denote discs in R^2 with centers x_1, x_2 and radius ϱ, where $\frac{1}{4} \leqq |x_1| \leqq \frac{1}{2}$, $\frac{1}{4} \leqq |x_2| \leqq \frac{1}{2}$ and $0 < 10\,\varrho < \min\{|x_1|, |x_2|, |x_1 - x_2|\}$. Let $E_i\,(i = 1, 2)$ be the event that there are k time instants $t_1, t_2, \ldots, t_k$ with $0 \leqq t_1 \leqq U$, $1 \leqq t_j - t_{j-1} \leqq U$ $(j = 2, 3, \ldots, k)$ such that*

$$X_{\alpha,2}(t_j) \in S_i \qquad (j = 1, 2, \ldots, k)\,.$$

Then

$$P(E_i) > c_{11}\varrho^{k(2-\alpha)}, \quad i = 1, 2\,;$$
$$P(E_1 \cap E_2) < c_{12}\,|x_1 - x_2|^{-k(2-\alpha)}\,\varrho^{2k(2-\alpha)}\,.$$

Proof. A lower bound for the probability that $X_{\alpha,2}(t)$ will hit S_i in $[0, U]$ is given by the lemma 4. If the process hits S_i in $[0, U]$, let t, be the first entry time. Since this is a stopping time, we can restart the process at $X(t_1)$ and apply the strong Markov property. The conditional probability of a further reentry to S_i in $[1, U]$ is then given by lemma 5. Repeating this argument $(k-1)$ times gives easily

$$P(E_i) > c_{11}\varrho^{k(2-\alpha)}\,.$$

If $\omega \in E_1 \cap E_2$, the process makes at least $2k$ entries $t_1 < t_2 < \ldots < t_{2k}$ into $S_1 \cup S_2$ in $[0, c_{10}]$ such that, for at least k of the integers $i\,(2 \leqq i \leqq 2k)$, $t_i - t_{i-1} \geqq \frac{1}{2}$. Let N_k be the number of ways of choosing k integers out of $2k$ and denote by

μ_1, the probability of entering at least one of S_1, S_2;

μ_2, the upper bound of the probability of entering S_j starting from a point of S_i $(i \neq j)$; and

μ_3, the upper bound of returning to S_i after time $\frac{1}{2}$ starting from a point $x \in S_i$.

It is clear that

$$P(E_1 \cap E_2) < N_k\,\mu_1\,\mu_3^{k-1}\,(\mu_2 + \mu_3)^k\,.$$

Now

$$\mu_1 \leqq 2c_1(5\varrho)^{2-\alpha}, \quad \text{by lemma 1};$$
$$\mu_2 \leqq c_1(2\varrho)^{2-\alpha}\,|x_1 - x_2|^{-2+\alpha}, \quad \text{by lemma 1};$$
$$\mu_3 \leqq c_3\,\varrho^{2-\alpha}, \quad \text{by lemma 3}.$$

Noticing that $|x_1 - x_2| \leqq 1$, it now follows easily that

$$P(E_1 \cap E_2) < c_{12}\,\varrho^{2k(2-\alpha)}\,|x_1 - x_2|^{-k(2-\alpha)}\,.$$

Lemma 9[1]. *Suppose k is a positive integer, and $2 > \alpha > 2(k-1)/k$. For each positive integer r consider the discs, S_ν centered at*

$$x_\nu = \left(\frac{1}{4} + \frac{\nu_1}{10r}, \frac{1}{4} + \frac{\nu_2}{10r}\right) \qquad \begin{array}{l} \nu_i = 1, 2, \ldots, r;\, i = 1, 2, \\ \nu = (\nu_1 - 1)r + \nu_2; \end{array}$$

of radius $\varrho_r = \lambda r^{-2/k(2-\alpha)}$.

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[1] I am indebted to Dr. J. TAKEUCHI for pointing out a computational error in an earlier version of lemmas 8, 9.

Let $E_\nu (\nu = 1, 2, \ldots, r^2)$ be the event that there exist time instants $t_1, t_2, \ldots, t_k$ with $0 \leqq t_1 \leqq U$, $1 \leqq t_j - t_{j-1} \leqq U (j = 2, \ldots, k)$ and that

$$X_{\alpha, 2}(t_j) \in S_\nu,\ j = 1, 2, \ldots, k.$$

Then, provided $\lambda = c_{13}$, $P\left(\bigcup_{\nu=1}^{r^2} E_\nu\right) \geqq c_9 > 0$ *for all r.*

Proof. It is clear that, at least for large r and small λ, all the conditions o lemma 8 are satisfied for each pair of discs $S_\nu, S_{\nu'}$. Using the estimates of lemma 8 gives

$$P\left(\bigcup_{\nu=1}^{r^2} E_\nu\right) \geqq \sum_{\nu=1}^{r^2} P(E_\nu) - \sum_{1 \leqq \nu < \nu' \leqq r^2} P(E_\nu \cap E_{\nu'})$$
$$> r^2 c_{11} \varrho_r^{k(2-\alpha)} - c_{12} \varrho_r^{2k(2-\alpha)} \sum_{1 \leqq \nu < \nu' \leqq r^2} |x_\nu - x_{\nu'}|^{k(2-\alpha)}.$$

Now, for fixed ν, the number of ν' such that

$$\frac{2^{s-1}}{r} \leqq |x_{\nu'} - x_\nu| \leqq \frac{2^s}{r}\ (s = 1, 2, \ldots, s_r)$$

is certainly less than 100.2^{2s+2}, $\left(2 > \frac{2^{s-1}}{r} \geqq 1\right)$. Hence

$$\sum_{1 \leqq \nu < \nu' \leqq r^2} |x_{\nu'} - x_\nu|^{-k(2-\alpha)} \leqq r^2 \sum_{s=1}^{s_r} 100.2^{2s+2} \left(\frac{2s-1}{r}\right)^{-k(2-\alpha)} < c_{14} r^4.$$

Hence

$$P\left(\bigcup_{\nu=1}^{r^2} E_\nu\right) > r^2 c_{11} \varrho_r^{k(2-\alpha)} - c_{12} c_{15} r^4 \varrho_r^{2k(2-\alpha)} > c_9 > 0$$

provided $c_{11} = 2 c_{12} c_{14} \lambda^2$.

Proof of Theorem 1 (i). We can now complete the proof apart from the special cases $\alpha = 1 = n$ and $\alpha = 2 = n$. Suppose now that $0 < \alpha < 2 = n$, and $\alpha > 2(k-1)/k$. Let Q_s be the event that there exists some disc D of radius $1/s$ and k time instants $t_1, t_2, \ldots, t_k$ with $0 \leqq t_1 \leqq U$, $1 \leqq t_j - t_{j-1} \leqq U$ $(j = 2, \ldots, k)$ such that all the $X(t_i)$, $i = 1, 2, \ldots, k$ are in D. Clearly, if r is large enough to ensure the $\varrho_r < 1/s$, the event $Q_s \supset \bigcup_{\nu=1}^{r^2} E_\nu$, using the notation of lemma 9. Thus

$$P(Q_s) \geqq c_9 > 0.$$

But now Q_s decreases as s increases so that, if $Q = \bigcap_{s=1}^{\infty} Q_s$,

$$P(Q) \geqq c_9 > 0.$$

Suppose now that $\omega \in Q$, so that $\omega \in Q_s$ for all s. For each s, pick a disk D_s and stopping times which satisfy

$$0 \leqq t_1^s \leqq U;\ 1 \leqq t_j^s - t_{j-1}^s \leqq kU,\ (j = 2, \ldots, k).$$

(To ensure t_i^s are stopping times it is important at each stage to choose the first entry or reentry time to D_s — for this reason we have had to relax the upper bound in the last inequality). Since all the values of t involved are in a bounded set we can choose a subsequence $\{s_i\}$ such that, for $j = 1, 2, \ldots, k$,

$$t_j^{s_i} \to \tau_j \quad \text{as} \quad i \to \infty ,$$

and the convergence is monotone. If $t_j^{s_i}$ decreases with i, then $X(t_j^{s_i}) \to X(\tau_j)$ as $i \to \infty$ since the sample functions are right continuous, and the same result holds if $t_j^{s_i}$ is increasing in i, since each element is a stopping time, by the quasi-left continuity. Hence if $\omega \in Q$, there exist values of the time $0 \leqq \tau_1 < \tau_2 < \cdots < \tau_k \leqq k\,U$ such that $\tau_j - \tau_{j-1} \geqq 1$ for $j = 2, 3, \ldots, k$ and

$$X(\tau_1) = X(\tau_2) = \cdots = X(\tau_k) .$$

Thus the probability of a k-multiple point for the path $L(0, k\,U; \omega)$ is at least $c_9 > 0$. By changing the scale, the probability of a k-multiple point for the path $L(0,1; \omega)$ is also at least $c_9 > 0$. By independence of distinct portions of the path, the probability of no k-multiple point in $L(0, n; \omega)$ is less than $(1 - c_9)^n$ which $\to 0$ as $n \to \infty$. Hence there is probability 1 that the path $L_{\alpha,2}(0, 1; \omega)$ has a k-multiple point provided $2 > \alpha > 2\,(k-1)/k$. It is clear that all the details of the proof work for $n = 1$ and 3 so that

(i) There are double points of $L_{\alpha,3}(0, 1; \omega)$ with probability 1 if $\alpha > \frac{3}{2}$;

(ii) There are k-multiple points of $L_{\alpha,1}(0, 1; \omega)$ with probability 1 if $1 > \alpha > (k-1)/k$.

It may be worth remarking that the techniques of the present section will also yield part (i) of the following theorem.

Theorem 5. *Suppose $\omega_1, \omega_2, \ldots, \omega_k$ are k independent realizations of a symmetric stable process of order α in R^n all starting from the same point at $t = 0$. Then these k realizations will have some point in common other than the starting point*

(i) *with probability* 1, *if* $2 \geqq \alpha > n\,(k-1)/k$;

(ii) *with probability* 0, *if* $\alpha \leqq n\,(k-1)/k$.

5. Hausdorff Measure of the Set of k-Multiple Points

In [*8*] we showed that the set of double points of the Brownian path in R^3 had σ-finite Λ^1-measure. We now use the estimates of section 2 to extend the methods of [*8*], obtaining the corresponding result for $L^k_{\alpha,n}\ (0, 1; \omega)$ — the set of k-multiple points of a stable process — in the case $k\alpha > n(k-1)$ where we know that k-multiple points exist. We again exclude the critical cases $\alpha = 1 = n$ and $\alpha = 2 = n$ where the upper bound for the dimension in theorem 3 is in any case trivial. There is also nothing interesting to say about the case $\alpha > 1 = n$, which is excluded from our statement of theorem 3.

In order to show that the Hausdorff measure of $L^k(0, 1; \omega)$ of dimension $\beta = k\alpha - n(k-1)$ is σ-finite we must divide the set into a countable number of pieces each of finite β-measure. For such a subset Q we must show that, for every $\delta > 0$, we can cover Q by a sequence $\{S_i\}$ of spheres whose diameters $\{d_i\}$ satisfy $\sum d_i^\beta < M < \infty$. It is sufficient, clearly, to show that

$$\bigcap_{j=1}^{k} L(r_{2j-1}, r_{2j}; \omega)$$

has finite β-measure for each sequence

$$0 \leqq r_1 < r_2 < \cdots < r_{2k} \leqq 1$$

of $2k$ distinct rationals in $[0, 1]$. This will follow by the same argument as the proof, which we now give, that

$$Q_k(\omega) = \bigcap_{j=1}^{k} L(2j-1, 2j; \omega)$$

has finite β-measure.

For any positive integer m, put $\varepsilon_m = m^{-1/\alpha}$ and consider the points of $L(1, 2; \omega)$ which are approached within ε_m by each of the pieces $L(2j-1, 2j; \omega)$. If we cover this set of near-ε_m returns then we will certainly have covered $Q_k(\omega)$. Split up the interval $[1, 2]$ into m equal pieces by the time points

$$t_i = 1 + i/m \qquad (i = 0, 1, \ldots, m).$$

The maximum displacement of $X(t) - X(t_i)$ in $t_i \leqq t \leqq t_{i+1}$ will be of the order of $m^{-1/\alpha}$. To be precise, put

$$Y_{i,m} = m^{1/\alpha} \sup_{t_i \leqq t \leqq t_{i+1}} |X(t) - X(t_i)|.$$

Then $Y_{i,m}$, $i = 0, 1, \ldots, m-1$ are random variables and, by lemma 6, for $\lambda > 8$,

$$P\{Y_{i,m} > \lambda\} < e^{-\lambda c_8}.$$

It is simpler for computational purposes to use a discrete random variable $\varrho_{i,m}$ defined as follows:

if $Y_{i,m} \leqq 8$, put $\varrho_{i,m} = 9m^{-1/\alpha}$;
if $2^s < Y_{i,m} \leqq 2^{s+1}$, put $\varrho_{i,m} = (1 + 2^{s+1}) m^{-1/\alpha}$, $s = 3, 4, \ldots$.

It is clear that if any point of $L(t_i, t_{i+1}; \omega)$ is to be a point of near-ε_m return, then each of the pieces $L(2j-1, 2j; \omega)$ must at least enter the sphere $S_{i,m}$ with center $X(t_i)$ and radius $\varrho_{i,m}$. We now find an upper bound for the probability that this happens.

Starting from $X(t_{i+1})$ which is a point of $S_{i,m}$ the probability of a return to $S_{i,m}$ for $3 \leqq t \leqq 4$ is, by lemma 3, less than $c_3 \varrho_{i,m}^{n-\alpha}$, since the relevant value of T is at least 1. If a return occurs in $[3, 4]$ choose the first return τ, which is a stopping time and repeat the argument $(k-1)$ times using the strong Markov property. Thus the probability $p_{i,m}$ of a return to $S_{i,m}$ in all of the intervals $[2j-1, 2j]$, $j = 2, 3, \ldots, k$ satisfies

$$p_{i,m} \leqq c_3^{k-1} \varrho_{i,m}^{(k-1)(n-\alpha)}.$$

Now put

$d_{i,m}(\omega) = 0$, if in at least one of $[2j-1, 2j]$, $j = 2, \ldots, k$ no return occurs;
$d_{i,m}(\omega) = 2\varrho_{i,m}(\omega)$ if all the $(k-1)$ returns occur;

$$l_m(\omega) = \sum_{i=0}^{m-1} [d_{i,m}(\omega)]^\beta.$$

Thus the random variable $l_m(\omega)$ is the sum of the β-th powers of the diameters of the spheres $S_{i,m}$ which are re-entered $(k-1)$ times in the intervals considered. The next step is to obtain an upper bound for the first moment of $l_m(\omega)$. Using the estimates just obtained,

$$E\{d_{i,m}^{\beta}\} \leqq c_{15} \sum_{s=3}^{\infty} P\{\varrho_{i,m} = (2^s+1)\, m^{-1/\alpha}\}[(2^s+1)\, m^{-1/\alpha}]^{\beta+(k-1)(n-\alpha)}$$
$$\leqq c_{15}[(9\, m^{-1/\alpha})^{\beta+(k-1)(n-\alpha)} + \sum_{s=4}^{\infty} \exp(-c_8 2^s)\{(2^s+1)\, m^{-1/\alpha}\}^{\beta+(k-1)(n-\alpha)}]$$
$$< \frac{c_{16}}{m},$$

on making use of lemma 6. Finally we have

$$E\{l_m(\omega)\} = m\, E\{[d_{i,m}(\omega)]^{\beta}\} < c_{16}.$$

We can now repeat the argument on page 861 of [8] to deduce that, with probability 1, there is a finite real number $M(\omega)$ such that, for a subsequence $m_1, m_2, \ldots$

$$l_{m_i}(\omega) \leqq M(\omega) \qquad i = 1, 2, \ldots.$$

Now, if we put

$$\max_{0 \leqq i \leqq m-1} \varrho_{i,m} = q_m,$$

it follows from lemma 6 that, for large m,

$$P\{q_m > \tfrac{1}{3}\,\delta\} < m \exp(-c_8 \tfrac{1}{3}\,\delta\, m^{1/\alpha}).$$

An application of the Borel Cantelli lemma now shows that only finitely many of the events $\{q_m > \frac{1}{3}\,\delta\}$ occur so we may assume all the covening spheres have diameter less than δ. It follows that

$$\Lambda^{\beta} Q_k(\omega) \leqq M(\omega) < \infty.$$

Going back to the argument in the second paragraph of the present section we have established that $L^k(0, T; \omega)$ has σ-finite β-measure.

Proof of Theorem 1 (ii). By the relationship between capacity and measure discussed in section 3, it follows that for every rational r, the β-capacity of $L^k(0, r; \omega)$ is zero. It follows that, if $\alpha \leqq nk/(k+1)$,

$$P\{X(t) \in L^k(0, r) \text{ for some } t > r\} = 0$$

by lemma 7. Hence, if $n\,(k-1)/k < \alpha \leqq nk/(k+1)$ the set $L^k(0, r; \omega) \cap L(r, \infty; \omega)$ is void for each rational r, and there cannot be any points of multiplicity $(k+1)$. This establishes theorem 1 (ii) for the transient case $\alpha < n$. When $\alpha \geqq n$ the theorem is vacuous.

A similar argument establishes theorem 5 (ii).

6. The Cauchy Process on the Line

The investigation of multiple points for Brownian paths in the plane was carried out by DVORETZKY, ERDÖS and KAKUTANI, [7] and [9]. They used certain estimates for the probability of return to a disc in given time intervals of a similar nature to our lemmas 1—5. If we show that the same estimates (apart from the values of the positive constants) are valid for the Cauchy process in R^1, then it is clear that the arguments of [7] and [9] will establish theorem 1 (i) for $\alpha = 1 = n$ and theorem 2, except for possible difficulties arising from discontinuities in the sample function (since theorem 2 clearly implies theorem 1 (i) for the Cauchy process it is sufficient to prove theorem 2). There are minor simplifications to the arguments of [7] and [9] since we are working in R^1 rather than R^2.

It is possible by rather a lot of hard work to estimate all the probabilities needed using the usual techniques of first passage time analysis and Laplace transforms. We prefer to use the amusing relationship between the Cauchy process in R^1 and Brownian motion in R^2 discovered by SPITZER [14]. This is given by the next lemma.

Lemma 10. *If* $X(t) + iY(t)$ *represents Brownian motion in* R^2 *and* $T(t) = \inf_{\tau \geqq 0}[\tau : Y(\tau) \geqq t]$ *is the first passage time across the line* $Y = t$, *then* $X[T(t)]$ *represents a Cauchy process in* R^1.

This means that if we observe the 1-dimensional Brownian process $X(t)$ only at those time instants t for which $Y(t) = \sup_{0 \leqq \tau \leqq t} Y(\tau)$, we obtain the sample path of the Cauchy process. Now (LÉVY [12], theorem 49.1) showed that the set of time instants where $Y(t) = \sup_{0 \leqq \tau \leqq t} T(\tau)$ is stochastically equivalent to the set of time instants τ for which $Y(\tau) = 0$. Since $X(t)$, $Y(t)$ are independent processes it follows that we can calculate probabilities of entering a linear set E by the Cauchy process by considering the planar set

$$E_0 = \{(x, y) : x \in E\,,\, y = 0\}$$

and calculating the probability that a planar Brownian path will enter E_0. We will only require these probabilities for intervals E, but even in this case the corresponding probabilities for E_0 have not been computed. However, under reasonable conditions, the probability that a Brownian path will hit $\{(x,y) : a \leqq x \leqq b, y = 0\}$ in a given time interval is of the same order of magnitude as the probability of hitting the disc with this segment as diameter — in fact one can prove that these probabilities as asymptotically the same as the length $b - a \to 0$. These mean that we can get the estimates we require from known results for the Brownian process in R^2. We obtain the results in a sequence of lemmas. Throughout this section $Z(t)$ will denote a planar Brownian process, and $C(t)$ will denote a linear Cauchy process.

Lemma 11. *If* $2 \geqq |z| \geqq \frac{1}{2}$, $0 < \varrho < \frac{1}{4}$, *then*

$$\frac{c_{17}}{\log 1/\varrho} < P_z\{\inf_{0 \leqq t \leqq 1} |Z(t)| < \varrho\} < \frac{c_{18}}{\log 1/\varrho}\,.$$

This can be obtained from lemmas 3 and 5 of [7] by a suitable change of scale.

Lemma 12. *For* $\frac{1}{2} \leqq \xi \leqq 1$, $0 < \varrho < c_{19}$,

$$\frac{c_{20}}{\log 1/\varrho} < P_\xi\{\inf_{0 \leqq t \leqq 1} |C(t)| < \varrho\} < \frac{c_{21}}{\log 1/\varrho}\,.$$

Proof. Because of the relationship in lemma 10, the probability

$$P_\xi\{\inf_{0 \leqq t \leqq 1} |C(t)| < \varrho\}$$

is precisely the probability that a planar Brownian process $Z(t)$ starting from 0 will enter the segment $\{\xi - \varrho \leqq x \leqq \xi + \varrho, y = 0\}$ in unit time. This is clearly less than the probability of entering the disc center $(\xi, 0)$, radius ϱ which in turn is less than $c_{18}/\log 1/\varrho$ by lemma 11.

In the other direction, the required probability is clearly greater than $p_1(p_2 - p_3)$ where

$p_1 = P\{Z(t)$ hits disc center $(\xi, 0)$ radius $\frac{1}{2}\varrho$ in $0 \leqq t \leqq \frac{1}{2}\}$;

$p_2 = P\{Z(t)$ starting from a point in disc radius $\frac{1}{2}\varrho$ hits a fixed diameter of a larger concentric disc of radius ϱ before exiting from larger disc$\}$;

$p_3 = P\{Z(t)$ starting from a point in disc radius $\frac{1}{2}\varrho$ remains in larger concentric disc of radius ϱ for $0 \leqq t \leqq \frac{1}{2}\}$.

Now p_2 is bounded away from zero for all starting points in the smaller disc (for a rigorous proof use the logarithmic potential theory) and $p_3 \to 0$ as $\varrho \to 0$ by lemma 7 so that $p_3 < \frac{1}{2}\inf p_2$. We can estimate p_1 from lemma 11 by a charge of scale. Combining these gives the required lower bound for $0 < \varrho < c_{19}$.

We also require an upper bound for the case $|\xi| < \frac{1}{2}$.

Lemma 13. *If* $0 < \varrho < |\xi| < 1$, *then*

$$P_\xi\{\inf_{0 \leqq t \leqq 1} |C(t)| = \varrho\} < c_{22}\frac{1 + \log 1/|\xi|}{\log 1/\varrho}.$$

This follows from lemma 4 of [7] by the argument used in the proof of our lemma 12.

Lemma 14. *If* $0 \leqq |\xi| \leqq \varrho < c_{23}$,

$$\frac{c_{24}}{\log 1/\varrho} < P_\xi\{\inf_{1/e \leqq t \leqq 1} |C(t)|\} < \frac{c_{25}}{\log 1/\varrho}.$$

Proof. This can also be deduced from the corresponding result for Brownian motion. However the case $\xi = 0$ is proved in lemma 1 of [*17*], and the methods used there immediately give the result as stated.

It is now clear that, by using the estimates of lemmas 12, 13, 14 instead of the corresponding ones for Brownian motion and the (slightly) simpler lattice of intervals in R^1 instead of discs in R^2, the proofs of Dvoreztky, Erdös, Kakutani in [*7*] and [*9*] go through. This means that theorem 2 is established apart from the difficulty about left hand limit points. This we now get around as follows. In the notation of [*9*] we obtain time points

$$t_n(j), \qquad j = 1, \ldots, 2^n$$

for which $C(t_n(j))$, $j = 1, \ldots, 2^n$ are all the same point ξ_n and choose a convergent subsequence $\xi_{n_p} \to \xi$ as $p \to \infty$. Put

$$A = \bigcap_{q=1}^{\infty} \overline{\bigcup_{p=q}^{\infty} \{t_{n_p}(j) : j = 1, 2, \ldots, 2^{n_p}\}}$$

where the bar denotes closure. Then A is a perfect set, and therefore of power $\boldsymbol{c}$. If we leave out of A all the points isolated on the right (not more than a countable set) we are left with a set B still of power $\boldsymbol{c}$. Now for any $t \in B$ we can find a sequence $q(m)$ increasing to infinity and integers j_m with $1 \leqq j_m \leqq 2^{q(m)}$ such that $t_{n(m)}(q(m))$ decreases to t as $m \to \infty$. Hence $C(t) = \xi$ since the paths are right continuous. This completes the proof of theorem 2.

7. The Dimension of the Set of Multiple Points

The results of section 5 already give us an upper bound for the dimension of the set of k-multiple points,

$$\dim L^k_{\alpha,n}(0, \infty; \omega) \leqq \beta = k\alpha - n(k-1),$$

when $\beta > 0$ (corresponding to the case where L^k is not void, by section 4). We now use theorem 4 with the methods of section 4 to show that the dimensional number of $L^k_{\alpha,n} \geqq \beta$. This will complete the proof of theorem 3. It is sufficient to show that, for any $\gamma > n - \beta$ there are points common to an $X_{\gamma,n}(t)$ process and the set $L^k_{\alpha,n}(0, 1; \omega)$ of k-multiple points of the an independent $X_{\alpha,n}(t)$ process. We deal first with the transient case $\alpha < n \leqq 2$. (Our method completely breaks down for $\alpha > \frac{3}{2}$, $k = 2$, $n = 3$). We can use the estimates of lemmas 1—5 to establish, by a proof very similar to that of lemma 8 and 9.

Lemma 15. *Suppose $n = 2$, $0 < \alpha < n$, $0 < \gamma < n$ and $k\alpha + \gamma > nk$. For each positive integer r consider the discs S_ν centered at*

$$x_\nu = \left(\frac{1}{4} + \frac{\nu_1}{10r}, \frac{1}{4} + \frac{\nu_2}{10r}\right) \qquad \nu_i = 1, 2, \ldots, r,\ i = 1, 2; \quad \nu = (\nu_1 - 1)r + \nu_2$$

of radius

$$\varrho_r = \lambda r^{-\mu} \text{ where } \mu = \frac{n}{(k+1)n - k\alpha - \gamma}, \quad \lambda = c_{26}.$$

Let E_ν $(\nu = 1, 2, \ldots, r^2)$ be the event that there are time instants $0 \leqq t_1 \leqq V$, $1 \leqq t_j - t_{j-1} \leqq V$, $j = 2, 3, \ldots, k$ and $0 \leqq t_{k+1} \leqq V$ such that

$$X_{\alpha,n}(t_j) \in S_\nu \quad \text{for} \quad j = 1, 2, \ldots, k$$

and

$$X_{\gamma,n}(t_{k+1}) \in S_\nu.$$

Then

$$P\left(\bigcup_{\nu=1}^{r^2} E_\nu\right) \geqq c_{27} > 0.$$

(In the above V is a fixed real number chosen large enough to ensure that the estimates are valid for both processes.)

The argument which follows lemma 9 can now be easily modified to establish.

Lemma 16. *If $X_{\alpha,n}(t)$ and $X_{\gamma,n}(t)$ are independent symmetric stable processes of orders α and γ in R^n where $k\alpha + \gamma > nk$, then with probability 1 there are time instants*

$$0 \leqq t_1 < t_2 < \cdots < t_k, \quad t_{k+1} > 0$$

such that

$$X_{\gamma,n}(t_{k+1}) = z = X_{\alpha,n}(t_i), \qquad i = 1, 2, \ldots, k.$$

This lemma means that $X_{\gamma,n}(t)$ hits the set $L^k_{\alpha,n}(0, \infty; \omega)$ at a positive time with probability 1. By theorem 4 we deduce that $\dim L^k_{\alpha,n}(0, \infty; \omega) \geqq \beta$. This completes the proof of theorem 3 in the transient case. The cases $\alpha = 1 = n$ of theorem 3 (ii) and $\alpha = 2 = n$ of theorem 3 (i) will follow a fortiori if we can establish theorem 3 (iii) and (iv). We write out the proof for theorem 3 (iv), since the remaining case follows by a similar argument.

We follow the sequence of arguments in [*9*] to give us the proof of theorem 3 (iv), giving the details only when there is an essential change in the argument.

The statements in [9] are labelled A, B, ... We need to replace B by the following lemma in which we compute the probability that at least one of r small intervals is entered by a stable process of order γ and is entered twice by each of k independent Cauchy processes. It is more difficult now to find the correct length ϱ_r for the intervals so that we have just enough independence to carry out the computation.

For small values of $s > 0$, $0 < \gamma < 1$ consider

$$y = f(s) = s^{1-\gamma}(\log 1/s).$$

This function is monotone increasing for small s, so it has an inverse $s = \psi(y)$ defined for small positive y.

Lemma 17. *Suppose $\omega_1, \omega_2, \ldots, \omega_k$ are k independent Cauchy processes starting (at $t = 0$) at $x_1, x_2, \ldots, x_k$ respectively, and ω_0 is an independent symmetric stable process of order $\gamma\,(0 < \gamma < 1)$ in R^1 starting at x_0 where $|x_i| < 1/100$ $(i = 0, 1, \ldots, k)$. Let*

$$\zeta_\nu = \frac{1}{4} + \frac{\nu}{4r}, \qquad \nu = 1, 2, \ldots, r;$$

$$\varrho_r = c_{28}\,\psi\left(\frac{1}{r}\right);$$

and denote by D_ν the event that there exist, for $j = 1, \ldots, k$ time instants t_j', t_j such that

$$0 < t_j \leqq 1, \quad \frac{1}{e} \leqq t_j' - t_j \leqq 1,$$

$$|C(t_j, \omega_j) - \zeta_\nu| < \varrho_r, \quad |C(t_j', \omega_j) - \zeta_\nu| < \varrho_r,$$

and a time instant t_0 such that $0 \leqq t_0 \leqq U$ and

$$|X_{\gamma,1}(t_0, \omega_0) - \zeta_\nu| < \varrho_r.$$

Then

$$P\Big(\bigcup_{\nu=1}^{r} D_\nu\Big) > c_{29} > 0,$$

for all integers $r \geqq r_0$.

Proof. Since the processes are all independent, lemmas 12, 14 and 4 give

$$P(D_\nu) > c_{30}\,\varrho_r^{1-\gamma}/(\log 1/\varrho_r)^{2k} = c_{30}\, f(\varrho_r);$$

$$P(D_\nu \cap D_{\nu'}) \leqq c_{31}\,\frac{\varrho_r^{2(1-\gamma)}(\log|\zeta_\nu - \zeta_{\nu'}|)^{4k}}{|\zeta_\nu - \zeta_{\nu'}|^{1-\gamma}(\log 1/\varrho_r)^{4k}}.$$

(Here we have used the strong Markov property and the same arguments as were used in the proof of lemma 8.) Now

$$\begin{aligned}
P\Big(\bigcup_{\nu=1}^{r} D_\nu\Big) &\geqq \sum_{\nu=1}^{r} P(D_\nu) - \sum_{1 \leqq \nu \leqq \nu' \leqq r} P(D_\nu \cap D_{\nu'}) \\
&> c_{30}\, r\, f(\varrho_r) - 2r[f(\varrho_r)]^2 c_{31} \sum_{\nu=2}^{r} \frac{(\log|\zeta_\nu - \zeta_1|)^{4k}}{|\zeta_\nu - \zeta_1|^{1-\gamma}} \\
&> c_{30}\, r\, f(\varrho_r) - 8c_{31}\, r^2[f(\varrho_r)]^2 \sum_{\nu=1}^{r} \frac{1}{4r}\left(\frac{4r}{\nu}\right)^{1-\gamma}\left(\log\frac{4r}{\nu}\right)^{4k} \\
&> c_{30}\, r\, f(\varrho_r) - c_{32}\, r^2[f(\varrho_r)]^2 \quad \text{for} \quad r \geqq r_1
\end{aligned}$$

since

$$\sum_{\nu=1}^{r} \frac{1}{4r}\left(\frac{4r}{\nu}\right)^{1-\gamma}\left(\log\frac{4r}{\nu}\right)^{4k} \sim \int_{0}^{1/4}\left(\frac{1}{x}\right)^{1-\gamma}\left(\log\frac{1}{x}\right)^{4k} dx$$

as $r \to \infty$. Choose c_{28} to satisfy

$$c_{28}{}^{1-\gamma} = \tfrac{1}{4}\, c_{30}/c_{32}$$

and note that, if $\varrho_r = c_{28}\,\psi\left(\frac{1}{r}\right)$, then

$$f(\varrho_r) \sim c_{28}{}^{1-\gamma}\cdot\frac{1}{r} \quad \text{as}\, r \to \infty\,.$$

For $r \geqq r_2$, we must have

$$P\Big(\bigcup_{\nu=1}^{r} D_\nu\Big) > \frac{2}{3}\, c_{28}{}^{1-\gamma}\cdot c_{30} - \frac{4}{3}\, c_{28}{}^{2-2\gamma}\cdot c_{32}$$
$$> c_{29} > 0\,.$$

This establishes the lemma.

The argument of [*9*] can now be continued with the obvious changes leading to the following lemma, which corresponds to (E) in [*9*].

Lemma 18. *Suppose* ω_0 *is a symmetric stable process of order* γ $(0 < \gamma < 1)$ *in* R^1 *and* $\omega_1, \omega_2, \ldots, \omega_k$ *are independent Cauchy processes. For any* $\varepsilon > 0$, *there is probability* 1 *that there is a point* $\zeta \in L_{\gamma,1}(0, \varepsilon; \omega_0)$ *which is a double point of* $L_{1,1}(0, \varepsilon; \omega_i)$ *for; each* $i = 1, 2, \ldots, k$.

Finally, using the modifications suggested for the proof of theorem 2 we obtain:

Lemma 19. *For* $0 < \gamma < 1$, *there is probability* 1 *that an independent stable process of order* γ *will hit the set* $L^{c}_{1.1}(0, 1; \omega)$ *of points of multiplicity* **c** *of a Cauchy process.*

If we apply theorem 4, we see that the set $L^{c}_{1,1}(0, 1; \omega)$ must have dimension 1 with probability 1. This completes the proof of theorem 3.

8. Further Problems

1. We remark again that we have been unable to compute the dimension of the set of double points $L^{2}_{\alpha,3}(0, 1; \omega)$ of a symmetric stable process of order α in R^3: the value suggested by our results is $(2\alpha - 3)$ for $\alpha > 3/2$.

2. It is clear that the methods we have used will solve other problems of a similar nature. For example, in R^2, if $0 < \alpha_i \leqq 2$, $i = 1, 2, \ldots, k$, $\sum_{i=1}^{k} \alpha_i > 2(k-1)$ and ω_i $(i = 1, \ldots, k)$ are independent stable process of order α_i in R^2, there is probability 1 that the set of points common to $L_{\alpha_i,2}(0, 1; \omega)$ will be non-void and have dimension $\sum_{i=1}^{k} \alpha_i - 2(k-1)$.

3. Our results suggest that one ought to be able to say something about the subset of points of a given fixed set E which are 'hit' by a stable process. We state a conjecture which seems to us to be plausible.

Conjecture. Suppose E is a Borel set in R^n such that $\dim[E \cap S(\varrho)] = \beta$ for all $\varrho > 0$, where $S(\varrho)$ is the sphere center 0, radius ϱ. Then, if $\alpha + \beta > n$, there is probability 1 that $\dim[E \cap L_{\alpha,n}(0, 1; \omega)] = \alpha + \beta - n$.

4. One can also ask questions of a more delicate nature about the Hausdorff measure of the sets $L^k_{\alpha,n}(0, 1; \omega)$ of k-multiple points. On would like to find the correct measure function $h(d)$ such that the h-measure of $L^k_{\alpha,n}$ is finite and positive with probability 1. This is known for the path set $L_{2,n}(0, 1; \omega)$ of Brownian motion (see [5] and [19]), and the methods used in [19] can be applied to $L_{\alpha,n}(0, 1; \omega)$ to show that, in order to measure the path set,

(i) for $\alpha < n$, the correct measure function is $d^\alpha \log\log \frac{1}{d}$,

(ii) for $\alpha = 1 = n$, the correct measure function is $d \log \frac{1}{d} \log\log\log \frac{1}{d}$.

I am unable to make a plausible onjecture about the correct measure function to measure $L^k_{\alpha,n}(0, 1; \omega)$ when $k \geqq 2$.

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