Z. Wahrscheinlichkeitstheorie verw. Gebiete
40, 125 – 145 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete
© by Springer-Verlag 1977

The Critical Measure Diffusion Process

D.A. Dawson*
Carleton University, Ottawa, Canada, K155B6

Summary. A multiplicative stochastic measure diffusion process is the continuous analogue of an infinite particle branching diffusion process. In this paper the limiting behavior of the critical measure diffusion process is investigated. Conditions are found under which a non-trivial steady state random measure exists and in this case a spatial central limit theorem is established.

1. Introduction

Multiplicative stochastic measure diffusion processes in R^d arise as the small particle limits of branching diffusion processes. This fact together with the basic construction of a stochastic measure diffusion process are obtained in [3]. The main objective of this paper is the description of the limiting behavior of measure diffusions. Although the methods introduced can be extended to the multitype case, we restrict our attention to the single type case.

One of the fundamental properties of the Galton-Watson branching process is that the critical process goes to extinction with probability one (c.f. Athreya-Ney [1; Chapter 1, Theorem 5.2]). On the other hand the effect of mixing a number of such populations is to counteract this tendency to extinction. In fact one of the principal results of this paper is that it is possible to have a mixing mechanism for an infinite collection of critical Galton-Watson processes so that a non-zero steady state exists. In the language of nonequilibrium thermodynamics, the mixing is a dissipative process whereas the multiplicative process is one which amplifies fluctuations.

Let $\mathscr{M}(R^d)$ denote the family of Borel measures on R^d. When $\mathscr{M}(R^d)$ is furnished with the topology of vague convergence, it becomes a Polish space. Let $\mathscr{B}(\mathscr{M}(R^d))$ denote the σ-field of Borel subsets of $\mathscr{M}(R^d)$. A *random measure* on R^d is an $\mathscr{M}(R^d)$-valued random variable, that is, one whose distribution is

* Supported in part by the National Research Council of Canada.

given by a probability measure on $(\mathscr{M}(R^d), \mathscr{B}(\mathscr{M}(R^d)))$. Let $C_K(R^d)$ denote the class of continuous functions on R^d with compact support. For $\nu \in \mathscr{M}(R^d)$, $\varphi \in C_K(R^d)$, let $\langle \nu, \varphi \rangle \equiv \int \varphi(x)\, \nu(dx)$. A probability measure, P, on $(\mathscr{M}(R^d), \mathscr{B}(\mathscr{M}(R^d)))$ is uniquely determined by the *characteristic functional* $L(.)$, where

$$L(f) \equiv \int_{\mathscr{M}(R^d)} \exp(i \int f(x)\, \nu(dx))\, P(d\nu) \qquad \text{for } f \in C_K(R^d). \tag{1.1}$$

The reader is referred to Jagers [8] for a review of random measures and characteristic functionals.

Throughout the paper we denote by μ the Lebesgue measure on R^d.

A *stochastic measure process* $\{X_t\colon t \geqq 0\}$ is an $\mathscr{M}(R^d)$-valued stochastic process. A Markov stochastic measure process with time homogeneous transition probabilities is uniquely determined (in the sense of finite dimensional distributions) by the characteristic functional of the initial distribution and the characteristic functional of the transition function;

$$L_{t,\nu}(f) \equiv E(\exp(i \int f(x)\, X_t(dx)) \,|\, X_0 = \nu) \tag{1.2}$$

for $\nu \in \mathscr{M}(R^d)$ and $f \in C_K(R^d)$.

In the case of multiplicative stochastic measure processes which are dealt with in this paper the characteristic functional plays the central role and can be explicitly computed.

2. The Multiplicative Stochastic Measure Diffusion Process

In this section we review the construction of the multiplicative stochastic measure diffusion process and refer the reader to Dawson [3; Section 6] for details. However for the purposes of this paper we restrict our attention to the critical case.

The *critical continuous branching process* is given by the solution of the Itô stochastic differential equation

$$dz(t) = (2\gamma\, z(t))^{\frac{1}{2}}\, db(t); \qquad z(0) = z_0 \tag{2.1}$$

where $\gamma > 0$ and $\{b(t)\colon t \geqq 0\}$ is a Wiener process.

Let $M(\theta, t)$, $\theta \in R$, $t \in R^+$, denote the characteristic function

$$M(\theta, t) \equiv E_{z_0}(\exp(i\theta z(t))) \tag{2.2}$$

where $E_{z_0}(.)$ denotes the expectation with respect to the probability measure induced by the solution of Equation (2.1). By a standard computation it can be verified that $M(.,.)$ satisfies the first order partial differential equation

$$\begin{aligned} &\partial M(\theta, t)/\partial t = i\gamma\, \theta^2\, \partial M(\theta, t)/\partial\theta \\ &M(\theta, 0) = \exp(i\theta z_0). \end{aligned} \tag{2.3}$$

Because of the multiplicative nature of the process, $\log(M(\theta, t))$ is of the form (refer to [1, 3] for details),

$$\log(M(\theta, t)) = i z_0 \psi(\theta, t) \tag{2.4}$$

where $\psi(\theta, t)$ is real-valued for each θ and t. $\psi(.,.)$ satisfies the functional equation

$$\psi(\theta, t+s) = \psi(\psi(\theta, t), s),$$
$$\psi(\theta, 0) = \theta. \tag{2.5}$$

It is also easy to verify that $\psi(.,.)$ satisfies the nonlinear evolution equation

$$\partial \psi(t)/\partial t = i \gamma \psi^2(t); \quad \psi(\theta, 0) = \theta. \tag{2.6}$$

Solving explicitly either Equation (2.3) or Equation (2.6), we obtain

$$M(\theta, t) = \exp(i \theta z_0/(1 - i \gamma \theta t)). \tag{2.7}$$

Using the moment generating properties of $M(\theta, t)$ we obtain

$$E_{z_0}(z(t)) = z_0 \quad \text{for all } t \geqq 0, \quad \text{and} \tag{2.8}$$

$$\mathrm{Var}_{z_0}(z(t)) = 2\gamma z_0 t \quad \text{for } t \geqq 0. \tag{2.9}$$

We now describe the corresponding Brownian measure process. A *Brownian measure process* $\{X_t : t \geqq 0\}$ is a Markov process with state space $\mathcal{M}(R^d)$ with time homogeneous transition probabilities and which satisfies the following conditions:

(Spatial independence) if $A \cap B = \emptyset$ and the pair $(X_0(A), X_0(B))$ are independent, then the pair $(X_t(A), X_t(B))$ are independent for all $t > 0$. [$X_t(A)$ denotes the random measure of the set A at time t] (2.10)

(Spatial homogeneity) if $X_0(A+x) = X_0(A)$, then $X_t(A+x) = X_t(A)$ in law for all $t > 0$, (2.11)

(Creation-free) if $X_0(A) = 0$, then $X_t(A) = 0$ for all $t > 0$. (2.12)

The process is said to be *multiplicative* if for all $\nu_1, \nu_2 \in \mathcal{M}(R^d)$, $P_{\nu_1 + \nu_2} = P_{\nu_1} * P_{\nu_2}$ where $*$ denotes convolution. In other words the distribution of the process with initial condition $\nu_1 + \nu_2$ is equal to the distribution of the sum of independent versions of the process with initial conditions ν_1 and ν_2, respectively.

Consider the family of mappings $\mathbf{T}_t$: $C_K(R^d) \to C_K(R^d)$ defined by

$$(\mathbf{T}_t \varphi)(x) = \psi(\varphi(x), t) \quad \text{for } t \geqq 0. \tag{2.13}$$

Then $\{\mathbf{T}_t : t \geqq 0\}$ is a semigroup of nonlinear operators on $C_K(R^d)$ with generator

$$(\Gamma \varphi)(x) = i \gamma \varphi^2(x). \tag{2.14}$$

It can be verified (c.f. [3]) that the $\mathcal{M}(R^d)$-valued Markov process in which the characteristic functional of the probability transition function is given by

$$L_{t,X_0}(\varphi) \equiv \exp(i \int \mathbf{T}_t \varphi(y) X_0(dy)) \tag{2.15}$$

for $X_0 \in \mathscr{M}(R^d)$ and $\varphi \in C_K(R^d)$ is a multiplicative Brownian measure process.

Proposition 2.2 *Let* $\{X_t: t \geqq 0\}$ *denote the Markov process defined by* (2.15). *If* $X_0 \in \mathscr{M}(R^d)$ *is non-atomic, then for each* $t>0$, X_t *is a completely random-measure with Kingman representation* (c.f. [8])

$$\log(L_{t,X_0}(\varphi)) = \int_{0+}^{\infty} \int (\exp(i\varphi(y)x)-1)\, \nu_t(dx,dy) \tag{2.16}$$

where ν_t *is the canonical measure on* $R \otimes R^d$ *and is given by*

$$\begin{aligned} \nu_t(dx,dy) &= (\gamma t)^{-2} \exp(-x/\gamma t)\, \mu(dx)\, X_0(dy), && x>0, \\ &= 0 &&, \quad x \leqq 0. \end{aligned} \tag{2.17}$$

Proof. Substituting (2.17) into (2.16) and integrating we obtain

$$\log(L_{t,X_0}(\varphi)) = \int (i\varphi(y)/[1 - i\gamma\varphi(y)t])\, X_0(dy).$$

The result then follows by the uniqueness property of the Kingman representation.

We complete the construction of a measure diffusion process by adding spatial diffusion to the Brownian measure process. Let $\{\mathbf{S}_t: t \geqq 0\}$ denote the semigroup of contraction operators on $C_K(R^d)$ which are associated with a conservative Markov process which lives on R^d. The Markov process associated with $\{\mathbf{S}_t: t \geqq 0\}$ will serve as the "spatial diffusion process" or "spatial motion process". We denote the infinitesimal generator of $\{\mathbf{S}_t: t \geqq 0\}$ by G. It was shown in [3: Section 6] using a nonlinear version of Trotter's product formula that

$$\mathbf{U}_t \equiv \lim_{n \to \infty} (\mathbf{T}_{t/n} \mathbf{S}_{t/n})^n, \qquad t \geqq 0, \tag{2.18}$$

defines a semigroup of nonlinear operators on $C_K(R^d)$ with infinitesimal generator $(\Gamma + G)$. It was also shown that $(\mathbf{U}_t: t \geqq 0\}$ satisfies

$$\|\mathbf{U}_t \varphi\| \leqq \|\varphi\| \tag{2.19}$$

where $\|.\|$ denotes the supremum norm. A similar argument implies that

$$\|\mathbf{U}_t \varphi\|_1 \leqq \|\varphi\|_1 \tag{2.20}$$

where $\|.\|_1$ denotes the L^1 norm.

Let $\mathscr{M}_I(R^d)$ denote the family of measures

$$\mathscr{M}_I(R^d) \equiv \{\lambda \in \mathscr{M}(R^d): 0 < \lim_{k \to \infty} (\lambda(A_k)/\mu(A_k)) < \infty\}$$

where A_k denotes the cube centered at the origin in R^d of side k.

Let $\mathbf{P}_I$ denote the family of probability measures on $\mathscr{M}(R^d)$ which are invariant under the transformations induced by translations in R^d. Henceforth we also assume that the spatial motion process is spatially homogeneous, that is,

$$G\,\Phi_x \varphi(.) = \Phi_x G \varphi(.)$$

where

$$\Phi_x \varphi(.) \equiv \varphi(.+x).$$

In this case the semigroup $\{\mathbf{U}_t\colon t \geqq 0\}$ then defines a Markov process which lives on $\mathcal{M}_I(R^d)$. The characteristic functional of the probability transition function is given by

$$L_{t,X_0}(\varphi) \equiv \exp(i \int \mathbf{U}_t \varphi(x) X_0(dx)) \tag{2.21}$$

for $X_0 \in \mathcal{M}_I(R^d)$, $\varphi \in C_K(R^d)$.

Note that (2.21) is for each $t>0$ the characteristic functional of an infinitely divisible random measure. The Markov process defined by (2.21) is known as the *critical multiplicative measure diffusion process.*

Since $\{\mathbf{U}_t\colon t \geqq 0\}$ is strongly continuous, the transition kernel is stochastically continuous and the process is characterized by a semigroup of contraction operators $\{\mathbf{W}_t\colon t \geqq 0\}$ acting on $C_b(\mathcal{M}_I(R^d))$, the space of bounded continuous functions on $\mathcal{M}_I(R^d)$. Let $\{\mathbf{W}_t^*\colon t \geqq 0\}$ denote the adjoint semigroup of operators acting on $\mathbf{P}_I$.

$P \in \mathbf{P}_I$ is said to be a *steady state random measure* for the measure diffusion process if

$$\mathbf{W}_t^* P = P \qquad \text{for all } t>0. \tag{2.22}$$

It is clear that the trivial random measure δ_0 is a steady state random measure for the Markov process defined by (2.21). In the following sections we investigate the possibility of the existence and properties of additional steady state random measures.

In the remainder of this paper we restrict our attention to the case in which the motion process is either a *Brownian motion* or a *symmetric stable process* in R^d. In other words we assume that

$$G = \Delta^{\alpha/2}, \qquad 0 < \alpha \leqq 2 \tag{2.23}$$

where $\Delta^{\alpha/2}$ is a shorthand for $-(-\Delta)^{\alpha/2}$, and where Δ is the Laplacian operator in R^d. In this case the distribution of a particle which starts at the origin and travels according to this motion process is given at time t by an infinitely divisible law with characteristic function

$$M(\theta) = \exp(-t\,\psi(\theta)) \tag{2.24}$$

for $\theta \in R^d$ and with

$$\psi(\theta) = |\theta|^\alpha, \qquad 0 < \alpha < 2,$$

or it is normally distributed with distribution $N(0, 2t)$ for the case $\alpha = 2$.

It is known that the symmetric stable process of index α in R^d is transient if and only if $d > \alpha$ and that the corresponding potential operator is the *Riesz potential of order* α, that is, for $f \in C_K(R^d)$,

$$U_\alpha f(x) \equiv \int_0^\infty \int p_\alpha(t, y-x) f(y)\, \mu(dy)\, dt$$
$$= \int g_\alpha(y-x) f(y)\, \mu(dy) \tag{2.25}$$

where

$$g_\alpha(y-x) \equiv \Gamma((d-\alpha)/2)(2^\alpha \pi^{d/2} \Gamma(\alpha/2))^{-1} |x-y|^{-(d-\alpha)}$$

and where $\Gamma(.)$ denotes the Gamma function and $p_\alpha(t,.)$ denotes the symmetric stable density function corresponding to the characteristic function given by (2.24). The reader is referred to [2] for details.

Remark 2.1. The referee has brought to my attention the earlier work of M. Jirina [9] on finite measure-valued branching processes. In particular, he extended to this general setting the basic fact that the total mass of a critical finite measure-valued branching process converges to zero as $t \to \infty$ with probability one.

3. The Recurrent Critical Case

The main result of this section is Theorem 3.1 in which we show that the critical multiplicative measure diffusion process goes to extinction in the recurrent critical case. Before stating the theorem we state and prove two technical lemmas.

The class of recurrent symmetric stable processes in R^d consist of the following:

$$d=2,\ \alpha=2, \text{ that is, the two dimensional Brownian motion,} \tag{3.1.a}$$

and

$$d=1, \quad 1 \leqq \alpha \leqq 2. \tag{3.1.b}$$

We require the following scaling properties of the densities of the symmetric stable processes in R^1.

$$p_\alpha(t,x) = t^{-1/\alpha} p_\alpha(1, x/t^{1/\alpha}) \tag{3.2}$$

$$p_\alpha(t,x) \leqq K t x^{-(\alpha+1)} \quad \text{for } x \geqq t^{1/\alpha}(\log t)^{1/\alpha} \tag{3.3}$$

for sufficiently large t (for proofs refer to Ibragimov and Linnik [7; Theorem 2.4.2]).

Lemma 3.1. *If $p_\alpha(t,x)$ denotes the density at time t of a recurrent symmetric stable process and $\nu \in \mathcal{M}_I(R^d)$, then*

$$\int_{\|x\| \geqq (t \log t)^{1/\alpha}} p_\alpha(t,x)\, \nu(dx) \to 0 \quad \text{as } t \to \infty. \tag{3.4}$$

Proof. Case 1. (Assume (3.1.a).) It suffices to show that

$$\int_{(t\log t)^{\frac{1}{2}}}^{\infty} (\exp(-r^2/t)/t)\,\bar{\nu}(dr) \to 0 \qquad \text{as } t\to\infty$$

where $\bar{\nu}([r,r+a))$ denotes the ν-measure of the annulus with inner radius r and outer radius $r+a$. But for sufficiently large t,

$$\int_{(t\log t)^{\frac{1}{2}}}^{\infty} (\exp(-r^2/t)/t)\,\bar{\nu}(dr) \leqq K \int_{(t\log t)^{\frac{1}{2}}}^{\infty} (t^{\frac{1}{2}}/r^3)\,\bar{\nu}(dr)$$

since $\exp(-x^2) < K/x^3$ for sufficiently large x.

Then

$$\int_{(t\log t)^{\frac{1}{2}}}^{\infty} (t^{\frac{1}{2}}/r^3)\,\bar{\nu}(dr) \leqq \sum_{k=K_t}^{\infty} (t^{\frac{1}{2}}/2^{3k})\,\bar{\nu}([2^k, 2^{k+1}))$$

where $K_t = [\log_2((t\log t)^{\frac{1}{2}})]$ and $[.]$ denotes "greatest integer in". Since $\nu \in \mathscr{M}_I(R^2)$, there exists L such that for $k \geqq L$,

$$\bar{\nu}([2^k, 2^{k+1})) \leqq c^{2k}$$

where c is a positive constant. Therefore,

$$\int_{(t\log t)^{\frac{1}{2}}}^{\infty} (t^{\frac{1}{2}}/r^3)\,\bar{\nu}(dr) \leqq K'/(\log t)^{\frac{1}{2}}$$

for sufficiently large t and the proof is complete.

Case 2. (Assume 3.1.b). If $\nu \in \mathscr{M}_I(R^1)$, then by (3.3),

$$\int_{(t\log t)^{1/\alpha}}^{\infty} p_\alpha(t,x)\,\nu(dx) \leqq K \int_{(t\log t)^{1/\alpha}}^{\infty} (t/x^{\alpha+1})\,\nu(dx)$$

$$\leqq K \sum_{k=K_t}^{\infty} (t/2^{k(\alpha+1)})\,\nu([2^k, 2^{k+1}))$$

where $K_t = [\log_2 (t\log t)^{1/\alpha}]$. But for sufficiently large k, $\nu([2^k, 2^{k+1})) \leqq c 2^k$, and therefore,

$$\int_{(t\log t)^{1/\alpha}}^{\infty} (p_\alpha(t,x))\,\nu(dx) \leqq K'/\log t$$

for sufficiently large t and the proof is complete.

Lemma 3.2 (a) *Given $\nu \in \mathscr{M}_I(R^1)$, there exists a constant $c>0$ such that for every monotone decreasing function $f(x) \geqq 0$ on $[0,\infty)$ such that f is constant on $[0,1]$,*

$$\int_0^t f(x)\,\nu(dx) \leqq c \int_0^t f(x)\,\mu(dx), \qquad t \geqq 1.$$

(b) *Given $\nu \in \mathscr{M}_I(R^2)$ there exists a constant $c>0$ such that for any spherically symmetric function, $f(x) \geqq 0$, monotone decreasing as a function of the distance* *from the origin in R^2 and such that $f(.)$ is constant on the unit circle,*

$$\int_{\|x\|\leqq t} f(x)\,\nu(dx) \leqq c \int_{\|x\|\leqq t} f(x)\,\mu(dx), \qquad t\geqq 1.$$

Proof. (a) Let $F(x)\equiv\int_0^x \nu(dx)$. Since by assumption $F(x)/x$ converges to a finite constant as $x\to\infty$,

$$\overline{\lim_{x\geqq 1}}\, F(x)/x<\infty.$$

Hence there exists a constant $c>0$ such that $F(x)\leqq c\,x$ for $x\geqq 1$. Then for $t\geqq 1$,

$$\begin{aligned}\int_0^t f(x)\,\nu(dx) &= F(t)f(t)-\int_0^t F(x)\,df(x)\\ &\leqq F(t)f(t)-c\int_0^t x\,df(x)\\ &= F(t)f(t)-c\,t\,f(t)+c\,f(1)+c\int_1^t f(x)\,\mu(dx)\\ &\leqq c\int_0^t f(x)\,\mu(dx).\end{aligned}$$

The proof of (b) follows in essentially the same way.

Theorem 3.1. (a) In the recurrent critical case there is no nontrivial steady state random measure.

(b) For every compact set $K\subset R^d$, $\varepsilon>0$, and $\nu\in\mathcal{M}_I(R^d)$,

$$\lim_{t\to\infty} P_\nu(X_t(K)>\varepsilon)=0. \tag{3.5}$$

Proof. Let $\varphi=\varphi_1+i\,\varphi_2\in C_K^c(R^d)$, the class of continuous complexvalued functions with compact support. Then,

$$\mathbf{T}_t\varphi(x)=\{\varphi_1(x)+i[\varphi_2(x)(1+\gamma\,\varphi_2(x))+\gamma t\varphi_1^2(x)]\}/(1+\gamma t\varphi_2(x))^2. \tag{3.6}$$

Hence $\mathbf{T}_t$ maps $i\,C_K(R^d)$ into itself and if $\varphi\in C_K(R^d)$, then

$$u(t,x)\equiv \operatorname{Im}(\mathbf{T}_t(i\,\varphi(x))=\varphi(x)/(1+\gamma\,t\,\varphi(x)). \tag{3.7}$$

Then

$$u(t,x)=u(0,x)/(1+\gamma\,t\,u(0,x)); \qquad u(0,x)=\varphi(x)\in C_K(R^d), \tag{3.8}$$

and

$$d/dt(u(t,x))=-\gamma\,u^2(t,x). \tag{3.9}$$

Note that by taking φ to be purely imaginary we are effectively replacing the characteristic functional by the Laplace transform.

In a similar way we define

$$v(t,x) \equiv \operatorname{Im}(\mathbf{U}_t(i\,\varphi(x)), \qquad v(0,x)=\varphi(x)\in C_K(R^d). \tag{3.10}$$

Then

$$\partial/\partial t(v(t,x)) = \Delta^{\alpha/2}\, v(t,x) - \gamma\, v^2(t,x). \tag{3.11}$$

To prove (3.5) it suffices to show that

$$\lim_{t\to\infty} \int v(t,x)\, \nu(dx) = 0. \tag{3.12}$$

Let

$$S_t \equiv \{x\in R^d\colon\ \|x\| \leqq ((K+t)\log(K+t))^{1/\alpha}\} \tag{3.13}$$

with $K>0$. Let $\hat{v}(t,x)$ denote the solution of the equation

$$\partial/\partial t(\hat{v}(t,x)) = \Delta^{\alpha/2}\, \hat{v}(t,x); \qquad \hat{v}(0,x)=\varphi(x). \tag{3.14}$$

Note that

$$\hat{v}(t,x) \geqq v(t,x).$$

Without loss of generality we can assume that φ is spherically symmetric and monotone decreasing as a function of the distance from the origin in R^d. We also note that $\hat{v}(t,x)$ can be dominated by a multiple of $p_\alpha(t+s,x)$ for some $s>0$ since $p_\alpha(t,x)>0$ and is monotone decreasing in $\|x\|$ for each $t>0$. Lemma 3.1 implies that

$$\int_{S_t^c} \hat{v}(t,x)\, \nu(dx) \to 0 \quad \text{as } t\to\infty, \tag{3.15}$$

and hence

$$\int_{S_t^c} v(t,x)\, \nu(dx) \to 0 \quad \text{as } t\to\infty.$$

Equation (3.11) implies that $\operatorname{Sup}_x v(t,x)$ is dominated by the solution of Equation (3.9) and therefore

$$\operatorname{Sup}_x v(t,x) \leqq K/t \tag{3.16}$$

for sufficiently large t. Hence for any $K'<\infty$,

$$\int_{\|x\|\leqq K'} v(t,x)\, \nu(dx) \to 0 \quad \text{as } t\to\infty. \tag{3.17}$$

We can apply Lemma 3.2 to conclude that it suffices to show that

$$\int_{S_t} v(t,x)\, \mu(dx) \to 0 \quad \text{as } t\to\infty.$$

We now construct a comparison function $w(t,x)$.

Let

$$w(t,x) \equiv \hat{v}(t,x) \quad \text{if } x \in S_t^c$$
$$\equiv W(t) \quad \text{if } x \in S_t, \tag{3.18}$$

where $W(t)$ is the solution of the differential equation

$$d/dt(W(t)) = -\gamma W^2(t) + (f'(t)/f(t))\, W(t) + g(t)\, f(t), \tag{3.19}$$

$$W(0) = \int_{S_0} \varphi(x)\, \mu(dx)/\mu(S_0),$$

and

$$f(t) \equiv (\mu(S_t))^{-1} = c((K+t)\log(K+t))^{-d/\alpha}$$

and

$$g(t) = -d/dt\Big(\int_{S_t^c} v(t,x)\, \mu(dx)\Big).$$

Note that

$$\int_0^\infty g(t)\, dt < \infty.$$

Now let

$$Z(t) \equiv W(t)^{-1} f(t) = (W(t)\, \mu(S_t))^{-1},$$
$$Z(0) = Z_0 = (W(0)\, \mu(S_0))^{-1}. \tag{3.20}$$

From (3.19) and (3.20) we obtain

$$d/dt(Z(t)) = \gamma f(t) - g(t)\, Z^2(t). \tag{3.21}$$

Hence,

$$Z(t) = Z_0 + \gamma \int_0^t f(t)\, dt - \int_0^t g(t)\, Z^2(t)\, dt. \tag{3.22}$$

Let us assume that $\int_0^\infty f(t)\, dt = \infty$. If $\operatorname{Sup}_{t>0} Z(t) < \infty$, then we easily conclude that the right hand side of (3.22) has infinite supremum thus yielding a contradiction. Therefore we conclude that $Z(t) \to \infty$ as $t \to \infty$.

Hence if $d \leqq \alpha$,

$$\int_0^\infty (\mu(S_t))^{-1}\, dt = c \int_0^\infty ((K+t)\log(K+t))^{-d/\alpha}\, dt = \infty$$

thus implying by the above argument that $W(t)\, \mu(S_t) \to 0$.

To complete the proof of (3.5) we must verify that

$$\int_{S_t} v(t,x)\, \mu(dx) \leqq W(t)\, \mu(S_t). \tag{3.23}$$

For $t=0$,

$$W(0)\,\mu(S_0)=\int_{S_0} v(0,x)\,\mu(dx). \tag{3.24}$$

To prove (3.23) it suffices to show that if

$$\int_{S_t} v(t_0,x)\,\mu(dx)=W(t_0)\,\mu(S_{t_0}),$$

then

$$d/dt\Big[\int_{S_t} v(t,x)\,\mu(dx)\Big]\Big|_{t=t_0}<d/dt(W(t)\,\mu(S_t))|_{t=t_0}. \tag{3.25}$$

Note that

$$d/dt(W(t)\,\mu(S_t))=-\gamma\,W^2(t)/f(t)+g(t)$$

whereas

$$\begin{aligned} &d/dt\Big[\int_{S_t} v(t,x)\,\mu(dx)\Big]\\ &\quad=\int_{S_t}\Delta^{\alpha/2}v(t,x)\,\mu(dx)+\partial/\partial y\Big(\int_{\|x\|\leqq y} v(t,x)\,\mu(dx)\Big)\Big|_{y=S_t}-\gamma\int_{S_t} v^2(t,x)\,\mu(dx)\\ &\quad<g(t)-\gamma\int_{S_t} v^2(t,x)\,\mu(dx)\end{aligned}$$

for $t>0$ since the stable semigroup is conservative.

But by Schwarz's inequality,

$$\int_{St} v^2(t,x)\,\mu(dx)\geqq\Big(\int_{St} v(t,x)\,\mu(dx)\Big)^2(\mu(S_t))^{-1}$$

with equality only if $v(t,x)$ is constant on S_t.

Hence

$$\begin{aligned} d/dt\Big[\int_{S_t} v(t,x)\,\mu(dx)\Big]\Big|_{t=t_0}&<g(t)-W^2(t_0)\,\mu(S_{t_0})\\ &=d/dt(W(t)\,\mu(S_t))|_{t=t_0}\end{aligned}$$

and the proof of (b) is complete.

Part (a) follows by noting that if X_0 is a steady state spatially homogeneous random measure with finite intensity, then the ergodic theorem implies that $X_0\in\mathcal{M}_I(R^d)$ with probability one.

Remark 3.1. An analogous result has been obtained by completely different methods by Liemant and Matthes [10] for infinitely divisible point processes which are generated by an iterated "cluster" or "shower" operation.

4. The Transient Critical Case

The principal result of this section is given by the following theorem.

Theorem 3.1. *Let $\{X_t\colon t\geqq 0\}$ denote a transient critical measure diffusion process. Then there is a parameter family $\{P_\rho\colon 0<\rho<\infty\}$ of probability measures on*

$\mathscr{M}_I(R^d)$ which are both steady state random measures for the process and also invariant under spatial translation. The parameter ρ corresponds to the mean density of the random measure and $P_{\rho_1+\rho_2}=P_{\rho_1}*P_{\rho_2}$ where $*$ denotes the convolution operation.

Proof. Recall that

$$L_{t,X_0}(\varphi)=\exp(i\int \mathbf{U}_t\,\varphi(x)\,X_0(dx)). \tag{4.1}$$

In this proof we set $X_0=\rho\,\mu$.

The central step in the proof is to show that

$$\lim_{t\to\infty}\int \mathbf{U}_t\,\varphi(x)\,\mu(dx)\equiv \mathbf{U}_\infty\,\varphi \tag{4.2}$$

exists for all $\varphi\in C_K(R^d)$ and that $\mathbf{U}_\infty\,\varphi\neq 0$ for at least some $\varphi\in C_K(R^d)$. Then

$$L_{X_\infty}(\varphi)\equiv\exp(i\,\rho\,\mathbf{U}_\infty\,\varphi) \tag{4.3}$$

is the characteristic functional of a steady state random measure, X_∞, for the measure diffusion process. To verify this note that for $s>0$,

$$\begin{aligned} L_{\mathbf{W}_s^* X_\infty}(\varphi) &= \lim_{t\to\infty}\exp(i\int \mathbf{U}_{t+s}\varphi(x)\,\rho\,\mu(dx)) \\ &= \exp(i\,\rho\,\mathbf{U}_\infty\,\varphi)=L_{X_\infty}(\varphi). \end{aligned} \tag{4.4}$$

The translation invariance of the random field X_∞ is a direct consequence of the translation invariance of G and of the Lebesgue measure in R^d.

We now proceed to prove (4.2). Since the semigroup $\{\mathbf{U}_t\colon t\geqq 0\}$ has infinitesimal generator $(\Gamma+G)$, we note that for $\varphi\in C_K(R^d)$,

$$\mathbf{U}_t\,\varphi=\mathbf{S}_t\,\varphi+i\,\gamma\int_0^t \mathbf{S}_{t-s}((\mathbf{U}_s\,\varphi)(\mathbf{U}_s\,\varphi))\,ds. \tag{4.5}$$

We introduce the family of *finite binary rooted trees*, $\mathscr{T}$, with binary composition operation denoted by $\circ$. τ_e denotes the tree consisting of just one vertex and $\tau_1\circ\tau_2$ denotes the rooted tree in which the root is connected to copies of τ_1 and τ_2 respectively. We denote the tree $\tau_e\circ\tau_e$ by τ_s. The *order*, $|\tau|$, of $\tau\in\mathscr{T}$ is defined as the number of boundary vertices, that is, those vertices having at most one neighbour.

We note that formally the solution of (4.5) can be written in the form

$$\mathbf{U}_t=\mathbf{S}_t+\sum_{\tau\in\mathscr{T}-\tau_e}(i\,\gamma)^{|\tau|-1}\,\mathbf{U}_t^\tau \tag{4.6}$$

where the $\{\mathbf{U}_t^\tau\colon \tau\in\mathscr{T}\}$ are defined inductively as follows:

$$\begin{aligned} &\mathbf{U}_t^{\tau_e}\equiv\mathbf{S}_t \\ &\mathbf{U}_t^{\tau_1\circ\tau_2}(\varphi)\equiv\int_0^t \mathbf{S}_{t-s}(\mathbf{U}_s^{\tau_1}(\varphi)\cdot\mathbf{U}_s^{\tau_2}(\varphi))\,ds. \end{aligned} \tag{4.7}$$

Hence the $\{\mathbf{U}_t^\tau\}$ can be explicitly in terms of $\{\mathbf{S}_t\colon t\geqq 0\}$.

Let

$$\mathbf{U}_\infty^\tau \varphi \equiv \lim_{t\to\infty} \int \mathbf{U}_t^\tau \varphi(x)\,\mu(dx). \tag{4.8}$$

We now explicitly compute $\mathbf{U}_\infty^{\tau_s}\varphi$ in the case $\alpha=2$.

Then

$$\begin{aligned}\mathbf{U}_\infty^{\tau_s}\varphi = \lim_{t\to\infty}\Big[&\textstyle\iint\int_0^t (2\pi(t-s))^{-d/2}\exp(-\frac{1}{2}|y-x|^2/(t-s))\\ &\cdot\left(\int (2\pi s)^{-d/2}\exp(-|u-y|^2/2s)\,\varphi(u)\,\mu(du)\int (2\pi s)^{-d/2}\right.\\ &\left.\cdot\exp(-|v-y|^2/2s)\,\varphi(v)\,\mu(dv)\right)\mu(dy)\,\mu(dx)\,ds\Big].\end{aligned} \tag{4.9}$$

Note that

$$|u-y|^2+|v-y|^2=\tfrac{1}{2}|u-v|^2+2\,|y-\tfrac{1}{2}(u+v)|^2. \tag{4.10}$$

Hence,

$$\begin{aligned}\mathbf{U}_\infty^{\tau_s}\varphi &= \lim_{t\to\infty}\Big[\textstyle\iint\int_0^t (2\pi(t-s))^{-d/2}\exp(-\frac{1}{2}|y-x|^2/(t-s))\\ &\qquad\cdot\left(\iint (4\pi s)^{-d/2}\exp(-|u-v|^2/4s)\,(\pi s)^{-d/2}\right.\\ &\qquad\left.\cdot\exp(-|y-\tfrac{1}{2}(u+v)|^2/s)\,\varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv)\right)\mu(dx)\,\mu(dy)\,ds\Big]\\ &= \lim_{t\to\infty}\tfrac{1}{2}\int_0^t (2\pi s)^{-d/2}\exp(-|u-v|^2/2s)\,\varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv)\,ds\\ &= \tfrac{1}{2}\iint g_2(u-v)\,\varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv).\end{aligned} \tag{4.11}$$

The last integral is convergent since we have assumed that the motion process is transient.

For $\alpha \neq 2$, note that (c.f. Blumenthal and Getoor [2; 2.20]),

$$p_\alpha(t,x)=\int_0^\infty p_2(u,x)\,\eta_t^{\alpha/2}(du) \tag{4.12}$$

where $\eta_t^{\alpha/2}$ is the distribution at time t of a one-sided stable process with index $\alpha/2$.

In the case $\alpha \neq 2$, we then have

$$\begin{aligned}\mathbf{U}_\infty^{\tau_s}\varphi &= \lim_{t\to\infty}\int_0^t\left(\textstyle\int_0^\infty\iint (2\pi r)^{-d/2}\exp(-|u-v|^2/2r)\eta_s^{\alpha/2}(dr/2)\,\varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv)\right)ds\\ &= 2^{-\alpha/2}\iint g_\alpha(u-v)\,\varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv).\end{aligned} \tag{4.13}$$

To verify that the formal solution (4.6) is actually a true solution we must investigate the convergence properties of the right hand side of (4.6).

First, note that $\mathbf{U}_\infty^\tau$ can be extended from $C_K(R^d)$ in such a way as to include indicator functions of compact subsets of R^d. We now assume that φ is the indicator function of a convex compact subset of R^d of diameter D. Let T denote the maximum expected sojourn time of a particle which satisfies the motion process with infinitesimal generator 2Δ in a sphere of radius D centered at the origin where the maximum is taken over the initial position of the particle.

From (4.9) and the fact that if two particles belong to a convex set so does their center of gravity, it follows that

$$\begin{aligned}\mathbf{U}_t^{\tau_s}\varphi(x) &= \int\int_0^t (2\pi(t-s))^{-d/2}\exp(-\tfrac{1}{2}|y-x|^2/(t-s))\\ &\quad\cdot\left(\int\int(4\pi s)^{-d/2}\exp(-|u-v|^2/4s)(\pi s)^{-d/2}\right.\\ &\quad\left.\cdot\exp(-|y-\tfrac{1}{2}(u+v)|^2/s)\,\varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv)\right)\mu(dy)\,ds\\ &\leqq T E_x(\varphi(Y_t))\end{aligned}\tag{4.14}$$

where $\{Y_t\colon t\geqq 0\}$ is the Markov process describing the motion of the center of gravity of two particles each independently performing a Markov motion process in R^d with infinitesimal generator Δ. Note that the center of gravity process moves "more slowly" than the original particle process and has infinitesimal generator $\frac{1}{2}\Delta$. Proceeding inductively, it then follows that in the case $G=\Delta$,

$$|\mathbf{U}_t^{\tau}\varphi(x)|\leqq T^{|\tau|-2}\,|\tau|\,|\mathbf{U}_t^{\tau_s}\varphi(x)|\tag{4.15}$$

and hence

$$|\mathbf{U}_\infty^{\tau}\varphi(x)|\leqq T^{|\tau|-2}\,|\tau|\,|\mathbf{U}_\infty^{\tau_s}\varphi|.\tag{4.16}$$

The factor $|\tau|$ arises as a result of the "slowdown" in the motion of the center of gravity of $|\tau|$ particles compared to the original particle motion.

Then

$$\begin{aligned}|\mathbf{U}_\infty^{\tau}(\theta\varphi)| &= |\theta^{|\tau|}\,\mathbf{U}_\infty^{\tau}\varphi|\\ &\leqq T^{|\tau|-2}\,\theta^{|\tau|}\,|\tau|\,|\mathbf{U}_\infty^{\tau_s}\varphi|.\end{aligned}\tag{4.17}$$

We now return to consider the convergence of the right hand side of (4.6). Note that

$$\mathbf{U}_\infty(\theta\varphi)=\lim_{t\to\infty}\Big[\theta\int\varphi(x)\,\mu(dx)+\sum_{\tau\in\mathscr{T}-\tau_e}\theta^{|\tau|}(i\gamma)^{|\tau|-1}\int\mathbf{U}_t^{\tau}\varphi(x)\,\mu(dx)\Big].$$

Therefore, for $\varphi\geqq 0$, $\theta\geqq 0$,

$$|\mathbf{U}_\infty(\theta\varphi)|\leqq\theta\int\varphi(x)\,\mu(dx)+\sum_{\tau\in\mathscr{T}-\tau_e}\theta^{|\tau|}\,\gamma^{|\tau|-1}\,T^{|\tau|-2}\,|\tau|\,|\mathbf{U}_\infty^{\tau_s}\varphi|.\tag{4.18}$$

Note that

$$\sum_{\tau\in\mathscr{T}-\tau_e}\gamma^{|\tau|-1}\,\theta^{|\tau|}\,T^{|\tau|-2}\,|\tau|\,|\mathbf{U}_\infty^{\tau_s}\varphi|\leqq\sum_{n=2}^{\infty}|\theta|^n K^n\,n\,C(n)\tag{4.19}$$

where K is a constant independent of θ, and $C(n)$ denotes the number of binary rooted trees of order n.

However $C(n)$ is also the coefficient of θ^n in the Taylor series expansion of $f(\theta)$ when $f(.)$ satisfies

$$f(\theta)=a\,\theta+\gamma f^2(\theta), \qquad f(0)=0, \tag{4.20}$$

that is,

$$f(\theta)=(1-(1-4\gamma\,\theta\,a)^{\frac{1}{2}})/2\gamma.$$

Therefore the power series in θ given by the right hand side of (4.18) has a non-zero radius of convergence. Hence $\mathbf{U}_\infty(\theta\,\varphi)$ is well defined for sufficiently small θ and is given by the power series (4.18) in θ. Using property (4.12), the analogue of (4.14) can be proved in the case $\alpha \neq 2$ and then the remainder of the proof follows in essentially the same way. The only difference is that the factor $|\tau|$ in (4.15), etc. must be replaced by the factor $|\tau|^{\alpha/2}$. Hence the result is valid for any transient symmetric stable case.

Consider the induced random measure X_∞ with distribution P_ρ which corresponds to the characteristic functional (4.3). Since μ is invariant under translation, $\mathbf{U}_\infty \Phi_x \varphi = \mathbf{U}_\infty \varphi$ and hence X_∞ is a spatially homogeneous random measure. Using (4.18) and the usual moment generating properties of the characteristic functional, it can be shown that P_ρ has mean density ρ. The condition $P_{\rho_1+\rho_2}=P_{\rho 1} * P_{\rho_2}$ follows immediately from Equation (4.3) and the proof is complete.

A random measure whose characteristic functional satisfies

$$\log L(\theta\,\varphi_K)=\sum_{k=1}^{\infty} a_k(K)\,\theta^k,$$

$$\varphi_K(x)\equiv\varphi(x/K), \qquad K>0,$$

with positive radius of convergence for θ in the complex plane, and such that

$$\overline{\lim_{K\to\infty}}\, a_k(K)/K^d<\infty, \tag{4.21}$$

for each $k\geqq 1$ is said to be *B-mixing.*

Corollary 4.1. (i) *X_0 is not B-mixing.*

(ii) *X_∞ satisfies the strong law of large numbers, that is, for $\varphi\in C_K(R^d)$,*

$$\lim_{K\to\infty} (\langle X_\infty, \varphi_K\rangle/K^d)=\int \varphi(x)\,\rho\,\mu(dx) \qquad \textit{almost surely } [P_\rho]. \tag{4.22}$$

(iii) *The invariant distribution P_ρ is the unique invariant ergodic measure for the critical measure diffusion process with mean density ρ.*

Proof. We first note that (c.f. proof of Theorem 5.1) that

$$\mathrm{Var}(\langle X_\infty, \varphi_K\rangle)\sim K^{d+\alpha} \qquad \text{as } K\to\infty. \tag{4.23}$$

Hence (i) follows immediately.

But then,

$$\mathrm{Var}(\langle X_\infty, \varphi_K\rangle/K^d)\to 0 \qquad \text{as } K\to\infty$$

and therefore we obtain the weak law of large numbers, that is,

$$\langle X_\infty, \varphi_K\rangle / K^d \to \rho \int \varphi(x)\,\mu(dx)$$

in probability as $K\to\infty$. But since $P_\rho \in \mathbf{P}_I$, the pointwise ergodic theorem implies that

$$\langle X_\infty, \varphi_K\rangle / \left(\int \varphi(x)\,\mu(dx)\cdot K^d\right)$$

converges almost surely with respect to P_ρ as $K\to\infty$ to a random variable, R. But this together with the weak law of large numbers implies that $R=\rho$ almost surely $[P_\rho]$ and the proof of (ii) is complete.

Now let X denote an invariant spatially homogeneous random measure with distribution P and which satisfies the strong law of large numbers with mean density ρ. Then for $\varphi \in C_K(R^d)$,

$$L_{\mathbf{W}_t * X}(\varphi) = E\left(\exp\left(i \int \mathbf{U}_t\, \varphi(x)\, X(dx)\right)\right).$$

But $u(t,x) = \mathbf{U}_t\,\varphi(x)$ and hence from (4.5),

$$\begin{aligned}\int u(t,x)\,X(dx) &= \int \mathbf{S}_t\,\varphi(x)\,X(dx) + i\gamma \int_0^t \mathbf{S}_{t-s}\, u^2(s,x)\,ds\,X(dx)\\ &= \int \varphi(x)(p_\alpha(t)*X)(dx) + i\gamma \int_0^t u^2(s,x)(p_\alpha(t-s)*X)(dx)\,ds.\end{aligned}$$

In view of (3.17) and Lemma 3.2 it follows that

$$\int_{\frac{1}{2}t}^{t} u^2(s,x)(p_\alpha(t-s)*X)(dx)\,ds \to 0, \qquad \text{a.s. } [P]. \tag{4.24}$$

Let $\mathscr{I}$ denote the σ-subfield of $\mathscr{B}(\mathscr{M}(R^d))$ of events which are P-almost surely invariant under the transformations induced by spatial translations. The results of Debes et al. [4; Satz 1.6] imply that

$$\int \varphi(x)(p_\alpha(t)*X)(dx) \to E\left[\int \varphi(x)\,X(dx) \,|\, \mathscr{I}\right]$$

in $L^1(P)$ as $t\to\infty$ and also

$$\int_0^{\frac{1}{2}t} u^2(s,x)(p_\alpha(t-s)*X)(dx)\,ds \to E\left[\int_0^{\frac{1}{2}t} u^2(s,x)\,X(dx)\,ds \,|\, \mathscr{I}\right]$$

in $L^1(P)$ as $t\to\infty$.

But

$$E\left[\int \varphi(x)\,X(dx)\,|\,\mathscr{I}\right] = \rho \int \varphi(x)\,\mu(dx) \qquad \text{a.s. } [P]$$

and

$$E\left[\int_0^{\frac{1}{2}t} u^2(s,x)\,X(dx)\,ds\,|\,\mathscr{I}\right] = \rho \int_0^{\frac{1}{2}t} u^2(s,x)\,\mu(dx)\,ds \qquad \text{a.s. } [P].$$

Hence

$$\int \mathbf{U}_t\,\varphi(x)\,X(dx) \to \rho\,\mathbf{U}_\infty\,\varphi \tag{4.25}$$

in $L^1(P)$ as $t\to\infty$ and for an appropriate sequence $\{t_k\colon k \geqq 1\}$ the convergence is almost sure. Since X is an invariant random measure, this implies that

$$L_X(\varphi) = L_{X_\infty}(\varphi) = \exp(i\rho \mathbf{U}_\infty \varphi).$$

Hence $X = X_\infty$ in law and the uniqueness assertion is proved.

Finally, for $\varphi \in C_K(R^d)$ we have

$$E(\exp(i\langle X_\infty, \varphi\rangle)|\mathscr{I}) = \exp(i\rho \mathbf{U}_\infty \varphi) \quad \text{a.s. } [P_\rho].$$

Hence the conditional distribution of X_∞ given $\mathscr{I}$ is uniquely determined by ρ. Since distinct ergodic measures are mutually singular, this implies that there is no nontrivial ergodic decomposition of P_ρ and hence the proof of ergodicity is complete.

5. Renormalization Theory and the Spatial Central Limit Theorem

In this section we consider the renormalization theory for the steady state random measure X_∞ whose existence was established in 4. The limit theorem is a special type of functional central limit theorem which has a different form from the usual central limit theorem for B-mixing random measures.

We first introduce the renormalization transformation. Given $K > 0$ and $\varphi \in C_K(R^d)$ we define

$$\varphi_K(x) \equiv \varphi(x/K). \tag{5.1}$$

Consider a new random measure X_∞^K defined by

$$\langle X_\infty^K, \varphi\rangle \equiv \langle X_\infty, \varphi_K\rangle = \int \varphi(x/K)\, X_\infty(dx). \tag{5.2}$$

The effect of this transformation is to reduce the spatial dimensions by a factor of K, that is, the random measure assigned to the unit cube by X_∞^K equals the random measure assigned to the cube of side K by the X_∞-random measure.

The functional central limit theorem consists in finding constants a_K, b_K such that

$$(X_\infty^K - a_K)/b_K \tag{5.3}$$

converges in law to a limiting random field as $K \to \infty$.

Theorem 5.1. Let X_∞ denote the steady state random measure for a critical transient measure diffusion process in which the motion process is a Brownian motion or symmetric stable process. Then in the sense (5.3), X_∞ has as limiting random field a Gaussian random field with covariance kernel given by the potential kernel of the motion process.

Proof. We first describe the proof in detail in the case $\alpha = 2$ and then indicate how it can be modified in the case $\alpha \neq 2$.

We begin by investigating the effect of the renormalization transformation on the coefficients in the power series expansion (4.18). For computational convenience we do not work directly with $\mathbf{U}^{\tau}$ but rather consider the sequence of power series obtained by solving Equation (4.5) by iteration. Let

$$\begin{aligned} u_1^{(K)}(t,x) &\equiv \theta(2\pi t)^{-d/2}\int \exp(-\tfrac{1}{2}|y+x|^2/t)\,\varphi(y/K)\,\mu(dy) \\ &= \theta(2\pi t/K^2)^{-d/2}\int \exp(-\tfrac{1}{2}|y/K-x/K|^2/(t/K^2))\,\varphi(y/K)K^{-d}\mu(dy). \end{aligned}$$

Hence

$$u_1^{(K)}(t,x) = u_1(t/K^2, x/K). \tag{5.4}$$

Let us define recursively

$$\begin{aligned} u_{n+1}^{(K)}(t,x) \equiv u_1^{(K)}(t,x) + i\gamma \int_0^t\!\!\int (2\pi(t-s))^{-d/2}\exp(-\tfrac{1}{2}|y-x|^2/(t-s)) \\ \cdot [u_n^{(K)}(s,y)]^2\,\mu(dy)\,ds \end{aligned} \tag{5.5}$$

and

$$u_{n+1}(t,x) \equiv u_{n+1}^{(1)}(t,x).$$

Note that

$$u_n(s,y) = \sum_k \theta^k u_{n,k}(s,y). \tag{5.6}$$

We now prove that

$$u_n^{(K)}(s,y) = \sum_k \theta^k u_{n,k}(s/K^2, y/K)K^{2k-2}. \tag{5.7}$$

To prove (5.7) we use mathematical induction. We have shown in (5.4) that (5.7) is true for $n=1$.

But

$$\begin{aligned} u_{n+1}^{(K)}(t,x) &= u_1(t/K^2, x/K) \\ &\quad + i\gamma K^2 \int_0^t\!\!\int (2\pi(t-s)/K^2)^{-d/2}\exp(-\tfrac{1}{2}|y/K-x/K|^2/((t-s)/K^2)). \\ &[u_n^{(K)}(s,y)]^2 K^{-d}\mu(dy)K^{-2}\,ds \\ &= u_1(t/K^2,x/K) + i\gamma K^2 \int_0^t\!\!\int (2\pi(t/K^2-r))^{-d/2} \\ &\quad \cdot \exp(-\tfrac{1}{2}|u-x/K|^2/(t/K^2-r))\Big[\sum_k K^{2k-2}\theta^k u_{n,k}(r,u)\Big]^2\mu(du)\,dr \\ &= \sum_k \theta^k u_{n+1,k}(t/K^2,x/K)K^{2k-2} \end{aligned}$$

and hence the proof of (5.7) by induction is complete.

Let

$$u_\infty^{(K)}(t,x) \equiv \lim_{n\to\infty} u_n^{(K)}(t,x)$$
$$= \sum_{k=1}^{\infty} \theta^k u_{\infty,k}(t/K^2, x/K) K^{2(k-1)}. \tag{5.8}$$

Then

$$\int u_\infty^{(K)}(t,x)\,\mu(dx) = \int \left(\sum_{k=1}^{\infty} u_{\infty,k}(t/K^2, x/K) K^{2(k-1)} \theta^k\right) \mu(dx)$$
$$= \int \left(\sum_{k=1}^{\infty} u_{\infty,k}(t/K^2, u) K^{2(k-1)+d} \theta^k\right) \mu(du)$$

and hence

$$\lim_{t\to\infty} \int u_\infty^{(K)}(t,x)\,\mu(dx) = \sum_{k=1}^{\infty} \left[\lim_{t\to\infty} \int u_{\infty,k}(t,u)\,\mu(du)\right] K^{2(k-1)+d} \theta^k. \tag{5.9}$$

Now consider the case $\alpha \neq 2$. Recall that

$$p_\alpha(t,x) = \int_0^\infty p_2(u,x)\,\eta_t^{\alpha/2}(du).$$

Therefore

$$u_1^{(K),\alpha}(t,x) = \int_0^\infty \theta(2\pi u)^{-d/2} \int \exp(-\tfrac{1}{2}|y/K - x/K|^2/(u/K^2))\,\varphi(y/K)\,\eta_t^{\alpha/2}(du) K^{-d} \mu(dy)$$
$$= \int_0^\infty u_1^\alpha(u/K^2, x/K)\,\eta_t^{\alpha/2}(du)$$
$$= \int_0^\infty u_1^\alpha(v, x/K)\,\eta_{t/K^\alpha}^{\alpha/2}(dv)$$
$$= u_1^\alpha(t/K^\alpha, x/K).$$

Similarly we obtain,

$$u_n^{(K),\alpha}(s,y) = \sum_k \theta^k u_{n,k}(s/K^\alpha, y/K) K^{\alpha(k-1)}$$

and hence

$$u_\infty^{(K),\alpha}(t,x) = \sum_{k=1}^{\infty} \theta^k u_{\infty,k}(t/K^\alpha, x/K) K^{\alpha(k-1)}. \tag{5.10}$$

Hence as above,

$$\lim_{t\to\infty} \int u_\infty^{(K),\alpha}(t,x)\,\mu(dx)$$
$$= \sum_{k=1}^{\infty} \left[\lim_{t\to\infty} \int \mathrm{u}_{\infty,k}^\alpha(t,u)\,\mu(du)\right] K^{\alpha(k-1)+d} \theta^k. \tag{5.11}$$

Thus we have obtained the cumulant generating functional

$$\begin{aligned}\Psi_K(\theta) &\equiv \log E(\exp(i\langle X_\infty - E(X_\infty), \theta\varphi_K\rangle))\\ &= \sum_{k=2}^{\infty} a_k K^{\alpha(k-1)+d}\,\theta^k.\end{aligned}$$

For $k=2$, $\alpha(k-1)+d=\alpha+d$.
Hence

$$\Psi_K(\theta/K^{(\alpha+d)/2}) = \sum_{k=2}^{\infty} a_k K^{(d-\alpha)(1-\frac{1}{2}k)}\,\theta^k. \tag{5.12}$$

Since $d>\alpha$ and $k\geqq 2$,

$$\lim_{K\to\infty} \Psi_K(\theta/K^{(\alpha+d)/2}) = a_2\,\theta^2, \tag{5.13}$$

the cumulant generating functional of a Gaussian random field. The covariance kernel is given by

$$\begin{aligned}&\lim_{t\to\infty} [(\textstyle\int u_2^\alpha(t,x)\,\mu(dx) - \int \varphi(y)\,\mu(dy))/i\gamma]\\ &\quad = \lim_{t\to\infty} 2^{-\frac{1}{2}\alpha} \iint_0^t \left[\int_0^\infty (2\pi r)^{-d/2} \exp(-|u-v|^2/2r)\,\eta_s^{\alpha/2}(dr)\right]\\ &\qquad \cdot \varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv) = 2^{-\frac{1}{2}\alpha} \int g_\alpha(u-v)\,\varphi(u)\,\varphi(v)\,\mu(du)\,\mu(dv).\end{aligned}$$

Hence the covariance kernel is proportional to the potential kernel for the spatial motion process. Note that the limiting random field and its covariance kernel are invariant under the combined effect of the renormalization transformation and an appropriate scale change. Hence the proof is complete.

Remark 5.1. The type of limit theorem derived in this section is different from that which would be applicable in other situations such as in the case of the steady state random measure arising from a subcritical measure diffusion process with immigration or in the critical case at a finite time with uniform initial distribution. In these latter situations, the random measures are B-mixing and consequently the covariance kernel of the limiting Gaussian random field is given by the Dirac delta function. Limit theorems of the type considered in this section have recently received attention in the study of the renormalization theory of the Ising model at the critical temperature (c.f. Gallavotti and Martin-Löf [6]).

The Gaussian random fields whose covariance function is given by an inverse power law were first investigated by Whittle [11] in an attempt to describe empirical agricultural data. They are also characterized as the Gaussian random fields which are invariant under the combined effect of a renormalization and appropriate scale transformation. Of course it is reasonable to expect that a limiting random field would be a "fixed point" of the renormalization transformation. Therefore it is reasonable to expect that Gaussian random fields whose covariance kernels are given by a Riesz potential will serve as the limiting random fields for a wide class of homogeneous and isotropic random measures and fields. For example, in the context of this paper this is likely to be the case if the spatial motion process with infinitesimal generator G is in the domain of attraction of a symmetric stable law in an appropriate sense.

References

1. Athreya, K.B., Ney, P.: Branching Processes. Berlin-Heidelberg-New York: Springer 1972
2. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968
3. Dawson, D.A.: Stochastic Evolution Equations and related Measure Processes. J. multivariate Analysis **3**, 1–52 (1975)
4. Debes, H., Kerstan, J., Liemant, A., Matthes, K.: Verallgemeinerung eines Satzes von Dobrushin, I. Math. Nachr. **47**, 183–244 (1970)
5. Feller, W.: An introduction to Probability Theory and its Applications. Vol. II. New York: Wiley 1966
6. Gallavotti, G., Martin-Löf, A.: Block-spin distributions for shortrange attractive Ising models. Il Nuovo Cimento **25**, 425–444 (1975)
7. Ibragimov, I.A., Linnik, Yu.V.: Independent and stationary sequences of random variables. Groningen: Wolters-Noordhoff 1971
8. Jagers, P.: Aspects of Random Measures and Point Processes. Advances in Appl. Probability **3**, 179–238 (1974)
9. Jiřina, M.: Asymptotic behavior of measure-valued branching processes. Rozpravy Československé Akad. Věd. Řada Mat. Přirod. Věd. **76**, no. 1 (1966)
10. Liemant, V.A., Matthes, K.: Verallgemeinerung eines Satzes von Dobrushin, IV. Math. Nachr., **59**, 311–317 (1974)
11. Whittle, P.: Topographic correlation, power law covariance functions and diffusion. Biometrika **49**, 305–314 (1962)

Received January 10, 1977